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Stability of singular, asymmetric stationary states of the Vlasov equation

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Summary. — We present Vlasov equilibria characterized by discontinuous distribution functions of electrons and of finite mass ions and by asymmetric electric potential profiles. These profiles well reproduce double layers, phase space holes, solitary waves, sheaths near electrodes and near surfaces of airless bodies in Space. By means of the energy method, we show that the stability of the proposed equilibria is better than that of the steady-state solutions of the Vlasov equation based on continuous distribution functions and symmetric potential profiles.

PACS `52.35.Sb` – Solitons; BGK modes.
PACS `52.40.Kh` – Plasma sheaths.
PACS `52.65.Ff` – Fokker-Planck and Vlasov equation.
PACS `94.05.Jq` – Spacecraft sheaths, wakes, and charging.

1. – Introduction

Spatially asymmetric distributions of physical quantities (or their asymmetric time development) are observed in several states of matter and only in special circumstances they may be treated as perturbations of a symmetric basic state.

One particular source of asymmetry is dissipation. This affects the propagation of otherwise symmetric linear and nonlinear waves (such as solitary waves or shock waves) in most diverse physical conditions, from granular materials (fig. 1) to bubbly liquids [1, 2].

Another possible source of weak asymmetry are imperfections in a medium. These modify its dispersion properties and affect wave propagation through the medium [3], thus distorting their otherwise symmetric shape.

Collisionless plasmas offer a fertile setting for the observation and laboratory reproduction of nonlinear, strongly asymmetric, long-lived structures, which cannot be conceived as being either small or dissipative perturbations of any symmetric counterpart.

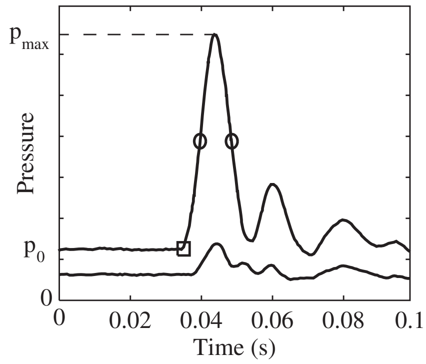

Fig. 1. – Pressure amplitude variations induced by wave propagation in a granular material. Reproduced from ref. [5] with permission.

Examples are the isolated electrostatic structures in the solar wind [4] and the plasma sheaths between electrodes [6-8] or facing the surface of sun-lit, airless bodies in Space [9].

A common problem facing the interpretation of the above phenomena is the reconstruction of the spatial distribution (or time development) of an observed physical quantity, *e.g.*, the electric potential in a plasma.

One approach to this problem is to specify the properties of the medium (*e.g.* the particle velocity distributions in a collisionless plasma) and then work out and solve the equations that govern the physical quantity of interest. In plasma Physics, this approach was used, *e.g.*, in refs. [9-11].

In some circumstances, however, the distribution of the potential in the plasma is itself the best datum emerging from observation, from which the particle distribution functions have to be reconstructed. This problem may be cast as an inverse integral problem and it is akin to that encountered, *e.g.*, in stellar dynamics [12].

To solve this inverse problem, a reasonable profile for the electric potential is needed. One possibility is to assume that this potential solves a given nonlinear partial differential equation and it may be specified as an expansion in terms of *ad hoc* functions: these may be simple transcendental functions, such as exp, tanh, sech [13], or functions solving suitable nonlinear equations such as the Riccati or the matrix Riccati equations (*e.g.* [14]).

In some cases, such as the one treated in this paper, however, not even the differential equation solved by the potential profile is actually known, and only information on its morphology is available. In sect. **3**, we shall see that the theory of elliptic functions [15] enables the reconstruction of the potential from even those qualitative properties.

To test the stability of steady plasma states, we face several difficulties. Some are due to the inhomogeneous and asymmetric nature of our equilibria, at variance with the periodic, symmetric equilibria of, *e.g.*, ref. [16].

More severe difficulties come from the singularity of the particle velocity distribution functions. This feature already appeared in refs. [10, 17], and it was recenly shown to be a necessary consequence of asymmetry [18], a result corroborated by sect. **4** below.

Even under homogeneous plasma conditions, singular, steady-state particle distributions are not subject to Penrose's stability criterion. A separate treatment revealed that homogeneous, singular electron distributions nevertheless have encouraging stability properties [18], at least against electrostatic perturbations.

The stability problem we consider in this paper tackles those difficulties arising from the inhomogeneous nature of the plasma state, while neglecting any dynamical character of the perturbations. This approach, known as the energy method, was adopted, *e.g.*, in ref. [9] and it is developed in sect. **5**.

2. – Notations, basic equations and boundary conditions

Let e be the elementary charge, n_0 a plasma density scale,

$$\hat{\phi} = \min \hat{\phi} + \phi_0 \phi,\ \phi_0 = \max \hat{\phi} - \min \hat{\phi} \tag{1}$$

the rescaled equilibrium electric potential and its scale,

$$\hat{x} = Lx,\ \hat{v} = v_0 v,\ L = \sqrt{[e\phi_0/(4\pi n_0 e^2)]},\ v_0 = \sqrt{(e\phi_0/m_\mathrm{e})} \tag{2}$$

the space and velocity coordinates and their respective scales, $\alpha = \mathrm{e}$ or $\alpha = \mathrm{i}$ a label denoting electron and ion quantities,

$$Z_\alpha e,\ \mu_\alpha = m_\alpha/m_\mathrm{e},\ -V_\mathrm{e} = Z_\mathrm{e}\phi,\ -V_\mathrm{i} = Z_\mathrm{i}(\phi - 1),\ w_\alpha = \mu_\alpha v^2/2 - V_\alpha \tag{3}$$

the particle charges, charge and mass ratios (of which $Z_\mathrm{e} = -1$, $\mu_\mathrm{e} = 1$), the scaled and normalized particle potential and total energies.

In the above notation, the one-particle equilibrium velocity distributions

$$\hat{f}_\alpha(\hat{x}, \hat{v}, \hat{t}) = n_0 F_\alpha(w_\alpha)/(v_0 Z_\alpha) \tag{4}$$

satisfy the steady-state Vlasov's equation, while Poisson's equation reads

$$-\phi'' = n_\mathrm{i} - n_\mathrm{e},\ n_\alpha = \int_{-\phi}^{\infty} \mathrm{d}w F_\alpha(w)/\sqrt{(w+\phi)}. \tag{5}$$

We seek solutions to eq. (5) subject to the following boundary conditions:

$$\phi(x_1) = a < 1,\ \phi(x_2) = 1,\ x_1 < x_2, \tag{6}$$

$$\phi_x(x_1) = 0 = \phi_x(x_2),\ -\phi_{xx}(x_1) = r \geq 0,\ -\phi_{xx}(x_2) = q \geq 0, \tag{7}$$

$$F_\mathrm{e}|_{x=x_2} = \sqrt{(\beta_\mathrm{e}/\pi)} e^{-\beta_\mathrm{e}(1+w_\mathrm{e})},\ F_\mathrm{i}|_{x=x_2} = \sqrt{(\beta_\mathrm{i}/\pi)} e^{-\beta_\mathrm{i} w_\mathrm{i}} \text{ for } w_\mathrm{i} \geq 0, \tag{8}$$

where a subscript x denotes $\mathrm{d}/\mathrm{d}x$ and $\beta_{\mathrm{e,i}}^{-1}$ are the particle temperatures at $x = x_2$ normalized to $e\phi_0$. In the following, the left ($x = x_1$) and right ($x = x_2$) boundaries will be referred to as the low and high potential boundaries, respectively. Only vanishing boundary electric fields and non-negative boundary charges will be considered in this paper (eq. (7)).

3. – Construction of the potential profile

We characterize the morphology of the electric potential spatial distribution $\phi(x)$ by the boundary conditions (eqs. (6) and (7)), and by assuming that it has one single minimum, that this minimum is quadratic and that it occurs at $x = 0$, where $\phi = 0$. Near the minimum,

$$\phi \sim x^2, \quad \phi_x \sim 2x \sim 2\,\mathrm{sign}(x)\sqrt{\phi}, \quad (\sqrt{\phi})_x \sim \mathrm{sign}(x). \tag{9}$$

Thus $\sqrt{\phi}$ has a cusp at $x = 0$, but $\mathrm{sign}(x)\sqrt{\phi}$ is smooth. Also, since ϕ has one single minimum, then $\mathrm{sign}(x)\sqrt{\phi}$ is a monotonic function of x. It is precisely this function that we wish to reconstruct.

To do so, we assume that, in some particular conditions (to be specified later), the potential profile can be represented by means of hyperbolic functions, *e.g.*, the tanh function. Since this function is the limit of Jacobi's elliptic sine as its elliptic modulus approaches 1, we conceive that the general potential profile may be constructed in terms of elliptic functions.

We set $u = k(x-x_2)$ (x_2 being the position of the right plasma boundary (eq. (6)) and k a positive constant) and, without loss of generality, we assume that, for $u < 0$, $\sqrt{\phi}$ is the restriction to the negative u-axis of an even function of u. Since $\sqrt{\phi}$ is also assumed to be elliptic, this even function necessarily is a rational combination of Weierstrass' function $\wp$ [15] and, being $\sqrt{\phi}$ monotonic, this combination necessarily has the form

$$\phi = y^2,\ -y = C - \frac{A\rho}{\wp u - e_1 + \rho},\ u = kx - u_2 < 0,\ u_2 = kx_2,\ k > 0, \tag{10}$$

$$\wp u = \wp(u; g_2, g_3). \tag{11}$$

Here, A, C, ρ are real constants, $e_1 = \wp\omega$, ω is the real half period of $\wp$ and $g_2,\ g_3$ are its elliptic invariants. Since, for any real u, $\wp u \geq e_1$, then, for ϕ to be non-singular, $\rho > 0$.

Next, owing to the homogeneity relation $\wp(\alpha u; \alpha^{-4} g_2, \alpha^{-6} g_3) = \alpha^{-2}\wp(u; g_2, g_3)$ and after a rescaling of the constants k and ρ in eqs. (10) and (11), we may set, without loss of generality,

$$g_2 = 12. \tag{12}$$

Denoting $\mathrm{d}/\mathrm{d}u$ by a prime, from eq. (10) we have

$$y_x = \frac{kA\rho\wp' u}{(\wp u - e_1 + \rho)^2}. \tag{13}$$

Since, as $u \to 0$, $\wp u \sim 1/u^2, \wp' u \sim -2/u^3$, and since $\wp'(\pm\omega) = 0$, we see that the location of the contiguous boundaries where $\phi_x = 2yy_x = 0$ (eq. (6)) are

$$u = 0 \quad i.e. \quad x = x_2 = u_2/k \quad \text{and} \quad u = -\omega \quad i.e. \quad x = x_1 = x_2 - \omega/k. \tag{14}$$

We now impose the boundary conditions (eqs. (6)) and require that the minimum of the potential occurs at $x = 0$ ($u = u_2$):

$$\phi(x_1) = a \Rightarrow A = 1 + \sqrt{a},\ \phi(x_2) = 1 \Rightarrow C = 1,\ \phi(0) = 0 \Rightarrow \wp u_2 = e_1 + \sqrt{a}\rho. \tag{15}$$

Here and in the following, it is understood that the square root operation precedes any of the four arithmetic operations.

By eqs. (5) and (13), the electric charge in the plasma is

$$-\phi_{xx} = -2[y_x^2 + yy_{xx}],\ y_{xx} = k^2 A\rho \left\{ \frac{\wp'' u}{(\wp u - e_1 + \rho)^2} - \frac{2(\wp' u)^2}{(\wp u - e_1 + \rho)^3} \right\}. \tag{16}$$

In particular, the boundary conditions (eq. (7)) imply that

$$r = -\phi_{xx}(x_1) = 4k^2 A\rho,\ q = -\phi_{xx}(x_2) = 12k^2 \sqrt{a}A(e_1^2 - 1)/\rho, \tag{17}$$

where we took into account that $\wp'' u = 6\wp^2 u - g_2/2$ and that $g_2 = 12$ (eq. (12)).

We now use the scaling invariance $k \mapsto \alpha k, \phi_{xx} \mapsto \alpha^2 \phi_{xx}$ built in eq. (17) to set, without loss of generality,

$$r = -\phi_{xx}(x_1) = 3\sqrt{a}A^3 R,\ q = -\phi_{xx}(x_2) = 3A^3 Q,\ \ k = A/2, \tag{18}$$

so that eq. (17) gives

$$\rho = 3Q,\ e_1 = \sqrt{}(1 + 3QR). \tag{19}$$

Once the value of e_1 is known from the charges at the boundaries, the values of $\wp$ at the imaginary half period ω', $\wp\omega' = e_3$, of $\wp(\omega + \omega') = e_2$, of the elliptic invariant g_3, and of the real half period ω are found from the relations $e_1 + e_2 + e_3 = 0$, $e_1e_2 + e_2e_3 + e_3e_1 = -g_2/4 = -3$, $g_3 = 4e_1e_2e_3$, and $\omega = \int_{e_3}^{e_2} \mathrm{d}u/\sqrt{}[4(e_1 - u)(e_2 - u)(e_3 - u)]$ and they are

$$e_2 = [-e_1 + \sqrt{3}\sqrt{}(1 - QR)]/2,\ e_3 = [-e_1 - \sqrt{3}\sqrt{}(1 - QR)]/2, \tag{20}$$

$$g_3 = 4\,(3QR - 2)\sqrt{}(1 + 3QR),\ \omega = \mathbf{K}(\kappa)/\sqrt{}(e_1 - e_3),\ \kappa = \sqrt{}[(e_2 - e_3)/(e_1 - e_3)], \tag{21}$$

where $\mathbf{K}(\kappa)$ is the complete elliptic integral of the first kind and κ its elliptic modulus.

The above relations completely determine the potential of eq. (10):

$$\phi = y^2,\ -y = 1 - \frac{(1 + \sqrt{a})3Q}{\wp u - e_1 + 3Q},\ u = k(x - x_2) < 0,\ k > 0 \tag{22}$$

in terms of a, the ratio of the potential values at the two boundaries, and of r, q, the values of the charges there (eqs. (6) and (7)).

A limitation of the above potential profile is that a vanishing charge at the high potential boundary ($Q = 0$) implies that $\phi \equiv 1$. To construct a potential non-trivially allowing for such possibility, we write

$$\phi = z^2,\ z = -D + \frac{B\sigma}{\wp u - e_1 + \sigma},\ u = kx + u_1 > 0,\ u_1 = -kx_1,\ k > 0, \tag{23}$$

where B, D, σ are constants and x_1 is the position of the left boundary (eq. (6)). The two contiguous boundaries where $\phi_x = 0$ now occur for

$$u = u_1 \ i.e.\ x = x_1 \quad \text{and} \quad u = \omega \ i.e.\ x = x_1 + \omega/k = x_2. \tag{24}$$

We require that

$$\phi(x_1) = a \Rightarrow D = \sqrt{a},\ \phi(x_2) = 1 \Rightarrow B = A,\ \phi(0) = 0 \Rightarrow \wp u_1 = e_1 + \sigma/\sqrt{a}, \tag{25}$$

where A was defined in eq. (15). The charges at the boundaries now are

$$r = -\phi_{xx}(x_1) = 4k^2 A\sqrt{a}\sigma,\ q = -\phi_{xx}(x_2) = 12k^2 A(e_1^2 - 1)/\sigma \tag{26}$$

and, choosing $\phi_{xx}(x_1), \phi_{xx}(x_2)$ and k as in eq. (18), we find

$$\sigma = 3R,\ e_1 = \sqrt{(1 + 3RQ)}. \tag{27}$$

The above relations completely determine the potential of eq. (23):

$$\phi = z^2,\ z = -\sqrt{a} + \frac{(1+\sqrt{a})3R}{\wp u - e_1 + 3R},\ u = k(x - x_1) > 0,\ k > 0. \tag{28}$$

According to eq. (18), the charge at the high potential boundary ($x = x_2$) may now vanish ($Q = 0$) and the potential ϕ remains non-trivial. The charge at the low potential boundary, however, remains finite ($R > 0$).

A third class of potential profiles may be constructed as follows. We use the property $\wp(u + \omega) = e_1 + (e_1 - e_2)(e_1 - e_3)/(\wp u - e_1)$ [15] and, taking e_2 and e_3 form eq. (20), we observe that the functions $Y(u) = y([u + u_2]/k)$ (eq. (22)) and $Z(u) = z([u - u_1]/k)$ (eq. (28)) are related by $Z(u) = Y(u + \omega)$.

This remarkable result shows that the functions $y(x)$ (taken for $x < 0$) and $z(x)$ (taken for $x > 0$), both vanishing at $x = 0$, are the smooth extension of each other. It is therefore legitimate to define the potential profile

$$\phi = y^2(x) \text{ for } x < 0, \quad \phi = z^2(x) \text{ for } x > 0. \tag{29}$$

When $e_1 = 1$, charges may vanish at one boundary (eqs. (22), (28)) or at both boundaries (eq. (29)). In these degenerate cases, $e_2 = e_1,\ e_3 = -2$, so that the elliptic modulus κ (eq. (21)) approaches 1 and the elliptic integral $\mathbf{K}$ (eq. (21)) and the half period ω diverge. The degenerate form of Weierstrass' function, in this case, is $\wp u = -2 + 3\coth^2(\sqrt{3}u)$, so that the elliptic potential profiles of eqs. (22), (28) and (29) reduce to the hyperbolic profiles reported in ref. [18].

Solutions with two vanishing boundary charges (a particular case of eq. (29)) may be conveniently written as follows. By means of the properties of elliptic functions we first reduce eq. (16) to

$$-\phi_{xx} = q\phi + (1 - \phi)(p + 12k^2\phi) - 4k^2 y(1 - y)(3b + 5y),\ y = \sqrt{\phi}\,\mathrm{sign}(x), \tag{30}$$

$$p = -4a - (qa - r)/A^5, \tag{31}$$

$$d = 1 - \sqrt{a} + (q\sqrt{a} + r)/(8\sqrt{a}A^5),\ b = \sqrt{a} + (1 - \sqrt{a})(p + 4a)/(8\sqrt{a}d), \tag{32}$$

where p is manifestly the charge at the potential minimum ($y = 0$): eq. (31) may thus be taken as a scaling law relating the charges at the two boundaries and that at the potential minimum.

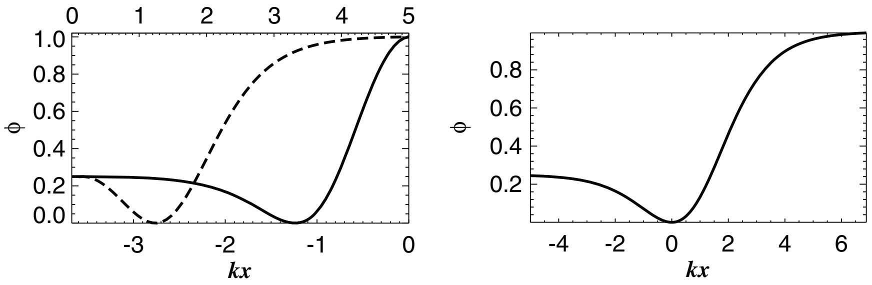

Fig. 2. – Left: the potentials of eq. (22) (with $Q = 1, R = 0.01$, solid line, lower x-scale) and eq. (28) (with $R = 1, Q = 0.01$, dashed line, upper x-scale). Right: the potential of eq. (33). $a = 0.25$ for all plots.

For vanishing boundary charges ($q = 0 = r$), we have $b = 1 - d = \sqrt{a}, p = -4a$ and eq. (30) admits the solution [18]

$$\phi = y^2,\ y = 2\sqrt{a}/[(1-\sqrt{a}) - (1+\sqrt{a})\coth(kx)]. \tag{33}$$

Examples of the potential profiles thus found are shown in fig. 2. The actual computation of Weierstrass' $\wp$ function was carried out by means of the rapidly converging series given in ref. [15].

4. – The particle distributions

The information we have gathered on the spatial distribution of the potential will now be used to reconstruct the steady state particle energy distributions F_e and F_i (eq. (4)), according to ref. [18]:

$$F_\alpha(-w) = G_\alpha(w) + \frac{1}{\pi}\int_0^\infty \mathrm{d}t \frac{\sqrt{w}}{\sqrt{t}}\frac{F_\alpha(t)}{t+w} + \frac{1}{\pi}\mathrm{P}\int_0^\infty \mathrm{d}t \frac{\sqrt{w}}{\sqrt{t}}\frac{F_\beta(t-\phi_\mathrm{b})}{t-w}. \tag{34}$$

In eq. (34), α denotes the ion species if β denotes the electrons' and vice-versa, w is the particle energy, ϕ_b is the maximum value of the potential on each side of the potential minimum ($\phi_\mathrm{b} = 1$ for $x > 0$, $\phi_\mathrm{b} = a$ for $x < 0$) and P denotes the principal value of an integral. The fractional charges [19] G_e and G_i are introduced in such a way that the charge in the plasma can be defined in the two equivalent forms [18]

$$-\phi_{xx} = p - \int_0^\phi \mathrm{d}t \frac{G_\mathrm{e}(t)}{\sqrt{(\phi - t)}} = q_\mathrm{b} + \int_\phi^{\phi_\mathrm{b}} \mathrm{d}t \frac{G_\mathrm{i}(\phi_\mathrm{b} - t)}{\sqrt{(t-\phi)}}. \tag{35}$$

In eq. (35), p is the charge at the potential minimum (where $\phi = 0$), $q_\mathrm{b} = q$ for $x > 0$, $q_\mathrm{b} = r$ for $x < 0$ and q, r are the boundary charges (eq. (7)).

Inverting Abel's equations (35) we find

$$G_\mathrm{e}(w) = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}w}\int_0^w \mathrm{d}\phi \frac{\phi_{xx} + p}{\sqrt{(w-\phi)}},\ G_\mathrm{i}(\phi_\mathrm{b} - w) = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}w}\int_w^{\phi_\mathrm{b}} \mathrm{d}\phi \frac{\phi_{xx} + q_\mathrm{b}}{\sqrt{(\phi - w)}}. \tag{36}$$

In eq. (36), ϕ_{xx} needs to be given as a function of ϕ as specified in eq. (30), and we have

$$G_\mathrm{e}(w) = 3d(2b - 5w) + \sqrt{w}(c_\mathrm{e} + 16w)\,\mathrm{sign}(x)/\pi, \tag{37}$$
$$G_\mathrm{i}(w) = 3d[2b - 5(1 - w)]\ln([1 + \sqrt{w}]/[1 - \sqrt{w}]) + \sqrt{w}[c_\mathrm{i} + 16(1 - w)]/\pi, \tag{38}$$
$$c_\mathrm{e} = 2[q - p - 12(1 + db) + 20d],\ c_\mathrm{i} = c_\mathrm{e} - 30d + 32, \tag{39}$$

where the constants b, d, p were defined in eqs. (31) and (32).

Inserting G_e in eq. (34) we see that, because of the asymmetry of the potential profile, that part of the electron distribution contributed by the electron fractional charge assumes different values on different sides of the potential minimum.

Physically this is made legitimate by the fact that the negative energy electrons that are on one side of the potential minimum are unable to overcome their electric potential barrier and they are thus secluded from the electrons on the other side of the barrier.

On the other hand, ions can freely move across the potential minimum and their distribution, being conserved along their trajectories, needs to be calculated only for, *e.g.*, $x > 0$.

5. – Stability

In sect. **3**, we established that several plasma steady state equilibria can be obtained, for the same values of the boundary charges, by varying the ratio a of the values that the potential has at the two boundaries.

It is of particular interest to establish which of these equilibria contains the least amount of potential energy. We assume that this particular equilibrium is the most stable, at least against very slow perturbations that introduce no appreciable time varying terms in Vlasov's equation. In these circumstances the potential energy to be considered is [9]

$$W = -\int_{x_1}^{x_2} \mathrm{d}x(\phi_x^2/2), \tag{40}$$

where the integral is extended to the whole plasma domain.

For the potential of eq. (33), the boundaries are at $x = \pm\infty$ and eq. (40) reduces to

$$\begin{aligned} W &= -4\,k\,a^2(1 + \sqrt{a})^2 \int_{-1}^{1} \mathrm{d}t(1 - t^2)/[(1 - \sqrt{a}) - (1 + \sqrt{a})t]^6 = \\ &k(3a^2 - 2\sqrt{a^3} + 2a - 2\sqrt{a} - 3)/120,\ k > 0. \end{aligned} \tag{41}$$

This quantity has its minimum at $a = 1/4$, as shown in the left panel of fig. 3. The electric potential profile for $a = 1/4$ was shown in the right panel of fig. 2.

A similar expression for W as a function of the boundary potential ratio a holds for the profile considered in eq. (10): now the boundaries may be set at $x = -\infty$ and $x = 0$ and we have

$$\begin{aligned} W &= -2k(1 + \sqrt{a})^2 \int_0^1 \mathrm{d}t(1 - t)[1 - (1 + \sqrt{a})t]^2\sqrt{[Qt + (1 - Q)t^2]} = \\ &-2\,k\,(1 + \sqrt{a})^2[I_0 - (3 + 2\sqrt{a})I_1 + (1 + \sqrt{a})(3 + \sqrt{a})I_2 + (1 + \sqrt{a})^2 I_3],\ k > 0, \end{aligned} \tag{42}$$

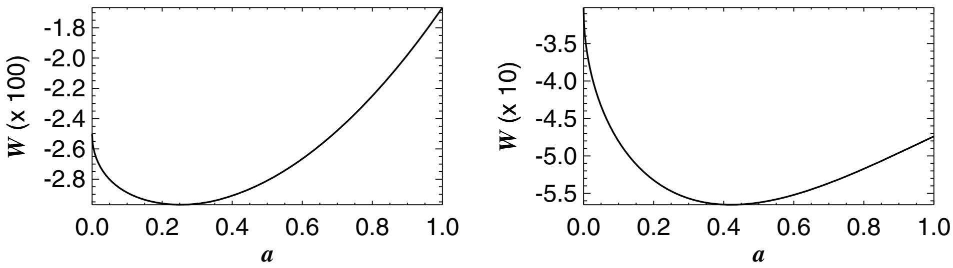

Fig. 3. – Left: the energy W of eq. (41), corresponding to the electric potential of eq. (33). Right: the same for eqs. (42)–(44), corresponding to the electric potential of eq. (22) in which $Q = 1.5, R = 0$.

where

$$(43)\quad 8(1-Q)I_0 = 2(2-Q) + Q^2 \arcsin([Q-1]/Q)/\sqrt{(Q-1)},$$

$$(44)\quad 6(1-Q)I_1 = (2-3QI_0),\ 8(1-Q)I_2 = (2-5QI_1),\ 10(1-Q)I_3 = (2-7QI_2).$$

Also in this case, a minimum of W appears, now at $a \simeq 0.42$, as shown in the right panel of fig. 3.

6. – Discussion and conclusion

In this article we determined the steady-state electrostatic potential, and the particle velocity distribution functions of electrons and ions sustaining it in a collisionless plasma, both quantities being subject to well-defined boundary conditions.

We started from only qualitative morphological information on the electric potential and, using the theory of elliptic functions, we uniquely reconstructed its space distribution, in both finite, semi-infinite and infinite domains, which are able to reproduce observed potential distributions (*e.g.* those in ref. [4]).

The particle velocity distribution functions were determined by solving an inverse integral problem and they are in agreement with those introduced, in a heuristic way, in ref. [18]. This shows that the singular part of these distributions arises under very general assumptions on the potential profile and it is intimately connected with the theory of elliptic functions.

To test the stability of the steady states thus found, we determined the potential energy of these states as the ratio of the electric potential's values on the plasma boundaries is varied in a quasi-static way. Our results clearly show that values of this parameter exist in which the potential energy has a minimum and that, for such values, the profile of the electric potential is asymmetric.

We conclude that collisionless, electrostatic plasma states endowed with asymmetric electric potential profiles and singular particle distributions have better stability properties than those of symmetric states, in agreement with ref. [9].

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