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Scattering of spinning bodies by a radiation field in Schwarzschild spacetime

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Summary. — We extend the analysis of Poynting-Robertson effect, *i.e.*, the deviation from geodesic motion of test particles due to scattering by a superposed radiation field to the Schwarzschild background, to the case of spinning bodies. The extra contribution of the deviation due to spin can be relevant for astrophysical systems like the binary pulsar system PSR J0737-3039 orbiting Sgr A*, but not for the Earth-Sun system.

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1. – Introduction

The motion of classical spinning test particles in a given gravitational background is described by the well-known Mathisson-Papapetrou (MP) model [1, 2]. Let $U^\alpha = \mathrm{d}x^\alpha/\mathrm{d}\tau$ be the timelike unit tangent vector to the "center-of-mass line" $\mathcal{C}_U$ of the spinning particle used to perform the multipole reduction, parametrized by the proper time τ. The equations of motion are

$$\frac{\mathrm{D}P^\mu}{\mathrm{d}\tau} = -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}U^\nu S^{\alpha\beta} \equiv F^{(\mathrm{spin})\mu}, \tag{1}$$

$$\frac{\mathrm{D}S^{\mu\nu}}{\mathrm{d}\tau} = P^\mu U^\nu - P^\nu U^\mu, \tag{2}$$

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where P^μ is the total 4-momentum of the particle and the antisymmetric tensor $S^{\mu\nu}$ denotes the spin tensor (intrinsic angular momentum) associated with it; both fields are defined only along this center-of-mass world line. This system of 10 equations is consistently completed by the supplementary conditions (see [3,4] and references therein)

$$S^{\mu\nu}P_\nu = 0. \tag{3}$$

The spin structure of the particle should produce very small deviations from geodesic motion; otherwise, the particle backreaction on the spacetime metric should be taken into account.

Let us consider a spinless test particle orbiting a star which emits radiation. The radiation pressure of the light emitted by the star exerts a drag force on the particle's motion, which usually causes the body to fall into the star unless it is so small that the radiation pressure pushes it away from the star: this phenomenon is called the Poynting-Robertson effect [5, 6] and has been fully investigated in the context of both Newtonian and Post-Newtonian gravity. Recently the Poynting-Robertson effect for a spinless test particle orbiting a black hole was studied in both the Schwarzschild and Kerr spacetimes [7, 8] without the restriction of slow motion, but ignoring the finite size of the radiating body. Here we generalize the above discussion to the more realistic case of a spinning test particle undergoing the Poynting-Robertson effect by including the radiation force in the Mathisson-Papapetrou model.

2. – Schwarzschild spacetime and Poynting-Robertson effect

Consider a Schwarzschild spacetime, whose line element written in standard coordinates is given by

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + N^{-2}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \tag{4}$$

where $N = (1-2M/r)^{1/2}$ denotes the lapse function, and introduce the usual orthonormal frame adapted to the static observers (or Zero Angular Momentum Observers, ZAMOs) following the time lines

$$e_{\hat t} = N^{-1}\partial_t, \qquad e_{\hat r} = N\partial_r, \qquad e_{\hat\theta} = \frac{1}{r}\partial_\theta, \qquad e_{\hat\phi} = \frac{1}{r\sin\theta}\partial_\phi, \tag{5}$$

where $\{\partial_t, \partial_r, \partial_\theta, \partial_\phi\}$ is the coordinate frame.

We limit our analysis to the equatorial plane $\theta = \pi/2$.

2˙1. *Test radiation field.* – Let a pure electromagnetic radiation field be superposed as a test field on the gravitational background described by the metric (4), with the energy-momentum tensor

$$T^{\alpha\beta} = \Phi^2 k^\alpha k^\beta, \qquad k^\alpha k_\alpha = 0, \tag{6}$$

where k is assumed to be tangent to an affinely parametrized outgoing null geodesic in the equatorial plane, *i.e.*, $k^\alpha\nabla_\alpha k^\beta = 0$ with $k^\theta = 0$.

We will only consider photons in the equatorial plane which are in outward radial motion with respect to the ZAMOs, namely with 4-momentum

$$k = E(n)(n + e_{\hat{r}}), \tag{7}$$

where $n = e_{\hat{t}}$ is the ZAMO 4-velocity and $E(n) = E/N$ is the relative energy of the photon as seen by the ZAMOs. Here $E = -k_t$ is the conserved energy associated with the timelike Killing vector field ∂_t. Note also that in this case $L = k_\phi = 0$.

Since k is completely determined, the coordinate dependence of Φ then follows from the conservation equations $T^{\alpha\beta}{}_{;\beta} = 0$. For photons on the equatorial plane we find

$$\Phi = \frac{\Phi_0}{r}. \tag{8}$$

Consider now a test particle moving in the equatorial plane $\theta = \pi/2$ accelerated by the radiation field, *i.e.* with 4-velocity

$$U = \gamma(U,n)[n + \nu(U,n)], \qquad \nu(U,n) \equiv \nu^{\hat{r}} e_{\hat{r}} + \nu^{\hat{\phi}} e_{\hat{\phi}} = \nu(\sin\alpha e_{\hat{r}} + \cos\alpha e_{\hat{\phi}}), \tag{9}$$

where $\gamma(U,n) = 1/\sqrt{1 - ||\nu(U,n)||^2} \equiv \gamma$ is the Lorentz factor and the abbreviated notation $\nu^{\hat{a}} \equiv \nu(U,n)^{\hat{a}}$ has been used. Similarly $\nu \equiv ||\nu(U,n)||$ and α are the magnitude of the spatial velocity $\nu(U,n)$ and its polar angle measured clockwise from the positive ϕ direction in the r-ϕ tangent plane, respectively.

A straightforward calculation gives the coordinate components of U

$$U^t \equiv \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\gamma}{N}, \qquad U^r \equiv \frac{\mathrm{d}r}{\mathrm{d}\tau} = \gamma N \nu^{\hat{r}}, \qquad U^\phi \equiv \frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{\gamma\nu^{\hat{\phi}}}{r}, \tag{10}$$

where τ is the proper time parameter along $\mathcal{C}_U$, and $U^\theta \equiv \mathrm{d}\theta/\mathrm{d}\tau = 0$. The force associated with the scattering of radiation is given by [5, 6, 9]

$$F^{(\mathrm{rad})\alpha} = -\sigma\mathcal{P}(U)^\alpha{}_\beta T^\beta{}_\mu U^\mu, \tag{11}$$

where σ is the scattering cross section and $\mathcal{P}(U)^\alpha{}_\beta = \delta^\alpha_\beta + U^\alpha U_\beta$ projects orthogonally to U. Explicitly

$$F^{(\mathrm{rad})} = \frac{mA}{N^2 r^2}\gamma^3(1-\nu^{\hat{r}})\left[(\nu^{\hat{r}} - \nu^2)n + \left(1 - \nu^{\hat{r}} - (\nu^{\hat{\phi}})^2\right)e_{\hat{r}} - (1-\nu^{\hat{r}})\nu^{\hat{\phi}} e_{\hat{\phi}}\right], \tag{12}$$

where we have used the notation $\sigma\Phi_0^2 E^2 = mA$, as in ref. [7]. For a later use it is convenient to introduce a friction parameter $f \equiv A/M$, which again should be small enough not to perturb the spacetime. The motion is then described by the equations

$$ma(U)^\mu \equiv m\frac{\mathrm{D}U^\mu}{\mathrm{d}\tau} = F^{(\mathrm{rad})\mu}, \tag{13}$$

and has been studied in detail in ref. [7].

2˙2. *Spinning particles undergoing PR effect.* – The most direct generalization of eq. (13) to the case of spinning test particles consists in including the radiation force term in eq. (1), so that one has

$$\frac{\mathrm{D}P^\mu}{\mathrm{d}\tau} = F^{(\mathrm{spin})\,\mu} + F^{(\mathrm{rad})\,\mu}, \tag{14}$$

plus the additional relations (2) and (3) involving the spin. Let us proceed to analyse the motion of spinning particles undergoing Poynting-Robertson effect in the equatorial plane of the Schwarzschild spacetime.

Let $P = mu$ be the 4-momentum for motion in the equatorial plane, with

$$u = \gamma_u[n + \nu_u(\sin\alpha_u e_{\hat r} + \cos\alpha_u e_{\hat\phi})], \qquad \gamma_u = 1/\sqrt{1-\nu_u^2}, \tag{15}$$

and introduce the spin vector associated with $S_{\mu\nu}$ by spatial duality

$$S^\beta = u_\alpha \eta^{\alpha\beta\mu\nu} S_{\mu\nu}, \tag{16}$$

where $\eta_{\alpha\beta\gamma\delta} = \sqrt{-g}\epsilon_{\alpha\beta\gamma\delta}$ is the unit volume 4-form and $\epsilon_{\alpha\beta\gamma\delta}$ ($\epsilon_{0123} = 1$) is the Levi-Civita alternating symbol. It is also useful to consider the scalar invariant $s^2 = \frac{1}{2}S_{\mu\nu}S^{\mu\nu}$, constant along $\mathcal{C}_U$ because of eqs. (2) and (3). Consistency of the model requires $|\hat s| \equiv |s|/(mM) \ll 1$ as stated above.

From the evolution equations for the spin it follows that the spin vector has a single nonvanishing and constant component along θ, *i.e.*,

$$S = -S^{\hat\theta} e_{\hat\theta} = -s e_{\hat\theta}. \tag{17}$$

In the absence of both spin and radiation we assume the geodesic motion of the test particle to be circular at $r = r_0$, *i.e.*,

$$U = U_K = \gamma_K(n \pm \nu_K e_{\hat\phi}), \tag{18}$$

where the Keplerian value of speed (ν_K) and the associated Lorentz factor (γ_K) and angular velocity (ζ_K) are given by

$$\nu_K = \sqrt{\frac{M}{r_0 - 2M}}, \qquad \gamma_K = \sqrt{\frac{r_0 - 2M}{r_0 - 3M}}, \qquad \zeta_K = \sqrt{\frac{M}{r_0^3}}. \tag{19}$$

The parametric equations of U_K are

$$t_K = t_0 + \Gamma_K\tau, \quad r = r_0, \qquad \theta = \frac{\pi}{2}, \qquad \phi_K = \phi_0 \pm \Omega_K\tau, \tag{20}$$

where now t_0, r_0 and ϕ_0 are constants and

$$\Gamma_K = \sqrt{\frac{r_0}{r_0 - 3M}}, \qquad \Omega_K = \frac{1}{r_0}\sqrt{\frac{M}{r_0 - 3M}}. \tag{21}$$

Corrections to geodesic circular motion are given by

$$U = U_K + fU_f + \hat{s}U_{\hat{s}}, \qquad u = U + fu_f + \hat{s}u_{\hat{s}}, \tag{22}$$

to first order in both parameters $\hat{s}$ and f. The geodesic value are thus modified as

$$t = t_K + ft_f + \hat{s}t_{\hat{s}}, \qquad r = r_0 + fr_f + \hat{s}r_{\hat{s}}, \qquad \phi = \phi_K + f\phi_f + \hat{s}\phi_{\hat{s}}. \tag{23}$$

The solution is straightforward (see ref. [10] for details)

$$\begin{aligned}
t_{\hat{s}} &= \mp\frac{6M^2}{r_0}\frac{\Omega_K^3}{\Omega_{\rm ep}^3}[\sin(\Omega_{\rm ep}\tau) - \Omega_{\rm ep}\tau],\\
t_f &= 4r_0\zeta_K\nu_K^2\frac{\Omega_K^3}{\Omega_{\rm ep}^4}\left\{[\cos(\Omega_{\rm ep}\tau) - 1] + \frac{\Omega_{\rm ep}}{2r_0\zeta_K\Omega_K}[\sin(\Omega_{\rm ep}\tau) - \Omega_{\rm ep}\tau] + \frac{3}{8}\gamma_K^2\Omega_{\rm ep}^2\tau^2\right\},\\
r_{\hat{s}} &= \pm 3r_0\frac{\Omega_K\zeta_K}{\Omega_{\rm ep}^2}[\cos(\Omega_{\rm ep}\tau) - 1],\\
r_f &= -r_0\zeta_K\frac{\Omega_K}{\Omega_{\rm ep}^2}\left\{[\cos(\Omega_{\rm ep}\tau) - 1] - 2r_0\zeta_K\frac{\Omega_K}{\Omega_{\rm ep}}[\sin(\Omega_{\rm ep}\tau) - \Omega_{\rm ep}\tau]\right\},\\
\phi_{\hat{s}} &= \pm\frac{\zeta_K}{\nu_K^2}t_{\hat{s}}, \qquad \phi_f = \pm\frac{\zeta_K}{\nu_K^2}t_f, \qquad \Omega_{\rm ep} = \sqrt{\frac{M(r_0 - 6M)}{r_0^3(r_0 - 3M)}},
\end{aligned} \tag{24}$$

where $\Omega_{\rm ep}$ is the well-known epicyclic frequency governing the radial perturbations of circular geodesics.

As an application we can estimate the amount of variation of the radial distance by taking the mean values over a period of the perturbed radius, *i.e.*,

$$\left\langle\frac{\delta r}{r}\right\rangle \equiv \frac{r - r_0}{r_0} = \Gamma_K\frac{\zeta_K^2}{\Omega_{\rm ep}^2}\left[\left(1 - 2\pi r_0\zeta_K\frac{\Omega_K}{\Omega_{\rm ep}}\right)f \mp 3M\hat{s}\gamma_K\zeta_K N_0\right]. \tag{25}$$

For instance, for the motion of the Earth about the Sun we find

$$\left\langle\frac{\delta r}{r}\right\rangle \approx 3\times 10^{-15} \mp 4\times 10^{-15} \approx 10^{-15}, \tag{26}$$

whereas for the binary pulsar system PSR J0737-3039 as orbiting Sgr A* [11]

$$\left\langle\frac{\delta r}{r}\right\rangle \approx 7.6\times 10^{-19} \mp 1.8\times 10^{-7} \approx 10^{-7}. \tag{27}$$

Therefore, in the latter case the effect of the spin on the orbit dominates with respect to the friction due to the radiation field.

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