IL NUOVO CIMENTO Vol. 36 C, N. 1 Suppl. 1 Gennaio-Febbraio 2013
DOI 10.1393/ncc/i2013-11484-7

Colloquia: 12th Italian-Korean Symposium 2011

Kinetic theory in curved spacetimes: Applications to black holes

D. Bini(1)(2)(3)(*) and D. Gregoris(4)(2)(**)

(1) *Istituto per le Applicazioni del Calcolo "M. Picone", CNR - I-00185 Roma, Italy*
(2) *ICRA, University of Rome I "La Sapienza" - I-00185 Roma, Italy*
(3) *INFN, Sezione di Firenze - I-50019 Sesto Fiorentino (FI), Italy*
(4) *University of Rome I "La Sapienza" - I-00185 Roma, Italy*

ricevuto il 9 Marzo 2012

Summary. — The equilibrium statistical moments of the Jüttner distribution function for a massive and a photon gas in an arbitrary spacetime are evaluated using a covariant approach and applications are considered to the case of a Schwarzschild black hole background spacetime. The motion of a massive test particle inside a photon gas is then studied to investigate drag effects on the particle motion due to radiation scattering, similarly to what happens for the so-called Poynting-Robertson effect.

PACS 04.20.Cv – Fundamental problems and general formalism.

1. – Covariant kinetic theory

Consider a Minkowski flat spacetime with line element written in standard Cartesian coordinates with $x^0 = t$ and $\eta_{\alpha\beta} = \mathrm{diag}[-1, 1, 1, 1]$(1)

$$\mathrm{d}s^2 = \eta_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta = -\mathrm{d}t^2 + \delta_{ab}\mathrm{d}x^a \mathrm{d}x^b \tag{1}$$

and let

$$P^\flat = -E\mathrm{d}t + p_a \mathrm{d}x^a, \qquad P = E\partial_t + p^a\partial_a, \tag{2}$$

(*) E-mail: binid@icra.it
(**) E-mail: danielegregoris@libero.it
(1) Here greek indices run from 0 to 3 whereas latin ones from 1 to 3. We also use geometrized units with $c = G = \hbar = 1$.

the fully covariant ($P^\flat$) as well as the contravariant (P) representations of the 4-momentum of a particle with nonzero rest mass m, *i.e.*

$$P^2 \equiv P \cdot P = -m^2 = -E^2 + \delta_{ab} p^a p^b \equiv -E^2 + \mathbf{p}^2. \tag{3}$$

Let us consider a gas of such particles all equal and point-like in equilibrium at the (absolute) temperature T. The Jüttner distribution function [1]

$$f = \alpha e^{-\beta\sqrt{\mathbf{p}^2+m^2}}, \qquad \alpha = \frac{n\beta}{4\pi\, m^2 K_2(m\beta)}, \tag{4}$$

is the correct special relativistic extension with respect to the metric (1) of the Maxwell-Boltzmann distribution function, accounting for a finite maximum speed of the particles. Here n is the particles density number, $\beta = 1/(k_B T)$, with k_B the Boltzmann constant, is the "inverse temperature" and $K_2(x)$ is the modified Bessel function of second kind of second order of argument x [2]. An equivalent manifestly covariant form of eq. (4) is

$$f = \alpha e^{\beta \xi_\mu P^\mu}, \tag{5}$$

where $\xi_\mu = (\partial_t)_\mu$ is the time-like Killing vector of the metric (1), associated with the temporal coordinate t and P^μ is given by eq. (2) (see, *e.g.*, [3-6] for a review of Kinetic theory in a curved spacetime). Recently [7] a covariant method to evaluate the statistical moments of f in the metric (1) has been introduced leading to the following expressions for the density current of particles and the stress-energy tensor:

$$N^\mu = 2\int f\,\delta^+(P^2+m^2)\,P^\mu\sqrt{-g}\mathrm{d}^4P = \frac{1}{\beta}\frac{\partial I}{\partial \xi_\mu}, \tag{6}$$

$$T^{\mu\nu} = 2\int f\,\delta^+(P^2+m^2)\,P^\mu P^\nu\sqrt{-g}\mathrm{d}^4P = \frac{1}{\beta^2}\frac{\partial^2 I}{\partial \xi_\mu \partial \xi_\nu}, \tag{7}$$

where functional generator I is given by

$$I = 2\int f\delta^+(P^2+m^2)\sqrt{-g}\mathrm{d}^4P = \int f\frac{\sqrt{-g}}{|p_t|}\mathrm{d}p^1\wedge\mathrm{d}p^2\wedge\mathrm{d}p^3. \tag{8}$$

Here $g = -1$ is the determinant of the metric (1) and the Dirac delta function takes into account the mass-shell condition $P_\mu P^\mu = -m^2$ (the overall factor of 2 is necessary for a P future-oriented). Using the following parametrization for P (so that the normalization mass-shell condition is automatically satisfied)

$$P = m\Big[\cosh\chi\,\partial_t + \sinh\chi\,\hat\nu\Big], \qquad \hat\nu = \sin\theta\cos\phi\partial_x + \sin\theta\sin\phi\partial_y + \cos\theta\partial_z, \tag{9}$$

with $0 \le \chi \le \infty,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi$, we can evaluate the integral I

$$I = \frac{4\pi\alpha m}{\beta}K_1(m\beta), \tag{10}$$

where $K_1(x)$ is the modified Bessel function of second kind of first order of argument x. The statistical moments follow easily due to the following property of the Bessel functions:

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{-n}K_n(x)] = -x^{-n}K_{n+1}(x), \tag{11}$$

leading to the following expressions:

$$N^\mu = N\xi^\mu, \qquad T^{\mu\nu} = \frac{N}{\beta}\Big(\eta^{\mu\nu} + \xi^\mu\xi^\nu J(m\beta)\Big), \qquad N = \frac{4\pi m^2\alpha K_2(m\beta)}{\beta}, \tag{12}$$

where $J(x) = xK_3(x)/K_2(x)$.

Similarly, in the case of a photon gas (a gas of particles with null 4-momentum P) we can perform the same integral I by changing the parametrization of P

$$P = \mathcal{E}\left(\partial_t + \hat{\nu}\right), \qquad \hat{\nu} = \sin\theta\cos\phi\partial_x + \sin\theta\sin\phi\partial_y + \cos\theta\partial_z, \tag{13}$$

where $\mathcal{E}$ is the photon energy as measured by the fiducial observers and $0 \leq \chi \leq \infty,\ 0 \leq \theta \leq \pi,\ 0 \leq \phi \leq 2\pi$ because in this case the mass-shell constraint in the momentum space is $P_\mu P^\mu = 0$. Now the (divergence-free and trace-free) stress-energy tensor is

$$T^{\mu\nu} = \frac{C}{3\beta^4}\Big[\eta^{\mu\nu} + 4\xi^\mu\xi^\nu\Big]. \tag{14}$$

where in our units $C = \frac{\pi^2}{15}$ has been determined comparing our result with the black-body theory.

Passing then to a generic spacetime with coordinates x^α and metric $\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta$, still admitting a killing timelike vector field ξ, it is easy to check that the above relations for a gas of massive particles gas can be extended by simply including a redshift factor for the temperature, that is

$$N^\mu = Nm^\mu, \qquad T^{\mu\nu} = \frac{N}{\beta\xi}\Big(g^{\mu\nu} + m^\mu m^\nu J(m\beta\xi)\Big), \qquad N = \frac{4\pi m^2\alpha K_2(m\beta\xi)}{\beta\xi}; \tag{15}$$

similarly, for a photon gas

$$T^{\mu\nu} = \frac{C}{3\beta^4\xi^4}\Big[g^{\mu\nu} + 4m^\mu m^\nu\Big], \tag{16}$$

where $N = \sqrt{-N_\mu N^\mu}$, $m^\mu = \xi^\mu/\xi$ (unitary and timelike) with $\xi = \sqrt{-\xi_\mu\xi^\mu}$ representing the fiducial congruence of observers. Written in this form both quantities N^μ and $T^{\mu\nu}$ can be obtained from the generating functional

$$\mathcal{I} = (4\pi\alpha m^2)\frac{K_1(m\beta\xi)}{m\beta\xi} \tag{17}$$

and satisfy the corresponding conservation laws, $\nabla_\mu N^\mu = 0$ and $\nabla_\mu T^{\mu\nu} = 0$. In the next section we will explicitly evaluate these quantities in the case of a Schwarzschild black-hole spacetime.

2. – Test gases on a Schwarzschild background

Let us consider as a background spacetime the Schwarzschild metric written in standard coordinates (t, r, θ, ϕ)

$$\mathrm{d}s^2 = -N(r)^2\mathrm{d}t^2 + N(r)^{-2}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2, \qquad N(r) = \sqrt{1-\frac{2M}{r}}, \tag{18}$$

(M is the mass of the black hole). The metric (18) is static, *i.e.* it admits the time-like Killing vector $\xi = \partial_t$. A family of fiducial observers with 4-velocity aligned with ∂_t has the following adapted orthonormal frame:

$$e_{\hat{t}} = N(r)^{-1}\partial_t, \qquad e_{\hat{r}} = N(r)\partial_r, \qquad e_{\hat{\theta}} = \frac{1}{r}\partial_\theta, \qquad e_{\hat{\phi}} = \frac{1}{r\sin\theta}\partial_\phi. \tag{19}$$

According to the general results of the previous section the energy-momentum tensor of a fluid of massive particles is characterized by

$$T = \frac{n\,K_2(m\beta N(r))}{\beta N^2(r)\,K_2(m\beta)}\left[(J(m\beta N(r)) - 1)\,e_{\hat{t}}\otimes e_{\hat{t}} + \delta^{\hat{a}\hat{b}}e_{\hat{a}}\otimes e_{\hat{b}}\right], \tag{20}$$

and

$$N = \frac{4\pi m^2\alpha K_2(m\beta N(r))}{\beta N(r)}, \tag{21}$$

whereas a photon gas corresponds to

$$T = \frac{C}{(\beta N(r))^4}\left[3e_{\hat{t}}\otimes e_{\hat{t}} + \delta^{\hat{a}\hat{b}}e_{\hat{a}}\otimes e_{\hat{b}}\right]. \tag{22}$$

2˙1. *Scattering by a radiation field.* – Let us assume that a test photon gas, described by the energy-momentum tensor (22), is superposed to the Schwarzschild background and let us consider a single massive test particle in motion with 4-velocity

$$U = \gamma\left(e_{\hat{t}} + \nu^{\hat{a}}e_{\hat{a}}\right), \qquad \gamma = \frac{1}{\sqrt{1-\delta_{\hat{a}\hat{b}}\nu^{\hat{a}}\nu^{\hat{b}}}}. \tag{23}$$

The particle is then accelerated by the radiation field. Denoting by $a(U)^\alpha = \nabla_U U^\alpha$ the particle's 4-acceleration, the equations of motion are given by

$$ma(U)^\alpha = -\sigma P(U)^\alpha{}_\mu T^{\mu\nu}U_\nu, \tag{24}$$

where σ is the cross section of the process (*e.g.*, Thomson scattering) and $P(U)^{\hat{\alpha}}{}_{\hat{\nu}} = \delta^{\hat{\alpha}}{}_{\hat{\nu}} + U^{\hat{\alpha}}U_{\hat{\nu}}$ projects orthogonally to U. Let us limit our analysis to equatorial motion of the test particle, *i.e.*, $\theta = \pi/2, \quad \nu^{\hat\theta} = 0$, allowed thanks to the spherical symmetry of the problem either in the presence of the radiation field. The equations of motion reduce to

$$
\begin{aligned}
\frac{\mathrm{d}\nu^{\hat r}}{\mathrm{d}\tau} &= -\frac{A\nu^{\hat r}}{N^4(r)} - N(r)\frac{\gamma}{r}\left[\nu_K^2(1-(\nu^{\hat r})^2) - (\nu^{\hat\phi})^2\right], \\
\frac{\mathrm{d}r}{\mathrm{d}\tau} &= \gamma N(r)\nu^{\hat r}, \\
\frac{\mathrm{d}\nu^{\hat\phi}}{\mathrm{d}\tau} &= -\frac{A\nu^{\hat\phi}}{N^4(r)} + \frac{\gamma N(r)}{r\gamma_K^2}\nu^{\hat\phi}\nu^{\hat r}, \\
\frac{\mathrm{d}\phi}{\mathrm{d}\tau} &= \frac{\gamma}{r}\nu^{\hat\phi},
\end{aligned} \tag{25}
$$

where we have introduced the Keplerian velocity ν_K with the Lorentz factor $\gamma_K = (1 - \nu_K^2)^{-1/2}$ and the coupling constant A between the test particle and the field:

$$
\nu_K^2 = \frac{M}{r-2M}\,, \qquad \gamma_K^2 = \frac{r-2M}{r-3M}\,, \qquad A = 8\frac{\sigma C}{m\beta^4}\,. \tag{26}
$$

The numerical integration of these equations shows that a spiral (inward) motion is the general feature; in particular a particle initially at $r = 6M$ (innermost stable circular orbit for the Schwarzschild metric) with initial null radial velocity and initial azimuthal velocity coinciding with the geodesic value ν_K falls anyway into the black hole. In addition, we see that there are not equilibrium orbits, irrespective of the value of σ which quantifies the intensity of the process.

This study broadens the one present in [8,9] where a different description of the photon field is considered. Comparing and contrasting more in detail with the above-mentioned works as well as with other related literature will be the object of a future work.

REFERENCES

[1] Jüttner F., *Ann. Phys. Chem.*, **34** (1911) 856; **35** (1911) 145; *Z. Naturforsch. A: Phys. Sci.*, **47** (1928) 542.

[2] Gradshteyn I. S. and Ryzhik I. M., *Tables of Integrals, Series and Products* (Academic Press, New York) 1980.

[3] Bernstein J., *Kinetic Theory in the Expanding Universe* (Cambridge University Press) 1988.

[4] Cercignani C. and Kremer G. M., *The Relativistic Boltzmann Equation: Theory and Applications* (Springer-Verlag, Birkhäuser, Basel) 2002.

[5] Ehlers J., Ford J., George C., Miller R., Montroll E., Schieve W. C. and Turner J. S., *Lectures in Statistical Physics* (Springer-Verlag, Berlin-Heidelberg-New York) 1974.

[6] Stewart J. M., *Non-Equilibrium Relativistic Kinetic Theory*, *Lect. Notes Phys.*, Vol. **10** (Springer) 1971.

[7] Chacon-Acosta G., Dagdug L. and Morales-Tecotl H. A., *Phys. Rev. E*, **81** (2010) 021126.

[8] Bini D., Jantzen R. and Stella L., *Class. Quantum Grav.*, **26** (2009) 055009.

[9] Bini D., Geralico A., Jantzen R., Semerák O. and Stella L., *Class. Quantum Grav.*, **28** (2011) 035008.