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Generic features of the cosmological evolution of density parameters

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Summary. — The evolution of various energy components with dark energy was examined. Recently many non-standard gravity models were suggested to explain the current observational data showing an accelerating phase since the recent past. All suggested models should mimic ΛCDM somehow, especially from the near past to the current epoch. However, most of them do not try to explain or predict what happens if their model were extended to the far past and/or the past. In this paper we want to address this point by analyzing the critical points of the evolution equations and their stability. Standard ΛCDM gives three critical points, radiation dominated, matter dominated, and cosmological constant dominated. Furthermore, the radiation-dominated point corresponds to the past stable point, the matter-dominated point to the saddle point, and the cosmological-constant–dominated point to the future stable point. This means that this model predicts that the universe starts from radiation domination then passes through a matter-dominated era and finally evolves into a cosmological-constant–dominated era, that is, the future de Sitter phase. We applied these creteria to few $f(R)$ gravity models to determine viable parameter ranges.

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1. – Introduction

Various recent observational data indicate that our universe is in an accelerating phase since the recent past [1-4]. The standard ΛCDM could explain these observational results within error bounds. However, this model has some critical problems including the cosmological constant problem. There are many models to overcome these problems including the $f(R)$ gravity model [5-9]. But most of the modified models concentrated in explaining the recent past acceleration and the current domination of the dark energy (or cosmologcal constant). There is also detailed analysis and classification for future evolutions [8, 10] and numerical works [11-13].

On the other hand, the statefinder model could explain the acceleration of the universe and the difference between the cosmological models in order to better fit the observational data [14]. Furthermore, the mapping of the cosmic expansion of the universe [15] is very crucial to understand the underlying physics. Hence, it is necessary to have observational data in the various redshift ranges.

In this paper, we want to make an analysis from the viewpoint of critical points and their stability [16]. Since we know that there exist at least radiation and (dark) matter, we concentrate on the behaviour of these components of the energy among others. The density parameters, which are defined as $\Omega_X = \frac{\rho_X}{\rho_c}$, have values between 0 and 1. Therefore, it is a natural choice to take them as dynamical variables for numerical and analytical analyses.

In sect. **2**, we will describe the relevants equations for the cosmological evolution. We assume the Friedmann-Lemaître-Roberston-Walker (FLRW) model for the metric in this paper. The critical points and their stability will be explained in sect. **3**. Some applications of our method to some cosmological models including the standard ΛCDM are presented in sect. **4**. Finally, we discuss the implications of our method in sect. **5**.

2. – Evolution equations

The standard procedure starting from the action for various gravity models leads to the following Hubble equation:

$$H^2 = \frac{8\pi G}{3}\left(\rho_{\rm m} + \rho_{\rm r} + \rho_{\rm X}\right), \tag{1}$$

where $\rho_{\rm m}$, $\rho_{\rm r}$, and $\rho_{\rm X}$ are dust matter densities, including dark matter density, radiation matter density, and densities of other types, respectively. Here $\rho_{\rm X}$ represents any other type of matter density collectively depending on the cosmological models. We also assume thre FLRW metric as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2\right), \tag{2}$$

where $a(t)$ is the scale factor. With this metric, the Ricci scalar is given by

$$R = 6\left(\dot{H} + 2H^2\right). \tag{3}$$

Now we define some useful variables as follows:

$$\rho_{\rm c} = \frac{3H^2}{8\pi G}, \tag{4a}$$

$$\Omega_{\rm m} = \frac{\rho_{\rm m}}{\rho_{\rm c}} = \frac{8\pi G}{3H^2}\rho_{\rm m}^0 e^{-3x}, \tag{4b}$$

$$\Omega_{\rm r} = \frac{\rho_{\rm r}}{\rho_{\rm c}} = \frac{8\pi G}{3H^2}\rho_{\rm r}^0 e^{-4x}, \tag{4c}$$

$$\Omega_{\rm X} = \frac{\rho_{\rm X}}{\rho_{\rm c}}, \tag{4d}$$

$$r = \frac{R}{12H^2} = \frac{1}{2H}\frac{\mathrm{d}H}{\mathrm{d}x} + 1, \tag{4e}$$

where ρ_c is the critical density, $x = \ln a$, r is the "reduced Ricci scalar", and we have assumed the standard equation of state for dust (dark) matter and radiation matter as 0 and 1/3, respectively.

The Hubble equation (1) can be rewritten as

$$1 = \Omega_m + \Omega_r + \Omega_X. \tag{5}$$

The evolution equations for the density parameters can be obtained by direct derivatives as

$$\frac{d\Omega_m}{dx} = -(4r-1)\,\Omega_m, \tag{6a}$$

$$\frac{d\Omega_r}{dx} = -4r\Omega_r, \tag{6b}$$

$$\frac{dr}{dx} = \frac{1}{12H^2}\frac{dR}{dx} - 4r\,(r-1)\,, \tag{6c}$$

$$\frac{dH}{dx} = 2\,(r-1)\,H. \tag{6d}$$

Equation (6c) could be redundant according to the model.

If we assume that any viable cosmological model does not evolve to big rip or any singularity [16], all density paarmeters should have a finite value during the cosmological evolution. Also the far-future and far-past state should not change with cosmological time, that is, it is a critical point of eqs. (6a) and (6b). These critical points might be also critical points for eqs. (6c) and (6d) depending on the underlying cosmological model. In the next section we will present a detailed analysis for critical points of eqs. (6) and their stability.

3. – Critical points and their stability

Mathematically, the critical point for a given function $f(x)$ is defined as a point, x, such that $f(x)$ is not differentiable or its first derivative is zero, $f'(x) = 0$ [17]. We extend this notion of critical point to apply the cosmological evolution problem. Usually, the differential equations of the cosmological evolution are implicit in the cosmological time, t or $x = \ln a$. Hence it is not possible to apply a mathematical definition for the cosmological evolution problem.

The system of differential equations is given by

$$h_i'(x) = H_i(h_1, h_2, \ldots; p_1, p_2, \ldots), \qquad i = 1, 2, \ldots, \tag{7}$$

where h_i denote the interesting dynamical variables and p_i the parameters of a certain cosmological model. The number of dynamical variables and model parameters are dependent on a certain cosmological model. We can define the critical points for the system of differential equations, eq. (7), as a set of dynamical variables $h_i^0, i = 1, 2, \ldots$, satisfying

$$H_i(h_1^0, h_2^0, \ldots; p_1, p_2, \ldots) = 0, \qquad i = 1, 2, \ldots. \tag{8}$$

These points too are called fixed points. The meaning of such critical points is the following: To find out the solution of eq. (7), we should supply some initial values for the dynamical variables, h_i. If we had set the initial values as one of the critical points, then all the dynamical variables will have the same value as the given initial values. That is, it does not evolve anymore. However, the initial state could not give the *exact* critical point, for some reasons. For this case, the dynamical variables evolve with cosmological time then finally approach one of the critical points or diverge, depending on the cosmological model and/or the initial state. In this case, the approaching critical point could be understood as an attarctor of the model.

In this sense the meaning of the critical point is different from its mathematical one. In the mathematical definition, the critical points are given as a point of the independent variable of the cosmological time, t or x, but for our case they are given as a set of specific dynamical variables. This is viable, since we want to find out a meaningful state of the universe not at a specific time but far future or far past to see its evolution. The decision for which one is the final and the initial state could be given by analyzing its stability as follows.

Substituting the linear perturbations $h_i' \to h_i' + \delta h_i'$ about the critical points into eq. (7), to the first order of perturbations, we can obtain the eigenvalues $\lambda_i, i = 1, 2, \ldots$. The number of eigenvalues depends on the number of independent equations, which depends on the cosmological model and the eigenvalues might depend on the model parameters p_j. Stability requires the real part of the eigenvalues to be negative. This means that a small deviation from the critical points having negative eigenvalues will disappear as it evolves. Hence the stable critical points are attractors for far-future evolution. Likewise, the critical points having positive eigenvalues are absolutely unstable for a small deviation. This means that the unstable critical points are stable if we evlove to the past instead of the future. Therefore, the positive eigenvalued critical points are attractors for a far-past evolution. Finally, the critical points having both positive and negative eigenvalues correspond to saddle points, which are intermediate states of the cosmological evolution.

In the next section, we will apply this idea to some specific cosmological models to find their viable parameter ranges to have far-futre and far-past cosmological states.

4. – Applications

In this section we will present the some applications of the previous sections for some well-known cosmological models.

4˙1. ΛCDM *case*. – For the famous ΛCDM model, the reduced Ricci scalar r is not an independent variable but is given as

$$r = 1 - \frac{3}{4}\Omega_{\rm m} - \Omega_{\rm r}, \tag{9}$$

which can be obtained by defferentiating the Hubble equation with the cosmologcal constant and substituting it into the definition of the reduced Ricci scalar, eq. ($4e$). Then the evolution equation, eqs. ($6c$) and ($6d$), can be rewriten as

$$\frac{{\rm d}r}{{\rm d}x} = \frac{9}{4}\left(1 - \Omega_{\rm m} - \frac{4}{3}\Omega_{\rm r}\right)\Omega_{\rm m} + 4\left(1 - \frac{3}{4}\Omega_{\rm m} - \Omega_{\rm r}\right)\Omega_{\rm r}, \tag{10}$$

$$\frac{{\rm d}H}{{\rm d}x} = -\left(\frac{3}{2}\Omega_{\rm m} + 2\Omega_{\rm r}\right)H, \tag{11}$$

respectively.

TABLE I. – *The critical points and their eigenvalues for the* ΛCDM *case.*

Label	$\Omega_{\rm m}$	$\Omega_{\rm r}$	r	λ_1	λ_2	Note
$P_{1\rm CDM}$	0	1	0	1	4	far-past attarctor
$P_{2\rm CDM}$	1	0	$\frac{1}{4}$	−1	3	intermediate state
$P_{3\rm CDM}$	0	0	1	−4	−3	far-future attractor

In this case, only eqs. (6*a*) and (6*b*) are independent. The evolution equations become

$$\frac{\mathrm{d}\Omega_{\rm m}}{\mathrm{d}x} = -\left(3 - 3\Omega_{\rm m} - 4\Omega_{\rm r}\right)\Omega_{\rm m}, \tag{12a}$$

$$\frac{\mathrm{d}\Omega_{\rm r}}{\mathrm{d}x} = -\left(4 - 3\Omega_{\rm m} - 4\Omega_{\rm r}\right)\Omega_{\rm r}. \tag{12b}$$

The critical points and eigenvalues are given in table I. Note that all three cirtical points are also the critical points for (10). However, only the critical point of $P_{3\rm CDM}$ is the critical point for (11). This means that the final state of ΛCDM is a de Sitter space with $R = 12H^2 =$ constant.

The critical points show that it is natural that the evolution of the universe starts from the radiation-dominated era ($\Omega_{\rm r} = 1, \Omega_{\rm m} = 0, \Omega_\Lambda = 0$), passing through the matter-dominated era ($\Omega_{\rm r} = 0, \Omega_{\rm m} = 1, \Omega_\Lambda = 0$) then finally approaches the dark-energy–dominated era ($\Omega_{\rm r} = 0, \Omega_{\rm m} = 0, \Omega_\Lambda = 1$).

4˙2. $f(R)$ *gravity*. – For this case and the subsequent two cases, we confine ourselves to the $f(R)$ model. In those cases, r is not a simple dependent varibale but should satisfy the differential equation (6*c*). This equation can be rewritten as [18]

$$\frac{\mathrm{d}r}{\mathrm{d}x} = \frac{F}{12H^2F'}\left(-1 + 2r + \frac{\Omega_{\rm m} + \Omega_{\rm r}}{F} - \frac{1}{6H^2}\frac{f}{F}\right) - 4r\left(r - 1\right), \tag{13}$$

where $F(R) = f'(R)$. We can obtain the critical points by solving eqs. (6*a*), (6*b*), and (13). In principle, one can find solutions at least numerically but it is unformidable. Hence in this paper we only consider to find the condition giving the far-future attractor as a standard ΛCDM case. Detailed analysis will be done elsewhere.

For the power law $f(R)$ form

$$f(R) = R + f_0 R^\alpha, \tag{14}$$

its derivatives with respect to R are given by

$$F(R) = 1 + \alpha f_0 R^{\alpha-1}, \qquad F'(R) = \alpha(\alpha - 1) f_0 R^{\alpha-2}. \tag{15}$$

The critical points, which mimic ΛCDM, are given in table II. The condition for the case of P_3 to be a far-future attractor as indicated in table II is the following:

$$f_0 = \frac{R_{\rm dS}}{R_{\rm dS}^\alpha(\alpha - 2)}, \tag{16}$$

TABLE II. – *The critical points and their eigenvalues for the power law case.*

Label	$\Omega_{\rm m}$	$\Omega_{\rm r}$	r	λ_1	λ_2	Note
P_1	0	1	0	0	1	indeterminate
P_2	1	0	$\frac{1}{4}$	-1	0	indeterminate
P_3	0	0	1	-4	-3	far-future attractor

where $R_{\rm dS} = 12H^2$ is the Ricci scalar for the final de Sitter space. This means that the power law $f(R)$ model could not choose the parameters arbitrarily to have a stable far-future cosmological state.

Now we wish to consider an exponential gravity. Exponential gravity was first considered in [19] and it was regarded as a realistic model which could explain the current accelerating universe [20]. Introducing

$$f(R) = R - \beta R_s \left(1 - e^{-R/R_s}\right), \tag{17}$$

its derivatives with respect to R are given by

$$F(R) = 1 - \beta e^{-R/R_s}, \qquad F'(R) = \frac{\beta}{R_s} e^{-R/R_s}. \tag{18}$$

In this case the critical points, which mimic ΛCDM, are the same as in table II. But the condition for the case P_3 to be a far-future attractor as indicated in table II is given as

$$\beta = \frac{1}{e^{-R_{\rm dS}/R_s} - 2\frac{R_s}{R_{\rm dS}}\left(1 - e^{-R_{\rm dS}/R_s}\right)}. \tag{19}$$

The Hu and Sawicki model takes the form

$$f(R) = R - \mu R_c \left[1 - \left(1 + \frac{R^2}{R_c^2}\right)^{-n}\right]. \tag{20}$$

Its derivatives with respect to R are given by

$$F(R) = 1 - 2\mu n \frac{R}{R_c}\left(1 + \frac{R^2}{R_c^2}\right)^{-(n+1)}, \tag{21}$$

$$F'(R) = -\frac{2\mu n}{R_c}\left[1 - (2n+1)\frac{R^2}{R_c^2}\right]\left(1 + \frac{R^2}{R_c^2}\right)^{-(n+2)}. \tag{22}$$

In this case the critical points, which mimic ΛCDM, are are the same as in table II. But the condition for the case P_3 to be a far-future attractor as indicated in table II is given as

$$\mu = \frac{R_{\rm dS}/R_s}{2\left[1 - \left\{1 + (n+1)\frac{R_{\rm dS}^2}{R_c^2}\right\}\left(1 + \frac{R_{\rm dS}^2}{R_c^2}\right)^{-(n+1)}\right]}. \tag{23}$$

In general, any $f(R)$ model should satisfy the equation

$$\frac{F}{R_{\text{dS}}F'}\left(1-\frac{2}{R_{\text{dS}}}\frac{f}{F}\right)=0, \tag{24}$$

which can be written as

$$\frac{R_{\text{dS}}}{2}=\frac{f(R_{\text{dS}})}{F(R_{\text{dS}})}, \tag{25}$$

to give a far-future de Sitter state. Any other model parameters not satisfying eq. (25), could not give the final attractor state.

5. – Discussions

We have shown that the evolution of various cosmological models can be described by the first-order differential equations of density parameters. Since the equations are of first order, they have critical points. The critical points can be far-past or far-future state of the universe depending on the stability of the critical point. Unfortunately, a full analysis has not been done with complexity. However, it was shown that all models could give the future de Sitter space as the final state with some constraints on the model parameters.

The analysis of the critical points with their stability can be used to find out a viable cosmological model out of various models. A full discussion for application and improvement of the method will be presented elsewhere.

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