IL NUOVO CIMENTO **38 C** (2015) 92
DOI 10.1393/ncc/i2015-15092-3

Colloquia: GIREP/MPTL 2014

How close can we get waves to wave functions, including potential?

Sergej Faletič(*)

Faculty of Mathematics and Physics, University of Ljubljana, Slovenia and
Poljane High School - Ljubljana, Slovenia

received 26 October 2015

Summary. — In the following article we show that mechanical waves on a braced string can have the same shapes as important wave functions in introductory quantum mechanics. A braced string is a string with additional transversal springs that serve as external "potential". The aim is not to suggest teaching quantum mechanics with these analogies. Instead, the aim is to provide students with some additional relevant experience in wave mechanics before they are introduced to quantum mechanics. We show how this experience can be used in a constructivist sense as the basis for building quantum concepts. We consider energy transfer along such string and show that penetration of a wave into a region with high "potential" is not unexpected. We also consider energy transfer between two such strings and show that it can appear point-like even though the wave is an extended object. We also suggest that applying quantization of energy transfer to wave phenomena can explain some of the more difficult to accept features of quantum mechanics.

1. – Introduction

Introductory quantum mechanics introduces students to describing particles with wave functions. But by this time they usually have barely any experience with wave mechanics. So not only do they have to learn that "particles" are not actually small balls, they also have to learn wave mechanics almost from the beginning. This naturally causes some problems.

Levrini *et al.* (2007) suggested from interviews with students that not understanding, but accepting quantum mechanics might be the biggest problem students face. We believe part of this is due to the fact that so far quantum object have been always described to them as particles. Now suddenly they have to describe them as waves, yet they are not really familiar with how waves behave. The topics of impedance and "potential" in the context of waves are barely ever discussed in introductory physics.

(*) E-mail: sergej.faletic@fmf.uni-lj.si

Hobson (2013) suggested using only fields to talk about quantum mechanics and not even talking about particles. He did not discuss the didactical value of this approach, but he showed that using particles is not necessary to build a consistent quantum picture. Yet, he has not attempted to provide a wave experience for students to draw from. We attempt to remedy this by showing those features of classical wave mechanics that are relevant for the behaviour of wave functions. We believe that, if students are already familiar with wave mechanics, many features of wave function mechanics will already be familiar to them and they can focus on what quantum mechanics, specifically quantization of energy transfer, adds to that.

In the first part we discuss how to produce behaviour similar to that of the wave function. We introduce the *braced string* or the *Klein-Gordon string* and discuss its properties. Gravel and Gauthier (2011) discussed a similar system in connection with the Klein-Gordon equation but not the Schroedinger equation and potential. Bertozzi (2010) compared different systems described by the Klein-Gordon equation, among which he mentioned this mechanical one, but he focused more on the electromagnetic application and on how they can help us give meaning to quantities in quantum field theory. We instead suggest how they can help students learn quantum concepts at introductory level. In the second part we discuss point-like interaction of waves and energy transfer. We believe this is the reason why we still talk about particles in introductory quantum mechanics; because they are point-like objects that we associate with point-like energy transfer. Mouchet (2008) used this kind of system to produce a mechanical model of scattering which is also a point-like interaction, but he focused on the technical part and did not suggest a didactical use for the system. In the last part we suggest how adding quantization to the picture might reconcile waves and point-like interactions.

2. – Theory

We started with the Schroedinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi = i\hbar\frac{\partial\psi}{\partial t}, \tag{1}$$

where $\hbar$ is the reduced Planck constant, m the mass of the particle being described, ψ is the wave function and V the potential. We asked ourselves what a similar form for a mechanical wave would be. The mathematical description would have to be of the following form:

$$T\frac{\partial^2 y}{\partial x^2}\mathrm{d}x - \mathrm{d}K(x)y = \mathrm{d}m\frac{\partial^2 y}{\partial t^2}. \tag{2}$$

We rewrite this equation in the form of differences because it is easier to understand and produce experimentally:

$$T\Delta x\frac{\Delta^2 y}{\Delta x^2} - K(x)y = \Delta m\frac{\partial^2 y}{\partial t^2}. \tag{3}$$

The continuous case of eq. (2) approximates well the discrete one of eq. (3), if the wavenumber is much smaller than $2\pi/\Delta x$. Here, y is the waveform (displacement from rest position), T, K and Δm are to be interpreted as generic parameters. For different systems the parameters represent different physical quantities, but in all cases T has to do with the restoring force and Δm has to do with inertia. In the case of a string, or a system of springs and beads, which is the one we will use in the description, T is the tension, Δm is the mass of a bead, and Δx is the distance between the beads. Since the right-hand side of eq. (3) represents mass times acceleration, all terms on the left-hand side are obviously forces. The second term is thus a force proportional to displacement. This can be a spring obeying Hooke's law and $K(x)$ is then the spring coefficient.

So all we have to do to introduce "potential" in wave mechanics is add transversal springs to our medium. This is also consistent with what "potential" means in quantum mechanics. It means that in this region part of the energy of the particle must be stored as potential energy due to some external field. Likewise in the case of waves, part of the energy of the medium must be stored as elastic potential energy of the external springs. This energy is returned to the medium when the springs relax, and it produces reflection effects. In fact, with such potential we can easily achieve energy reflection and produce standing waves.

Standing waves are particularly important, because they often form a basis and all other waves can be written as linear superpositions of the standing waves. Therefore, from here on we will only discuss standing waves. For standing waves the temporal part of the solution is known to be an oscillating term. The spatial part of eq. (2), on the other hand, is exactly the same as its quantum counterpart. Therefore, the solutions are the same as for the appropriate quantum case. The standing waves should be of the same shape as the wave functions of the stationary states in quantum mechanics. We will not derive them here, but one can find them in many books on introductory quantum mechanics, *e.g.*, Schwabl (2007) on pages 49, 65 and 72 and on the internet. More technical details on the system can be found in Faletič (2015).

2˙1. *The discriminating quantity.* – For the simplest case of $K(x)$ being piecewise constant we can derive the dispersion relation. This gives some further insight. We assume the solution on each piece of the medium to be of the form

$$y_i(x,t) = (A_i \cos(k_i x) + B_i \sin(k_i x)) \cos(\omega t). \tag{4}$$

We insert this in eq. (2) and we get

$$k(\omega) = \frac{\omega}{c_0}\sqrt{1-\left(\frac{\omega_1}{\omega}\right)^2}.$$

Here ω_1 represents the natural frequency of an oscillator made from one segment of the medium (Δm) and the spring attached to it. We will call this a *lone oscillator* and the medium can be described as a system of strongly coupled lone oscillators, but the description as a wave is more convenient for our purpose. It can be seen that ω_1/ω determines whether the wavenumber k will be real or imaginary. If $\omega_1 < \omega$ (the natural frequency of the potential is lower than the frequency of the wave), k is real, and we get a wave with a different wavelength, which is always longer than in a free medium (fig. 1(c)). If $\omega_1 > \omega$, k is imaginary, and we get an exponential shape. In fact, as figs. 1 and 2 show, the tunnelling waveform and the finite potential well waveform can be both reproduced.

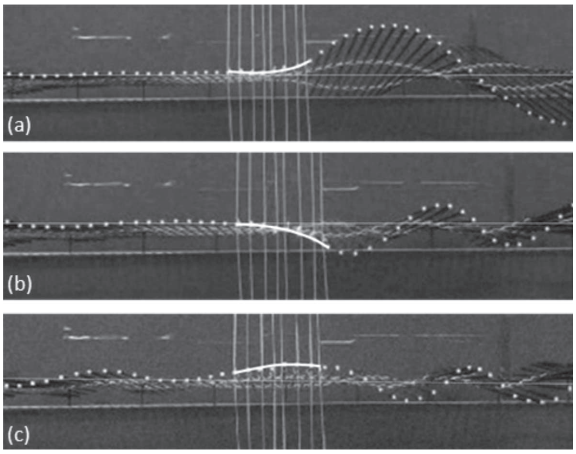

Fig. 1. – Waveforms in the "tunnelling" setup. (a) The waveform at low excitation frequency. The shape inside the potential is exponential and it drops to insignificance before it reaches the other side of the potential. (b) The shape at a higher frequency but with still an imaginary k inside the potential. The shape does not drop to insignificance before it reaches the other side of the potential, therefore a wave with very small amplitude is visible on the other side. The wave might be a little difficult to discern on the static picture, but the oscillations are clearly visible in the real experiment. (c) For an even higher frequency k becomes real. The shape inside the potential is sinusoidal and the energy is transferred to the other side.

3. – Experiments

We used a torsional wave-machine for the experiments. The wave equation in this case is somewhat different

$$D\Delta x^2 \frac{\Delta^2\varphi}{\Delta x^2} - b(x)r(x)^2\varphi = I_1 \frac{\partial^2\varphi}{\partial t^2}.$$

Here D is the torsional coefficient of the restoring torsional mechanism (in most cases of wave-machines a wire that connects the rods), $b(x)$ is the coefficient of the spring connected to the rod at position x, $r(x)$ is the distance between the central wire and the point on the rod where the spring is attached. I_1 is the moment of inertia of that rod. The expression $b(x)r(x)^2$ acts as the external potential. $r(x)\varphi(x)$ is the vertical displacement at distance r, $b(x)r(x)\varphi(x)$ is the force due to the external spring and $b(x)r(x)^2\varphi(x)$ is the torque. The natural frequency of one oscillator of this form can be easily derived from Newton's law of motion

$$\omega_1(x)^2 = \frac{b(x)r(x)^2}{I_1}.$$

The freedom in choosing $r(x)$ allows us to use the same springs and change ω_1 by changing where they are attached to the rod. We used elastic strings (but we will continue to call them springs throughout this article) instead of springs. We are aware that they do not strictly follow Hooke's law, but they are far easier to acquire and work with. We will show that the results do not suffer from the choice.

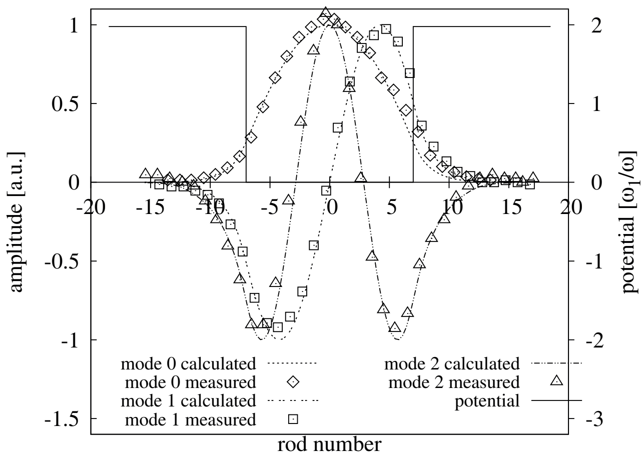

Fig. 2. – The shape of the experimental wave (points) and the theoretical prediction (lines) for the first three stationary states of a finite potential well. The shape of the potential is also shown.

3˙1. *Special waveforms.* – By positioning a number of springs on subsequent rods in the middle of the wave-machine we achieved a narrow potential barrier. We varied the excitation frequency so that ω_1/ω was either greater than or lower than one. Figures 1(a) through 1(c) show the results. As expected, for an imaginary k we get an exponential shape. For low frequencies all energy is reflected inside the potential (fig. 1(a)). For a higher frequency but still below the threshold of imaginary k some energy is "tunnelled" through the potential and a sinusoidal waveform with smaller amplitude appears on the other side (fig. 1(b)). For a frequency above the threshold the shape inside the potential is sinusoidal with a longer wavelength (fig. 1(c)).

Besides these waveforms, we also tested the finite potential well by leaving the centre of the wave-machine without springs and adding springs to the sides of the wave-machine. In fig. 2 we can see that the waveforms match well with the theoretical predictions, which are the same as the spatial parts of solutions of the Schroedinger equation. We also see the exponential tails penetrating the "potential".

After realizing that the "potential" is made from springs, we believe it is intuitive that some energy still gets transferred even into the region where displacement is hindered, but with each next segment less and less energy can be transferred, so the displacement approaches zero, while the energy gets reflected and produces a standing wave. From this, it is intuitive to conclude that only infinitely strong springs would prevent any displacement at all within the region.

The freedom in parameter $r(x)$ allows us to easily produce a parabolic potential by linearly increasing $r(x)$ with x. We did this, too, and produced waveforms consistent with wave functions of the quantum harmonic oscillator (fig. 3).

We attempted the experiment with a different system, made of coupled pendula as the medium. The natural frequency of each pendulum provides the potential, while the coupling provides the waveform. On fig. 4 we only show pictures to prove that the waveform of the finite potential well can be easily achieved also with this kind of system. Out of 10 pendula, three on each side had approximately half the length of the rest. The exponential tails are clearly visible.

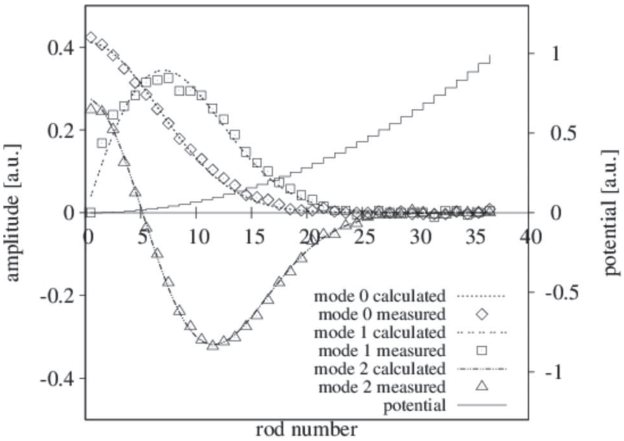

Fig. 3. – The shape of the experimental wave (points) and the theoretical prediction (lines) for the first three stationary states of a wave inside a parabolic potential (the shape of the piecewise constant approximation used is also shown). The waveforms are the same as the spatial part of the wave function in a quantum harmonic oscillator.

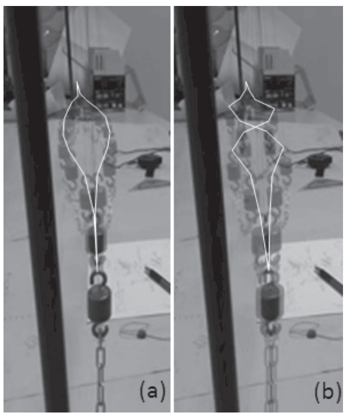

Fig. 4. – Two extreme positions of the standing wave, superimposed on the same picture for (a) the first and (b) the second stationary state of the finite potential well. The white lines are added to emphasize the shape of the waveform.

4. – Energy transfer

We have shown that waves behave very similarly to wave functions in quantum mechanics. We believe that with this experience it also becomes intuitive why in a finite potential well the energy of the particle still penetrates the "wall" to a degree, and why it is necessary to make infinitely strong springs (an infinite potential well), if we want the wave function to become zero at the "wall". With this, it is a natural question why do typical courses in quantum mechanics still talk about particles. We believe the reason is that when we observe interactions, which is all we can observe, they appear point-like, which is a feature more readily associated with particles than waves.

Here we show that waves can also exhibit point-like energy transfer. When we produce a wave, we typically do it by exciting only one segment of the medium and let the wave

Fig. 5. – Coupled pendula can be used as a type of braced medium. The medium is made of pendula on strings coupled by a chain hanging from one pendulum to the next (A). On the bottom right side is the driving pendulum (B). A string (C) connects the string of the driving pendulum and one of the strings of the medium, and serves as coupling.

propagate. This interaction is very limited in space, point-like. Yet, it excites the whole wave. We, therefore, know that energy can be transferred to a wave in a point-like manner. Can it also be extracted from the wave this way? Yes. Especially with a wave-machine we frequently use viscous damping at the end to extract energy from the wave so it would appear as if it travels to infinity. To do this we often use a damper connected to the last rod of the wave-machine. Again, a very spatially bound interaction that yet extracts all the energy from the wave.

We have done experiments with the coupled pendula medium. We connected the whole medium to another pendulum via weak coupling as shown on fig. 5. We observed and filmed the beats as the whole energy of the pendulum is transferred at one point-like interaction to the medium and back. We have thus proven that an energy transfer can occur between a pendulum and a wave at a specific point, and we believe there is no reason to doubt that it can occur in the same way between two waves. We have shown this with simulations and we intend to show it with an experiment. There is really no theoretical reason why it should not be possible. This suggests that we can expect point-like energy transfer also between two entities described with wave functions.

Does this point-like energy transfer between waves contain any further similarity with the quantum case? Yes. For example, it cannot be done in the nodes of the standing wave. This is similar to the fact that we cannot detect an interaction (find a particle) where its wave function (and consequently probability density) is zero. But, it can be done anywhere else. We hypothesized that it would be most efficient where the amplitude is the highest. To that end, we measured the beat frequency when the coupling was far from the peak of the antinode (at the third weight out of ten), and close to the peak of the antinode (at the fifth weight). For the first case, we got $(5.8 \pm 0.2) \cdot 10^{-3}\,\text{Hz}$, and for the second $(8.5 \pm 0.2) \cdot 10^{-3}\,\text{Hz}$. We took care to keep the coupling strength the same as much as possible. We also verified that there is only a slight dependence of beat frequency on initial amplitude of the pendulum, on the order of 5% or $0.4 \cdot 10^{-3}\,\text{Hz}$, which is not enough to account for the measured difference. We still tried to keep the initial amplitude the same. We, therefore, conclude that the energy is indeed transferred faster (more energy transfer per unit time) when the coupling is closer to the peak of the antinode than when it is further from the peak.

As a side experiment we also confirmed that, if the frequency of the driving pendulum is set to the frequency of the second stationary state, it is in fact the second stationary state that is produced on the medium during energy transfer.

4·1. *Quantization.* – To use this experience with quantum phenomena, we must introduce quantization of energy transfer, which is the most important feature of quantum mechanics that sets it apart from classical mechanics. In classical wave mechanics the energy is transferred over time, and we can transfer different amounts of energy at different positions. This allows us to sample the shape of the waveform. In the quantum world, the energy of one entire quantum must be transferred at once. Once the quantum is transferred, the state of the particle changes to a state with one less quantum of energy. The wave function changes accordingly. Therefore we cannot sample it by transferring energy from different positions, because as soon as one quantum is transferred the wave function changes and we would not be sampling the same wave function again. The only way to get statistics out of the situation and sample the entire wave function is to prepare a great number of identical wave functions and sample each once, hoping that the transfer will not occur all the time at the same position. That is exactly what quantum mechanics describes.

The quantum point-like transfer of energy means that it is also transferred in only one of all the possible positions (but never from the impossible ones —the nodes). How to account for different rates of transfer at different positions that we encountered in the classical case, if the energy of one entire quantum must be transferred at once? Through statistics. Over a longer time, it should occur less often that the energy is transferred at positions far from the peaks of the antinodes, and more often near the peaks. So, on average, there will be less energy transferred from places with lower amplitude. Thus, efficiency translates to probability. Since lower energy cannot be achieved with smaller packages, the only alternative is to make packages less frequent.

We end our discussion of quantization here. We are aware that this comparison probably has limitations, and we do not pretend to be able to show that all quantum behaviour, except quantization, is also exhibited by classical waves. We just wanted to show that much of it is, and set those features apart from features that are specifically quantum.

5. – Conclusions and discussion of didactical value

We have shown that shapes such as we encounter with wave functions in quantum mechanics are inherent to wave mechanics, even classical. We have introduced external potential in the form of external springs that affect the shape of the waveform and the energy flux of the wave. We have shown that the discriminating quantity that determines the energy transfer through a potential and, therefore, the shape of the waveform is the ratio between the natural frequency of a segment of the medium, if it is considered as an independent oscillator, and the frequency of the wave. We have shown that waves can exhibit point-like interaction. We have compared all these features to features of quantum wave functions, but we do not suggest using them as an analogy to teach quantum mechanics. Instead, we suggest to use them as a foundation in a constructivist sense, while building a consistently quantum picture. We only hope to minimize the cognitive conflicts that arise from comparing particle behaviour to wave behaviour by showing that much of the differences between particles and waves are already present in classical physics. We have shown that classical waves already exhibit penetration into a potential that is too high and tunnelling. This is not what is new in the quantum world. We have also shown that classical waves already exhibit point-like interaction. This is also not what is new. The real novelty comes from quantization. We, therefore, believe that the emphasis when teaching quantum mechanics should be on quantization: what are the changes that quantization brings to what we already know is wave behaviour. We believe this can help students understand the features specific to the quantum world better, and set them apart from what are features of all waves.

The described system is used as only a part of a larger short course on quantum mechanics for high school students. The course consists of other experiments and topics, but we will report here on the parts that are done with the system described here. We have tested this approach on two different groups of high school students: one group were second year students who chose "optional physics" at the Poljane High School. These had two years of physics in primary school and were in the second year of a 70-hour-per-year course at high school. The other group were those who attend preparations for the International Physics Olympiad (IPhO). We have not yet any quantifiable results, so we can only report on our observations. We introduced the mechanical model after we showed the electron scattering experiment which we used to justify describing electrons with wave-like entities. They saw that with low frequencies the exponential tails occur and also that the tail gets longer with higher frequency and shorter with increased potential. After this they were able to predict what would be necessary to remove the tails: infinite potential. They have shown no surprise at the fact that tunnelling occurs with mechanical waves. This is encouraging since in our experience it causes big surprise when discussing it in the context of particles. On the topic of quantization, they were introduced to wave interference and then compared it to single photon interference. They learn in the context of the photoelectric effect that each photon transfers its entire energy to one electron. To reconcile the extended wave description of the photon with a point-like interaction, we show them the experiment with the pendulum transferring energy to the wave, and other point-like interaction, such as driving the wave at one end, which are very common for waves. After this, both groups were able to reinterpret the interference pattern as a probability pattern. The different beat frequency experiment described in *Energy transfer* was used to reinforce this interpretation: more joules per second must mean more photons per second, which must mean greater probability to interact at that point. We had the impression that students had no difficulty accepting the wave phenomena on their own. Quantization was a new concept to them, but they were able to use it and integrate it at least to a certain degree in the wave description. We view these points as positive achievements of the approach, although rigorous pedagogical study still remains to be done. We have collected some feedback, but it has not been analyzed yet. Apart from the IPhO group being able to learn faster, we have noticed no discernable differences between the groups. Individuals in both groups were able to use the acquired knowledge to make predictions about new phenomena. We presume the percentage of those able to do this would be higher in the IPhO group, but there were some in both groups. Again, a more rigorous study might show further differences.

We have presented a system that can set apart the inherently wave properties of wave functions from their strictly quantum properties and their interpretations. We have shown some positive observations in class and hope to do a more thorough study in the future.

REFERENCES

[1] Bertozzi E., *Eur. J. Phys.*, **31** (2010) 1499.
[2] Faletič S., *Eur. J. Phys.*, **36** (2015) 035023.
[3] Gravel P. and Gauthier C., *Am. J. Phys.*, **79** (2011) 447.
[4] Mouchet A., *Eur. J. Phys.*, **29** (2008) 1033.
[5] Hobson A., *Am. J. Phys.*, **81** (2013) 211.
[6] Levrini O., Fantini P. and Pecori B., "The problem is not understanding the theory, but accepting it": a study on students' difficulties in coping with quantum physics. *Proceedings of the GIREP-EPEC Conference 2007, Frontiers of physics education* (Zlatni rez, Rijeka) 2007, pp. 319–324.
[7] Schwabl F., *Quantum mechanics*, 4th edition (Springer-Verlag, Berlin, Heidelberg) 2007.