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Electromagnetic waves in gravitational wave spacetimes

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Summary. — We have considered the propagation of electromagnetic waves in a space-time representing an exact gravitational plane wave and calculated the induced changes on the four-potential field A^μ of a plane electromagnetic wave. By choosing a suitable photon round-trip in a Michelson interferometer, we have been able to identify the physical effects of the exact gravitational wave on the electromagnetic field, *i.e.* phase shift, change of the polarization vector, angular deflection and delay. These results have been exploited to study the response of an interferometric gravitational wave detector beyond the linear approximation of the general theory of relativity. A much more detailled examination of this problem can be found in our paper recently published in *Classical and Quantum Gravity* (**28** (2011) 235007).

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1. – Introduction

The spacetime metric of an exact gravitational plane wave with a single polarization state (+ state) can be written in the "Rosen form" as follows [1, 2]:

$$g = -\frac{1}{2}(\mathrm{d}u \otimes \mathrm{d}v + \mathrm{d}v \otimes \mathrm{d}u) + F(u)^2 \mathrm{d}x \otimes \mathrm{d}x + G(u)^2 \mathrm{d}y \otimes \mathrm{d}y, \tag{1}$$

where the coordinates (u, v, x, y) are adapted to the spacetime symmetries; *i.e.* ∂_v, ∂_x, ∂_y are all Killing vectors. The two null coordinates u and v are related to a standard temporal coordinate t and a spatial coordinate z (the direction of propagation of the wave) by the transformation $u = t - z$ and $v = t + z$. The vacuum Einstein field equations associated with eq. (1) reduce to the single equation $R_{uu} = 0$, *i.e.*

$$\frac{F''(u)}{F(u)} + \frac{G''(u)}{G(u)} = 0, \tag{2}$$

where a prime denotes differentiation with respect to u. The wave is then propagating along the positive z-axis with axes of polarization aligned with the coordinate axes x and y. In the following we will consider a sandwich-wave solution, *i.e.* a curved-spacetime region in the interval $u \in [0, a^2/\tau]$ between two Minkowskian regions, where the constant parameters a and τ have been introduced, with τ representing the duration of the interaction of particles or fields with the wave and $1/a$ the overall curvature of the wave region. A possible choice of metric functions is the following [2]:

$$F(u) = \begin{cases} 1, & u \le 0, & \text{(I)} \\ \cos(u/a), & 0 \le u \le a^2/\tau, & \text{(II)} \\ \alpha + \beta u, & a^2/\tau \le u, & \text{(III)} \end{cases} \tag{3}$$

$$G(u) = \begin{cases} 1, & u \le 0, & \text{(I)} \\ \cosh(u/a), & 0 \le u \le a^2/\tau, & \text{(II)} \\ \gamma + \delta u, & a^2/\tau \le u, & \text{(III)} \end{cases}$$

where labels I, II and III refer to In-zone, Wave-zone and Out-zone, respectively. The constants α, β, γ and δ can be found by requiring C^1 regularity conditions at the boundary of the sandwich, $u = 0$ and $u = a^2/\tau$, that is

$$\alpha = \cos\left(\frac{a}{\tau}\right) + \frac{a}{\tau}\sin\left(\frac{a}{\tau}\right), \qquad \beta = -\frac{1}{a}\sin\left(\frac{a}{\tau}\right), \tag{4}$$

$$\gamma = \cosh\left(\frac{a}{\tau}\right) - \frac{a}{\tau}\sinh\left(\frac{a}{\tau}\right), \qquad \delta = \frac{1}{a}\sinh\left(\frac{a}{\tau}\right).$$

2. – Scattering of electromagnetic waves by the gravitational wave

Maxwell's equation in the Lorenz gauge, $\Box A_\alpha \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu A_\alpha = 0$, $\nabla_\mu A^\mu = 0$, can be easily solved for the vector potential A, in this case leading to

$$A^\flat = \frac{A_0}{\sqrt{FG}}\, e^{i\phi}\, e^\flat. \tag{5}$$

This solution represents a field, which is not a wave in general, propagating in a direction associated with positive v, x, y coordinates. Since it is a wave in region I (see [3]), we will refer to ϕ as the phase and e_μ as the polarization vector of the field also in the other regions. In general, the phase ϕ is given by

$$\phi = \left(\int^u p_u \mathrm{d}u\right) + p_v v + p_x x + p_y y, \qquad p_u = p_u(u) = \frac{1}{4p_v}\left(\frac{p_x^2}{F^2} + \frac{p_y^2}{G^2}\right), \tag{6}$$

and the contravariant polarization vector by

$$e = -\frac{1}{p_v}\left(\frac{p_x}{F}\cos(\vartheta) + \frac{p_y}{G}\sin(\vartheta)\right)\partial_v + \frac{\cos(\vartheta)}{F}\partial_x + \frac{\sin(\vartheta)}{G}\partial_y. \tag{7}$$

The explicit expression for the "phase" in region III is generically a function of the coordinates u, v, x, y and can be re-xpressed in Cartesian coordinates U, V, X, Y by a transformation $U = u$, $X = F(u)x$, $Y = G(u)y$, $V = v + F(u)F'(u)x^2 + G(u)G'(u)y^2$. It is dominated by its value along the null geodesics, namely

$$\phi_{III(\mathrm{d})} = \frac{Q_x^2 + Q_y^2}{4Q_v}U + Q_vV + Q_xX + Q_yY + \tilde{C}_{III} = Q_\alpha X^\alpha + \tilde{C}_{III}, \tag{8}$$

In fact, let us consider the "phase" given by eq. (6) in region III and along a generic curve $X^\alpha = X^\alpha(\lambda)$ as a function of the parameter λ along that curve, and require its variation $\mathrm{d}\phi_{III}/\mathrm{d}\lambda$ to be vanishing in order to determine the dominant part. We find that this extremal condition is satisfied exactly by the null geodesics in region III (see [3]). In addition, we can say that even if our general solution for the electromagnetic field after the passage of the gravitational wave is not exactly a plane wave, it is dominated by a plane wave with the wave vector aligned with that of a null geodesic of the background, with the phase given by eq. (8). Similarly to what happens for the phase the "polarization" vector is also dominated by the corresponding value along the null geodesics (see [3]), in the sense that the e^X_{III} and e^Y_{III} components do not depend on the curve, while the e^V_{III} component reaches its extremal value on the null geodesics, namely

$$e_{III(\mathrm{d})} = -\frac{1}{Q_v}\left[Q_x\cos(\vartheta) + Q_y\sin(\vartheta)\right]\partial_V + \cos(\vartheta)\partial_X + \sin(\vartheta)\partial_Y, \tag{9}$$

where the $Q_\alpha = \mathrm{const}$ are the components of the dominant wave vector as emerging after the scattering by the gravitational wave (analogous to the components p_x, p_y, p_v of the four-momentum of an unperturbed photon).

We will now consider the variation in the properties of the electromagnetic wave, by comparing the dominant parts of the solutions before and after the passage of the gravitational wave. The contravariant polarization vector has a variation only in the v-component, namely

$$\begin{aligned}\Delta e_v = &\left(\left[1-\cos\left(\frac{a}{\tau}\right)\right]\frac{p_x}{p_v} - 2\sin\left(\frac{a}{\tau}\right)\frac{x_0}{a}\right)\cos(\vartheta)\\ &+\left(\left[1-\cosh\left(\frac{a}{\tau}\right)\right]\frac{p_y}{p_v} + 2\sinh\left(\frac{a}{\tau}\right)\frac{y_0}{a}\right)\sin(\vartheta).\end{aligned} \tag{10}$$

After the passage of the gravitational wave and in terms of the dominant mode analysis discussed above, the phase of the electromagnetic wave is shifted by

$$\Delta\phi = -\frac{a}{4p_v}\left[p_x^2\tan\left(\frac{a}{\tau}\right) + p_y^2\tanh\left(\frac{a}{\tau}\right)\right] + p_v(v_s - v_0), \tag{11}$$

where the x_0, y_0, v_0 relate to a generic starting point $P_s = (u_s, v_s, x_s, y_s)$. Note that the transformed coordinates (U, V, X, Y) are Cartesian, so that the new metric functions are such that $F_{III} = G_{III} = 1$. As a consequence, the amplitude of the dominant part of the electromagnetic field is unaffected by the passage of the gravitational wave.

3. – Photon moving along one axis

Let us now consider the motion of photons along x or y axes which represent the direction of the arm of a Michelson interferometer with the beam splitter in the origin [4]. The photons start at the beam-splitter (denoted by $*$) and are reflected once by an end mirror at a distance L from the origin, denoted by small s (we regard the mirrors as fixed and therefore do not consider the timelike geodesics associated with them). At the start of proper time, $\lambda = 0$, the photon is assumed at the generic point $P_{s,x}$ or $P_{s,y}$ on the mirror (where $x_s = L$ and $y_s = 0$ or $x_s = 0$ and $y_s = L$), where the momentum is p_x or p_y in negative x- or y-direction (towards the origin). In $P_{s,x}$ and $P_{s,y}$ is $v_s = u_s$, ensuring $z_s = 0$. The momenta p_x and p_y are constrained by demanding $z_* = 0$ in the origin, namely $p_x = 2p_v$, $p_y = 2p_v$, $p_v < 0$. The choice of negative momentum p_v ensures that u increases with λ.

We consider two photons (or photon beams), making the round trip through the interferometer along the x- and y-axis, respectively, and arriving again at the beam splitter afterwards. The photons start from the origin at $u^* = u_s + L$, and we can foresee two possible scenarios: I) The photons travel from the origin to the mirror unperturbed and are reflected by the mirror (in $P'_{s,x}$ or $P'_{s,y}$) at $u'_s = u_s + 2L$. On the return trip they encounter the gravitational wave, and return to the origin at $\tilde{U}^{*\prime}_x$ or $\tilde{U}^{*\prime}_y$ respectively. II) The photons encounter the gravitational wave on the way to the mirror, where they are reflected at $\tilde{U}'_{s,x}$ or $\tilde{U}'_{s,y}$ (depending on the interferometer arm we consider). They return to the origin in the post-wave region and arrive there again at $\tilde{U}^{*\prime}_x$ or $\tilde{U}^{*\prime}_y$ respectively. The two photons emerging from the gravitational wave experience a deflection in Z-direction of

$$\tilde{Z}^{*\prime}_x = \frac{1}{2}\left[\left(\frac{Q_x}{p_v} + 1\right) L + u_s - \tilde{U}^{*\prime}_x\right], \qquad \tilde{Z}^{*\prime}_y = \frac{1}{2}\left[\left(\frac{Q_y}{p_v} + 1\right) L + u_s - \tilde{U}^{*\prime}_y\right], \tag{12}$$

respectively, and arrive at the origin at a coordinate time

$$\begin{aligned} \tilde{U}^{*\prime}_x &= \frac{a^2}{\tau} + a\frac{2L + (L + u_s)\cos\left(\frac{a}{\tau}\right) - a\sin\left(\frac{a}{\tau}\right)}{(L + u_s)\sin\left(\frac{a}{\tau}\right) + a\cos\left(\frac{a}{\tau}\right)}, \\ \tilde{U}^{*\prime}_y &= \frac{a^2}{\tau} - a\frac{2L + (L + u_s)\cosh\left(\frac{a}{\tau}\right) - a\sinh\left(\frac{a}{\tau}\right)}{(L + u_s)\sinh\left(\frac{a}{\tau}\right) - a\cosh\left(\frac{a}{\tau}\right)}. \end{aligned} \tag{13}$$

We can regard the path of a single photon travelling through the interferometer as the center of a photon beam. The deflection decreases the intensity of the interference pattern of the two photon beams, but the magnitude of the deflection compared to the cross section of the photon beam is very small, whereas the change in the interference pattern of two photon beams due to their shifted phase is a much greater effect. It should also be noted that the expressions for the deflection and delay of photons arriving at the origin after the passage of the wave are the same for scenarios I and II. They are distinguished by the relation between u_s and L. In scenario I the photons have to leave the origin at $L < |u^*| < 2L$ to meet the wave on the return trip towards the origin, while in scenario II the photons leave from the origin at $0 < |u^*| < L$ in order to encounter the wave on the way to the mirror. (The above computations required an examination of the geodesics detailled in [3].)

When the two photon beams arrive at the beam splitter again after the round trip, they have a relative phase shift

$$\Delta\phi = \Delta\phi_x - \Delta\phi_y = -a\tan\left(\frac{a}{\tau}\right) + a\tanh\left(\frac{a}{\tau}\right) \tag{14}$$

and a relative change in polarization

$$\begin{aligned}\Delta e_v = {} & 2\left[1-\cos\left(\frac{a}{\tau}\right) - \frac{u_s+L}{a}\sin\left(\frac{a}{\tau}\right)\right]\cos(\vartheta) \\ & -2\left[1-\cosh\left(\frac{a}{\tau}\right) + \frac{u_s+L}{a}\sinh\left(\frac{a}{\tau}\right)\right]\sin(\vartheta).\end{aligned} \tag{15}$$

The scenario considered (I or II) is distinguished by the choice of u_s in terms of L as above, yielding different expressions for the relative change in polarization. The relative phase shift depends only on the dimension of the gravitational wave, not on the construction of the interferometer.

4. – Concluding remarks

We have considered the propagation of a test electromagnetic field on the background of an exact gravitational plane wave with single (+) polarization, and have extended the recent analysis of Finn in ref. [5], in which the gravitational wave has been considered in the linear approximation. The existence of an exact solution for the phase shift between the In-zone and Out-zone, where special relativity holds and there is no residual gauge freedom of general relativity, makes us more confident about the physical interpretation of the response of a gravitational interferometer. In this respect, we should also mention that, in the limit of weak gravitational wave ($a \gg 1$), we recover the results of Rakhmanov in ref. [6]. In this limit mention should also be given to the pioneering work of Mashhoon and Grishchuk [7]. It is worth noticing that the solution for the electromagnetic field consists of a plane wave in the In-zone and a non singular field in the Out-zone which emerges after the interaction. We found that the emerging field is dominated by plane wave behaviour where the wave vector is aligned with the null geodesics of the background. Furthermore, we determined the phase shift between the ingoing electromagnetic wave and the dominant part of the outgoing field as the significant response of a Michelson interferometer to the presence of an exact gravitational wave. In addition, we have calculated the change of the polarization vector, and the angular deflection and delay of photon beams making the round trip in the interferometer. No matter how small these effects are, they could be measured by using different detection methods.

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