IL NUOVO CIMENTO **38 C** (2015) 106
DOI 10.1393/ncc/i2015-15106-2

Colloquia: GIREP/MPTL 2014

Chaotic behaviour of Zeeman machines at introductory course of mechanics

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received 26 October 2015

Summary. — Investigation of chaotic motions and cooperative systems offers a magnificent opportunity to involve modern physics into the basic course of mechanics taught to engineering students. In the present paper it will be demonstrated that Zeeman Machine can be a versatile and motivating tool for students to get introductory knowledge about chaotic motion via interactive simulations. It works in a relatively simple way and its properties can be understood very easily. Since the machine can be built easily and the simulation of its movement is also simple the experimental investigation and the theoretical description can be connected intuitively. Although Zeeman Machine is known mainly for its quasi-static and catastrophic behaviour, its dynamic properties are also of interest with its typical chaotic features. By means of a periodically driven Zeeman Machine a wide range of chaotic properties of the simple systems can be demonstrated such as bifurcation diagrams, chaotic attractors, transient chaos and so on. The main goal of this paper is the presentation of an interactive learning material for teaching the basic features of the chaotic systems through the investigation of the Zeeman Machine.

1. – Introduction

This work is organically linked to our website [1] where the electronic materials (reading PDF files, simulation programs and videos) could be downloaded (the elements of the electronic material will be denoted by #). The theoretical background of the simulations will be summarized here only shortly because it can be found in our previous paper [2]. In the present paper it will be demonstrated that the Zeeman Catastrophe Machine can be a versatile and motivating tool for students to get introductory knowledge about chaotic

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motion via interactive simulations. For the numerical investigations we have used the *Dynamics Solver* which can be downloaded freely from website [3] and on our website [1] a short description of the program is also available (# *ds_brief_tutorial.pdf*).

Ever since Edward Lorenz has discovered that simple nonlinear systems can produce inherently unpredictable behaviour, which is called chaotic motion, the interest in the theory of it has risen rapidly, and much effort has been invested in integrating it into the graduate as well as the undergraduate curricula. Excellent introductory monographs are available which explain the basic ideas and concepts [4], and in which a wide variety of simple mechanical systems producing chaotic behaviour are deployed [5-7]. Maybe, in the light of these, it seems to be a superfluous effort to increase the number of examples of the simple chaotic systems. Although the scepticism is reasonable, we think that the Zeeman Machine got exceptional advantages as a teaching material. It will be proven that in spite of its simplicity, by investigating it, a broad range of characteristics of chaotic motion can be covered which generally needs the discussion of several different systems. The machine was originally prepared for the demonstration of the catastrophe phenomenon which is a result of a quasi-static process. However, applying a periodic driving force it produces chaotic motion which can be easily studied both theoretically and experimentally. Initial enthusiasm and motivation of students are often lost, when they are unable to understand the theory behind chaotic behaviour. The Zeeman Machine provides an easily understandable theoretical background of various chaotic features, and gives an insight into the dynamics of chaotic motion. Simulations of the motion help us to avoid too mathematical or abstract teaching, and interactive programs support the exploratory activities of the students. Therefore a very important requirement for the electronic material is that the software should be easily usable by the student. Software available for the simulations of the dynamics of physical systems can be classified into three categories:

- high level programming languages: Pascal/Delphi, C/C++, java, python, etc.,
- programs for general purpose: Maple, MathCad, Mathematica, MatLab, etc.,
- user programs for special purpose (in our case programs which are modelling dynamic systems): Dynamics Solver, E&F Chaos, Phaser, XPP, Pyndamics, etc.

Searching the adequate program for the simulation of the Zeeman system programs were investigated according to the following point of view:

- availability (and expenditure),
- programming skills needed and the estimated programing time of the simulation,
- validity and reliability,
- speed and accuracy.

To illustrate this procedure software was chosen from all three groups and with the use of them the same problem was solved. (A stroboscopic map was produced for a damped pendulum the suspension point of which is moving uniformly on a vertical circle.)

Table I shows the extremely good properties of the Dynamics Solver. These properties are combined with high flexibility in modelling physical systems.

We have applied the Dynamics Solver both in university teaching and in our research work successfully. The only disadvantage of it is that it can be used only at Windows

TABLE I. – *Comparison of the programs.*

	Pascal	Maple	Dynamics Solver
Availability	free downloadable exists (*e.g.* FreePascal)	very expensive	free
Preliminary knowledge	advanced programming knowledge needed, to write a simulation program requires more hours	basic level programming skills needed, to produce a worksheet requires some hours.	programming skills are not necessary, one or two hours are sufficient to produce a simulation
Validity	the program is based on the programmer's own routines therefore the validity cannot be decided generally.	built in programs guarantee the high level	high level, validity and reliability are.
Speed and accuracy	It is very speedy the accuracy should be checked due to the number representation used in it	very slow, but it is extremely accurate due to the number representation used	very rapid and accurate.

platform (although it runs at Linux, Unix and Mac platforms at the free WINE compatible platform which is in our opinion is a reliable possibility).

2. – Catastrophe phenomenon and Zeeman's Machine

Those systems whose temporal evolution (dynamics) is defined uniquely by rules are called deterministic systems. *Catastrophe theory* deals with the description of the quasi-static motion of those deterministic systems in which small, continuous changes in one or more parameters cause abrupt, discontinuous, dramatic changes in the equilibrium state of the system. The Catastrophe Machine bearing his name was created by E. C. Zeeman in the 1970's to illustrate and study catastrophe phenomena [8]. The device is very simple (anyone can build it) and can easily be studied quantitatively. Fix a flat disk of radius R with an axle at a point of a rigid sheet. Take two identical rubber strings with an unstretched length of L_0, fasten one end of one of the rubber strings to the circumferential point $\boldsymbol{P}$ of the disk and the other end of the string, slightly stretching it, in point $\boldsymbol{A}(-A,0)$ of the sheet. Fasten one end of the other string at point $\boldsymbol{P}$ of the disk, and let the other end hang free. In our experiments, we will move this end $\boldsymbol{B}$ in the plane of the sheet. For the quantitative description, establish a coordinate system in the plane of the sheet, whose origin is the centre of the disc, its X-axis is a straight line through points $\boldsymbol{A}$ and $\boldsymbol{O}$ (fig. 1), and its Y-axis is perpendicular to this straight line.

Let us study the behaviour of the system by slowly moving the end $\boldsymbol{B}$ of the L_2 rubber string parallel to the Y-axis towards the X-axis starting from different $B(X_0, Y_0)$ points. In most cases, we find that the position of the end $\boldsymbol{B}$ clearly defines the Φ angle, which changes constantly as $\boldsymbol{B}$ is moved. By nearing the string end $\boldsymbol{B}$ towards the X-axis, then crossing it and moving away from it in the opposite direction, the absolute value of Φ decreases, changes sign when crossing the axis and increases while

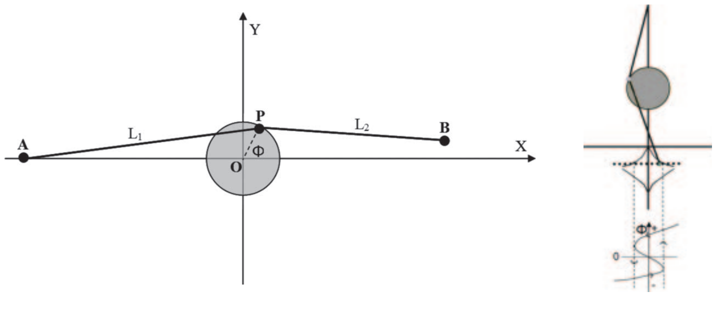


Fig. 1. – (a) Zeeman's catastrophe machine. (b) Bifurcation area and hysteresis.

moving away from it. However, a strange area bounded by four curved lines, which can be well designated experimentally, constitutes an exception. At any internal point of this area, the sign of the angle Φ which belongs to the equilibrium can be either positive or negative. This area is called *bifurcation area* (fig. 1(b)). The experiments show that in the bifurcation area, the change in the angle determining the equilibrium is direction-dependent according to the movement of the point $\boldsymbol{B}$, the change happens differently when going from right to left than in the other way around. The change in the angle exhibits *hysteresis*. Hysteresis is an essential feature in the behaviour of nonlinear systems.

To begin the use of the Dynamics Solver let us start the *#zeeman_animation.ds* simulation file. The theoretical background of the Zeeman catastrophe machine and a suggested way for the use of the simulation can be found in the reading file *#1_prologue_Zeeman_catastrophe-machine.pdf.*

3. – Dissipative chaos

In catastrophe theory, we study the quasi-static properties of the Zeeman Machine and the abrupt changes in its equilibrium state [9]. However, the dynamics of the machine moving due to an external force also produce results showing very interesting chaotic properties [10].

The equation of motion of the Zeeman Machine can be derived from the Lagrangian equation [11]. The phase space of the system is only two-dimensional (with the variables angle and angular velocity), which is known to be too "tight" for the chaotic motion to emerge. Much more interesting type of motion can be created if a periodic driving force is applied at the $\boldsymbol{B}(X,Y)$ end of the second rubber string. The excitation of the system has been investigated using a driving force of period T_p applied at the $\boldsymbol{B}$ end of the second string (we define $\Theta = \frac{2\pi}{T_p}t$ phase variable of driving). The dissipative system described by these equations starting from arbitrary initial condition (Φ_0, ω_0) will reach an equilibrium position due to the continuous energy loss. Henceforth each quantities with dimension of length will be expressed with the R radius of the disc, that is dimensionless variables denoted by lowercase letters will be used: $A = a \cdot R,\ X = x \cdot R,\ Y = y \cdot R,\ L_0 = l_0 \cdot R,\ L_1 = l_1 \cdot R,\ L_2 = l_2 \cdot R$, and $c = \frac{I \cdot R^2 \cdot k}{\gamma^2}$ is a dimensionless parameter. The introduction of the periodic driving force increases dimension of the phase space from two to three, and as it is well known in systems with three dimensional phase space chaotic motion can be occurred.

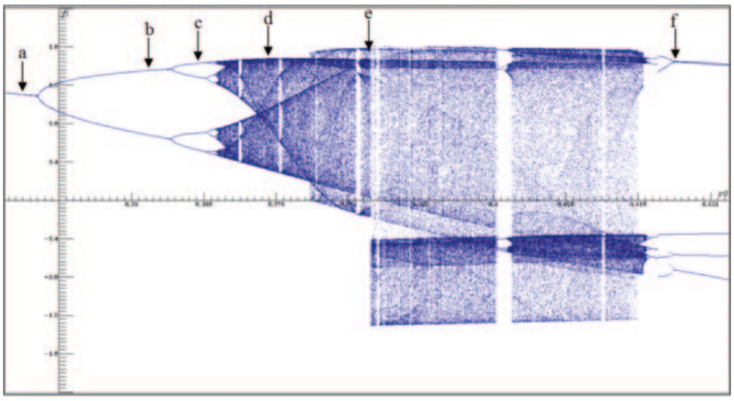


Fig. 2. – The bifurcation diagram of the system with $(\Phi_0 = 0, \omega_0 = 0, \Theta_0 = 0)$.

As we show in Appendix of the *#2_dissipative_chaos.pdf* the equations of motion are

$$\text{(1)} \quad \begin{cases} \dfrac{\mathrm{d}\Phi}{\mathrm{d}t} = f_1(\Phi,\omega,\Theta) = \omega, \\ \dfrac{\mathrm{d}\omega}{\mathrm{d}t} = f_2(\Phi,\omega,\Theta) = c \cdot \left[\dfrac{(l_1 - l_0)}{l_1} \cdot a \cdot \sin\Phi + \dfrac{(l_2(\Theta) - l_0)}{l_2(\Theta)} \right. \\ \qquad\qquad \left. \times \left(y(\Theta) \cdot \cos\Phi - x(\Theta) \cdot \sin\Phi\right) \right] - \omega, \\ \dfrac{\mathrm{d}\Theta}{\mathrm{d}t} = f_3(\Phi,\omega,\Theta) = \dfrac{2\pi}{T_p}, \end{cases}$$

where:

$$\begin{cases} l_1 = \sqrt{(\cos\Phi + a)^2 + (\sin\Phi)^2}, \\ l_2(\Theta) = \sqrt{(x(\Theta) - \cos\Phi)^2 + (y(\Theta) - \sin\Phi)^2} \end{cases}$$

and where the angular velocity $\omega = \mathrm{d}\Phi/\mathrm{d}t$.

As it was mentioned earlier the $\boldsymbol{B}$ end of the second rubber string undergoes simple harmonic motion in the direction of the Y-axis with a centre at $(x_0, 0)$ point. The period and amplitude are T_p, and y_0, respectively. It means that in eqs. (1) the variables of $x(\Theta)$ and $y(\Theta)$, should be replaced by x_0 and $y_0 \cdot \sin(\Theta)$, respectively. The system was investigated at $c = 10$; $a = 6$; $l_0 = 3$; $y_0 = 0.6$; $T_p = 3$ fixed parameters as a function of the x_0 control parameter.

The bifurcation diagram of the system and the trajectories in the phase space can be displayed by simulation files *#harmonically_driven_zeeman_bifurcation_diagram.ds* and *#harmonically_driven_zeeman_phase_space.ds*, respectively. The suggested way for the use of the simulations can be found in the reading file *#2_dissipative_chaos.pdf*.

The bifurcation map shown in fig. 2 is a typical example for that of chaotic systems. The sequence denoted by **a** corresponds to a limit cycle consisting of only one point on the stroboscopic map. The sequence denoted by **b** represents a limit cycle consisting of two points on the stroboscopic map; range **c** represents a four point limit cycle and so on. The development of such type of bifurcation series is a typical precursor of the chaotic behaviour. Bifurcation diagram at **d** and **e** refers to a chaotic zone of motion belonging

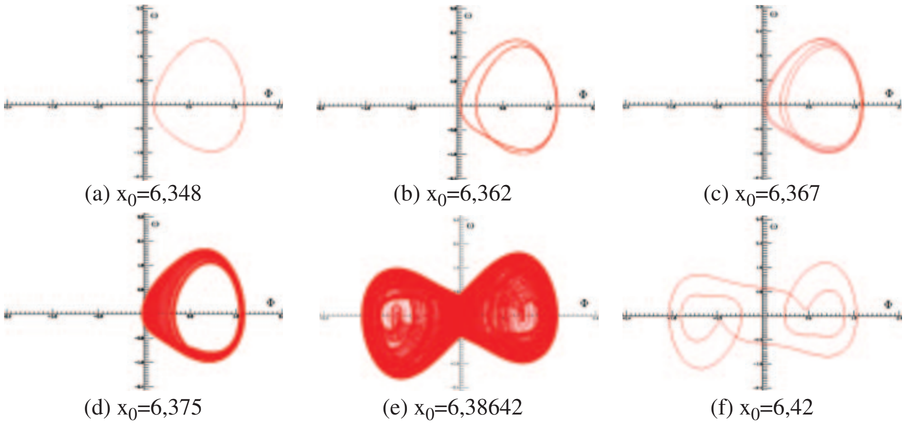


Fig. 3. – The trajectories of motion in phase plane with $(\Phi_0 = 0, \omega_0 = 0, \Theta_0 = 0)$.

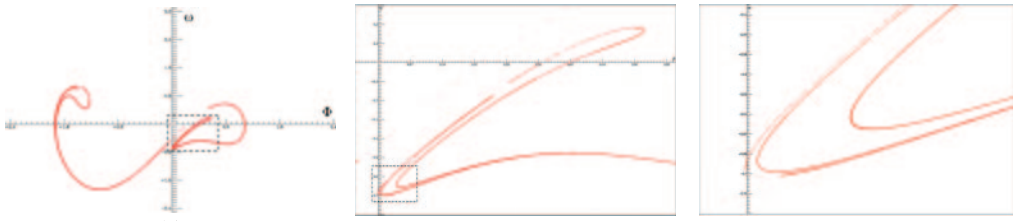
to the control parameters given. Without going in the details of the chaotic zone it is only mentioned that the chaotic zone is a finite one and after it the periodic behaviour appears again (**f**).

Using the bifurcation diagram we can find proper values of the control parameter where the trajectories are worth depicting. The diagrams in fig. 3 show the trajectories in the Φ-ω phase plane, and are labelled by the same letters as the corresponding regions of the bifurcation diagram in fig. 2.

Figure 3 shows a very strange and important feature of the motion of the Zeeman Machine. Attractors shown in figs. 3(e) and 3(f) are centrally symmetric to the origin of the Φ-ω phase plane, while those represented in figs. 3(a)–3(d) are not. The Zeeman Machine itself has also a symmetry axis (the X-axis in fig. 1) and its equation of motion also holds this symmetry consequently eq. (1) is invariant to the change of the variables of $\Phi \to -\Phi$, $y \to -y$ and $\omega \to -\omega$. It is very strange that there are equilibrium positions of the system which do not keep the symmetry of the equation of motion. This behaviour is called *spontaneous symmetry breaking*. This is one of the most exciting phenomena of modern physics, inter alia, it is the basis of the Higgs mechanism by which the mass of the elementary particles in the standard model is interpreted [12,13]. The symmetry breaking and the Psychological conditioning give an obvious possibility for the manipulation of our brain. On website [14] there is a spinning-cat animation, which can be seen rotate clockwise or anticlockwise randomly. Video #*ypm_manip.avi* [15] presents a brilliant and amusing example of how our brain can be governed with symmetry breaking to form a predetermined opinion in an important question.

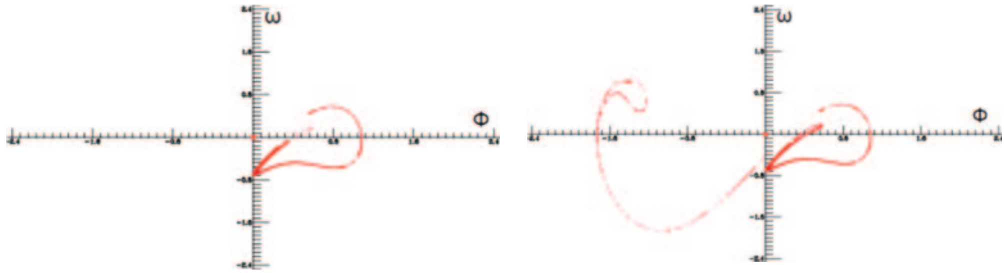
A more exact picture of the chaotic attractor can be obtained by applying a stroboscopic mapping with using #*harmonically_driven_zeeman_stroboscopic.ds* (the suggested way for the use of the simulation can be found in the reading file #*2_dissipative_chaos.pdf*).

Figure 4 shows stroboscopic representation of the attractor exhibited in fig. 3(e) at $\varphi_s = 0$. In figs. 4(b) and 4(c) the magnification of the territory bordered by dashed line in figs. 4(a), and 4(b), can be seen, respectively (the procedure of the magnification can be found #*ds_brief_tutorial.pdf* file of the folder Tools of our electronic material). These diagrams demonstrate perfectly the *scale property* of the chaotic attractor and show its *Cantor fiber*-like structure.



(a) (b) (c)

Fig. 4. – The fractal-structure of chaotic attractor ($x_0 = 6,38642$).



(a) after t=30 steps (b) after t=300 steps

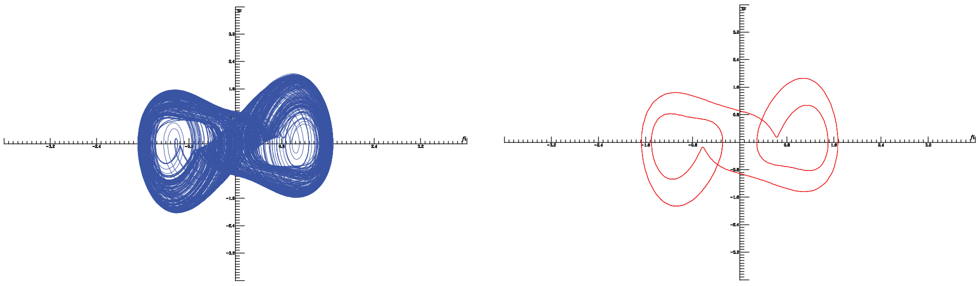
Fig. 5. – Deviation of 10000 neighbouring phase points ($x_0 = 6,38642$).

The sensitivity of the time evolution of a system on the initial conditions is also investigated customarily in chaotic systems. To study this effect let us start a system with different but very close initial values and follow the deviation of the trajectories. This study can be visualized by the *phase drop methods*.

In figs. 5(a) and 5(b) the position of 10000 neighbouring phase points located initially in a square domain of size 0.01 around the origin of the Φ-ω plane can be seen after 30 and 300 time steps, respectively (the control parameter is $x_0 = 6,38642$). This phenomenon can be investigated by program *#harmonically_driven_zeeman_phase_domain.ds* (the suggested way for the use of the simulation can be found in the reading file *#2_dissipative_chaos.pdf*).

It is well observable that points which are originally of immediate vicinity of each other are diverging very quickly. The small initial "phase drop" (phase domain) spreads out strongly already after 30 time steps, and after 300 time steps it covers essentially the whole chaotic attractor.

Edward Lorenz expressed this extreme sensitivity with an example taken from meteorology. His famous question: "Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?" was a title one of his lectures in 1972. This question led to name the extreme sensitivity of a system on initial conditions as butterfly effect, which is a very equivocal and remiss notion. The video *#butterfly-effect_Lorenz.flv* [16] shows the butterfly effect in the famous Lorenz model, while video *#butterfly-effect_parody.mov* [17] gives a caricature of this frequently cited effect.



(a) t=0 - 5200 (b) t=5200 - 10400

Fig. 6. – The transient chaos.

(a) t=0 - 5200 (b) t=5200 - 10400

Fig. 7. – Stroboscopic map of the transient chaos.

4. – Transient chaos

Permanent chaos studied in the previous chapter can exist for an arbitrarily long time period. Contrast to it transient chaos is interim chaotic motion occurring at finite time interval after which the motion becomes periodic. Of course in case of transient chaos chaotic attractors do not exist but a non attracting chaotic saddle can be found which has zero measure and which can be approached arbitrarily closely by trajectories. Trajectories can permanently stay at its neighbour. Chaotic saddle is embodied by those trajectories which exhibit chaotic behaviour for a relatively long time. The basic new feature here is the finite lifetime of chaos.

The theoretical background and a suggested way for the use of the simulation files can be found in the reading file *#3_transient_chaos.pdf*. Figure 6 shows the trajectory of the system. In fig. 6(a) the first 5200 time steps, in fig. 6(b) the second 5200 ones can be seen. It can be observed that at the first stage the motion is chaotic, but at the later second stage it is periodic. It means that the motion of these parameter exhibits the transient chaos.

Figure 6(b) clearly shows a limit cycle (closed curve), but what is the order of it? The stroboscopic map reflects a more demonstrative way these features of the motion: an n-cycle appears as a set of n distinct points mapped onto each other, and each one returns to its initial position after n steps. Stroboscopic map of the first stage of the motion outlines almost perfectly the chaotic attractor, while subsequent stage displays five discrete points corresponding to a five-cycle motion (see fig. 7).

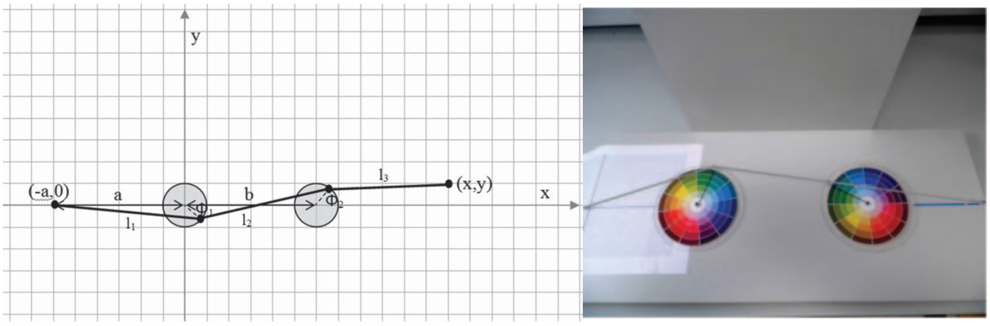


Fig. 8. – The coupled Zeeman Machines.

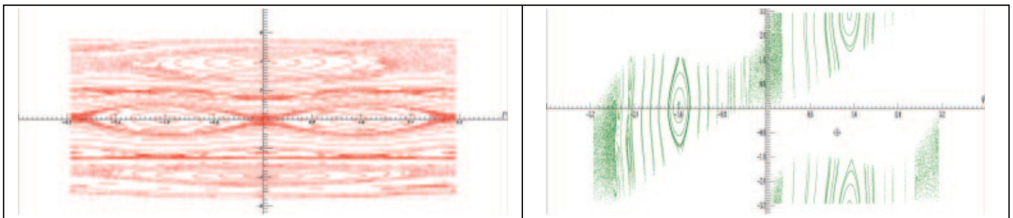
The stroboscopic pictures can be investigated with the file *#harmonically_driven_zeeman_transient_strob.ds* (a suggested way for the use of the simulation files can be found in the reading file *#3_transient_chaos.pdf.*).

5. – Conservative chaos

Zeeman machine discussed in previous chapters is a dissipative system so it is not appropriate for studying the chaotic behaviour of *conservative systems*. Although the original version of the Zeeman machine is frictionless and therefore conserves the energy its two dimensional phase space is not wide enough to produce chaotic motion. An external driving force had to be introduced to increase the number of dimension of the system. However, it seems to be a plausible idea, it is still a novelty, to use *coupled Zeeman machines* for the investigation of chaotic behaviour of conservative systems. Figure 8 shows two coupled frictionless Zeeman Machines which forms a conservative system with four dimension phase space.

The theoretical background of the coupled Zeeman catastrophe machines and a suggested way for the use of the simulation files can be found in the reading file *#4_conservative_chaos.pdf.*

In conservative systems there are no attractors the character of motion depends on the initial conditions [4]. In order to get an overview of the system's behaviour Poincaré maps belonging to the same energy, but corresponding to different initial conditions should be plotted. From the four initial conditions $(\Phi_{10}, \Phi_{20}, \omega_{10}, \omega_{20})$ only three can be chosen freely, the fourth one should be determined from the expression of the energy



(a) Phase plane $\Phi_1 - \omega_1$ (b) Phase plane $\Phi_1 - \Phi_2$

Fig. 9. – Poincaré maps with $(l_0 = 3, a = 6, b = 6), (x = 12, y = 1)$ and energy $e = 60$.

$e = \omega_1^2 + \omega_2^2 + (l_1 - l_0)^2 + (l_2 - l_0)^2 + (l_3 - l_0)^2$, where the sum of the kinetic and potential energy e scaled in units of $\frac{1}{2}kR^2$.

In fig. 9 Poincaré maps of the motion are shown on the Φ_1-ω_1 and Φ_1-Φ_2 phase planes at a given energy value. The maps are representing well the *conservative chaos* and they show *fat fractal* like big chaotic areas with *periodic isles* in them.

6. – Conclusion

It was demonstrated that in spite of its simplicity the Zeeman Catastrophe Machine can be a versatile and motivating tool for students to get introductory knowledge about chaotic motion via interactive simulations. Studying the dynamics of the machine a broad range of characteristics of chaotic motion can be covered which generally needs the discussion of several different systems. The work is organically linked to our website [1] where the electronic materials (reading PDF files, simulation programs and videos) could be freely downloaded.

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[17] `http://www.youtube.com/watch?v=1YzvbmmJiOc`.