Kinetic theory in curved spacetimes: Applications to black holes

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Summary. — The equilibrium statistical moments of the Jüttner distribution function for a massive and a photon gas in an arbitrary spacetime are evaluated using a covariant approach and applications are considered to the case of a Schwarzschild black hole background spacetime. The motion of a massive test particle inside a photon gas is then studied to investigate drag effects on the particle motion due to radiation scattering, similarly to what happens for the so-called Poynting-Robertson effect.

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1. – Covariant kinetic theory

Consider a Minkowski flat spacetime with line element written in standard Cartesian coordinates with $x^0 = t$ and $\eta_{\alpha\beta} = \text{diag}[-1, 1, 1, 1](1)$

\begin{equation}
(1) \quad ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta = -dt^2 + \delta_{ab} dx^a dx^b
\end{equation}

and let

\begin{equation}
(2) \quad P^\beta = -E dt + p_a dx^a, \quad P = E \partial_t + p^a \partial_a,
\end{equation}

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(*) Here greek indices run from 0 to 3 whereas latin ones from 1 to 3. We also use geometrized units with $c = G = \hbar = 1$. © Società Italiana di Fisica
the fully covariant \( (P^\mu) \) as well as the contravariant \( (P) \) representations of the 4-momentum of a particle with nonzero rest mass \( m \), i.e.

\[
P^2 \equiv P \cdot P = -m^2 = E^2 + \delta_{ab} p^a p^b \equiv -E^2 + p^2.
\]

Let us consider a gas of such particles all equal and point-like in equilibrium at the (absolute) temperature \( T \). The Jüttner distribution function [1]

\[
f = \alpha e^{-\beta \sqrt{P^2 + m^2}}, \quad \alpha = \frac{n\beta}{4\pi m^2 K_2(m\beta)},
\]

is the correct special relativistic extension with respect to the metric (1) of the Maxwell-Boltzmann distribution function, accounting for a finite maximum speed of the particles. Here \( n \) is the particles density number, \( \beta = 1/(k_B T) \), with \( k_B \) the Boltzmann constant, is the “inverse temperature” and \( K_2(x) \) is the modified Bessel function of second kind of second order of argument \( x \) [2]. An equivalent manifestly covariant form of eq. (4) is

\[
f = \alpha e^{\beta \xi_\mu P^\mu},
\]

where \( \xi_\mu = (\partial_t)_\mu \) is the time-like Killing vector of the metric (1), associated with the temporal coordinate \( t \) and \( P^\mu \) is given by eq. (2) (see, e.g., [3-6] for a review of Kinetic theory in a curved spacetime). Recently [7] a covariant method to evaluate the statistical moments of \( f \) in the metric (1) has been introduced leading to the following expressions for the density current of particles and the stress-energy tensor:

\[
N^\mu = 2 \int f \delta^+(P^2 + m^2) P^\mu \sqrt{-g} d^4P = \frac{1}{\beta} \frac{\partial I}{\partial \xi_\mu},
\]

\[
T^{\mu\nu} = 2 \int f \delta^+(P^2 + m^2) P^\mu P^\nu \sqrt{-g} d^4P = \frac{1}{\beta^2} \frac{\partial^2 I}{\partial \xi_\mu \partial \xi_\nu},
\]

where functional generator \( I \) is given by

\[
I = 2 \int f \delta^+(P^2 + m^2) \sqrt{-g} d^4P = \int f \frac{\sqrt{-g}}{|p^1|} dp^1 \wedge dp^2 \wedge dp^3.
\]

Here \( g = -1 \) is the determinant of the metric (1) and the Dirac delta function takes into account the mass-shell condition \( P_\mu P^\mu = -m^2 \) (the overall factor of 2 is necessary for a \( P \) future-oriented). Using the following parametrization for \( P \) (so that the normalization mass-shell condition is automatically satisfied)

\[
P = m \left[ \cosh \chi \dot{t} + \sinh \chi \dot{\nu} \right], \quad \dot{\nu} = \sin \theta \cos \phi \dot{x} + \sin \theta \sin \phi \dot{\nu} + \cos \theta \dot{z},
\]

with \( 0 \leq \chi \leq \infty, \ 0 \leq \theta \leq \pi, \ 0 \leq \phi \leq 2\pi \), we can evaluate the integral \( I \)

\[
I = \frac{4\pi \alpha m}{\beta} K_1(m\beta),
\]
where \( K(x) \) is the modified Bessel function of second kind of first order of argument \( x \). The statistical moments follow easily due to the following property of the Bessel functions:

\[
\frac{d}{dx}[x^{-n}K_n(x)] = -x^{-n}K_{n+1}(x),
\]

leading to the following expressions:

\[
N^\mu = N\xi^\mu, \quad T^{\mu\nu} = \frac{N}{\beta}\left(\eta^{\mu\nu} + \xi^\mu \xi^\nu J(m\beta)\right), \quad N = \frac{4\pi m^2\alpha K_2(m\beta)}{\beta},
\]

where \( J(x) = xK_3(x)/K_2(x) \).

Similarly, in the case of a photon gas (a gas of particles with null 4-momentum \( P \)) we can perform the same integral \( I \) by changing the parametrization of \( P \)

\[
P = \mathcal{E}(\partial_t + \hat{v}), \quad \hat{v} = \sin \theta \cos \phi \partial_x + \sin \theta \sin \phi \partial_y + \cos \theta \partial_z,
\]

where \( \mathcal{E} \) is the photon energy as measured by the fiducial observers and \( 0 \leq \chi \leq \infty, \ 0 \leq \theta \leq \pi, \ 0 \leq \phi \leq 2\pi \) because in this case the mass-shell constraint in the momentum space is \( P_\mu P^\mu = 0 \). Now the (divergence-free and trace-free) stress-energy tensor is

\[
T^{\mu\nu} = \frac{C}{3\beta^4}\left[\eta^{\mu\nu} + 4\xi^\mu \xi^\nu\right].
\]

where in our units \( C = \frac{\pi^2}{15} \) has been determined comparing our result with the black-body theory.

Passing then to a generic spacetime with coordinates \( x^\alpha \) and metric \( ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta \), still admitting a killing timelike vector field \( \xi \), it is easy to check that the above relations for a gas of massive particles gas can be extended by simply including a redshift factor for the temperature, that is

\[
N^\mu = N m^\mu, \quad T^{\mu\nu} = \frac{N}{\beta \xi}\left(g^{\mu\nu} + m^\mu m^\nu J(m\beta\xi)\right), \quad N = \frac{4\pi m^2\alpha K_2(m\beta\xi)}{\beta \xi};
\]

similarly, for a photon gas

\[
T^{\mu\nu} = \frac{C}{3\beta^4 \xi^4}\left[g^{\mu\nu} + 4m^\mu m^\nu\right],
\]

where \( N = \sqrt{-N_\mu N^\mu}, \ m^\mu = \xi^\mu / \xi \) (unitary and timelike) with \( \xi = \sqrt{-\xi_\mu \xi^\mu} \) representing the fiducial congruence of observers. Written in this form both quantities \( N^\mu \) and \( T^{\mu\nu} \) can be obtained from the generating functional

\[
I = (4\pi m^2)^2 \frac{K_1(m \beta \xi)}{m \beta \xi}
\]

and satisfy the corresponding conservation laws, \( \nabla_\mu N^\mu = 0 \) and \( \nabla_\mu T^{\mu\nu} = 0 \). In the next section we will explicitly evaluate these quantities in the case of a Schwarzschild black-hole spacetime.
2. – Test gases on a Schwarzschild background

Let us consider as a background spacetime the Schwarzschild metric written in standard coordinates \((t, r, \theta, \phi)\)

\[
ds^2 = -N(r)^2 dt^2 + N(r)^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,
\]

\(N(r) = \sqrt{1 - \frac{2M}{r}},\)

\((M\) is the mass of the black hole). The metric (18) is static, \(i.e.\) it admits the time-like Killing vector \(\xi = \partial_t\). A family of fiducial observers with 4-velocity aligned with \(\partial_t\) has the following adapted orthonormal frame:

\[
e^\hat{t} = N(r)^{-1} \partial_t, \quad e^\hat{r} = \frac{1}{r} \partial_r, \quad e^\hat{\theta} = \frac{1}{r \sin \theta} \partial_\theta, \quad e^\hat{\phi} = \frac{1}{r} \sin \theta \partial_\phi.
\]

According to the general results of the previous section the energy-momentum tensor of a fluid of massive particles is characterized by

\[
T = \frac{n K^2(m \beta N(r))}{\beta N^2(r) K^2(m \beta)} \left[(J(m \beta N(r)) - 1) e^\hat{t} \otimes e^\hat{t} + \delta^{\hat{a}}_{\hat{b}} e^\hat{a} \otimes e^\hat{b}\right],
\]

and

\[
N = \frac{4 \pi m^2 \alpha K^2(m \beta N(r))}{\beta N(r)},
\]

whereas a photon gas corresponds to

\[
T = \frac{C}{(\beta N(r))^4} \left[3 e^\hat{t} \otimes e^\hat{t} + \delta^{\hat{a}}_{\hat{b}} e^\hat{a} \otimes e^\hat{b}\right].
\]

2’.1. Scattering by a radiation field. – Let us assume that a test photon gas, described by the energy-momentum tensor (22), is superposed to the Schwarzschild background and let us consider a single massive test particle in motion with 4-velocity

\[
U = \gamma \left(e^\hat{t} + \nu^\hat{a} e^\hat{a}\right), \quad \gamma = \frac{1}{\sqrt{1 - \delta_{\hat{a}\hat{b}} \nu^\hat{a} \nu^\hat{b}}}.
\]

The particle is then accelerated by the radiation field. Denoting by \(a(U)^\alpha = \nabla_U U^\alpha\) the particle’s 4-acceleration, the equations of motion are given by

\[
ma(U)^\alpha = -\sigma P(U)^\alpha_\mu T^{\mu\nu} U^\nu,
\]

where \(\sigma\) is the cross section of the process (\(e.g.\), Thomson scattering) and \(P(U)^\hat{a}_{\hat{\nu}} = \delta^\hat{a}_{\hat{\nu}} + U^\hat{a} U_{\hat{\nu}}\) projects orthogonally to \(U\). Let us limit our analysis to equatorial motion of
the test particle, i.e., $\theta = \pi/2, \quad \nu^\theta = 0$, allowed thanks to the spherical symmetry of the problem either in the presence of the radiation field. The equations of motion reduce to

$$
\frac{d\nu^\nu}{d\tau} = - \frac{A\nu^\nu}{N^4(r)} - N(r)\frac{\gamma}{r} \left[ \nu_K^2 (1 - (\nu^r)^2) - (\nu^\phi)^2 \right],
$$

$$
\frac{dr}{d\tau} = \gamma N(r)\nu^r,
$$

$$
\frac{d\nu^\phi}{d\tau} = - \frac{A\nu^\phi}{N^4(r)} + \frac{\gamma N(r)}{r \nu_K^2} \nu^\theta \nu^r,
$$

$$
\frac{d\phi}{d\tau} = \frac{\gamma}{r} \nu^\phi,
$$

where we have introduced the Keplerian velocity $\nu_K$ with the Lorentz factor $\gamma_K = (1 - \nu_K^2)^{-1/2}$ and the coupling constant $A$ between the test particle and the field:

$$
\nu_K^2 = \frac{M}{r - 2M}, \quad \gamma_K^2 = \frac{r - 2M}{r - 3M}, \quad A = 8 \frac{\sigma C}{m^2 \beta T}.
$$

The numerical integration of these equations shows that a spiral (inward) motion is the general feature; in particular a particle initially at $r = 6M$ (innermost stable circular orbit for the Schwarzschild metric) with initial null radial velocity and initial azimuthal velocity coinciding with the geodesic value $\nu_K$ falls anyway into the black hole. In addition, we see that there are not equilibrium orbits, irrespective of the value of $\sigma$ which quantifies the intensity of the process.

This study broadens the one present in [8,9] where a different description of the photon field is considered. Comparing and contrasting more in detail with the above-mentioned works as well as with other related literature will be the object of a future work.

REFERENCES


