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Hadronic light-by-light scattering and the muon $g - 2$

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Summary. — The largest uncertainties in the Standard Model calculation of the anomalous magnetic moment of the muon $(g-2)_\mu$ come from hadronic contributions. In particular, it can be expected that in a few years the subleading hadronic light-by-light (HLbL) contribution will dominate the theory uncertainty. We present a dispersive description of the HLbL tensor, which is based on unitarity, analyticity, crossing symmetry, and gauge invariance. Such a model-independent approach opens up an avenue towards a data-driven determination of the HLbL contribution to the $(g-2)_\mu$.

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PACS `13.40.Em` – Electric and magnetic moments.
PACS `13.75.Lb` – Meson-meson interactions.

1. – Introduction

The anomalous magnetic moment of the muon $(g-2)_\mu$ has been measured [1] and computed to very high precision of about 0.5 ppm (see, *e.g.*, [2]). For more than a decade, a discrepancy has persisted between the experiment and the Standard Model prediction, now of about 3σ. Forthcoming experiments at FNAL [3] and J-PARC [4] aim at reducing the experimental error by a factor of 4.

The main uncertainty of the theory prediction is due to strong interaction effects. At present, the largest error arises from hadronic vacuum polarisation, which, however, is expected to be reduced significantly with the help of new data from e^+e^- experiments [2].

In a few years, the subleading(¹) hadronic light-by-light (HLbL) contribution will dominate the theory error. In the present calculations of the HLbL contribution, systematic errors are difficult, if not impossible, to quantify. A new strategy is required to avoid the model dependence as far as possible, to provide a solid estimate of the theory uncertainties and to reduce them. It is not clear yet if lattice QCD will become competitive and can achieve this goal [7-9]. In [10, 11], we have presented the first dispersive description of the HLbL tensor(²). By making use of the fundamental principles of unitarity, analyticity, crossing symmetry, and gauge invariance, we provide a model-independent approach that will allow a more data-driven determination of the HLbL contribution to the $(g-2)_\mu$.

Here, we report on an improvement of our dispersive approach [13, 14]. We have constructed a generating set of Lorentz structures that is free of kinematic singularities and zeros. This simplifies significantly the calculation of the HLbL contribution to the $(g-2)_\mu$. Our dispersive formalism defines both the pion-pole and pion-box topologies in an unambiguous way. By constructing a Mandelstam representation for the scalar functions, we prove that the box topologies are equal to the scalar QED (sQED) contribution multiplied by pion vector form factors. The new formalism also allows us to consistently include D-waves in the $\pi\pi$-rescattering contribution.

2. – Lorentz structure of the HLbL tensor

In order to study the HLbL contribution to the $(g-2)_\mu$, we need a description of the HLbL tensor, the hadronic Green's function of four electromagnetic currents, evaluated in pure QCD:

$$(1)\quad \Pi^{\mu\nu\lambda\sigma}(q_1,q_2,q_3) = -i\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{d}^4z\,e^{-i(q_1\cdot x+q_2\cdot y+q_3\cdot z)}\langle 0|T\{j^\mu_{\mathrm{em}}(x)j^\nu_{\mathrm{em}}(y)j^\lambda_{\mathrm{em}}(z)j^\sigma_{\mathrm{em}}(0)\}|0\rangle .$$

Gauge invariance requires the HLbL tensor to satisfy the Ward-Takahashi (WT) identities

$$(2)\qquad \{q_1^\mu, q_2^\nu, q_3^\lambda, q_4^\sigma\}\Pi_{\mu\nu\lambda\sigma}(q_1,q_2,q_3) = 0,$$

where $q_4 = q_1 + q_2 + q_3$. The HLbL tensor can be written *a priori* in terms of 138 basic Lorentz structures built out of the metric tensor and the four-momenta [15]. Our first task is to write the HLbL tensor in terms of Lorentz structures that satisfy the WT identities, while at the same time the scalar functions that multiply these structures must be free of kinematic singularities and zeros. A recipe for the construction of these structures has been given by Bardeen, Tung [16], and Tarrach [17] for generic photon amplitudes. Gauge invariance imposes 95 linear relations between the 138 initial scalar functions. A generating set(³) consisting of 43 elements can be constructed following Bardeen and Tung [16]. However, as it was shown by Tarrach [17], such a set is not free

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(¹) Even higher-order hadronic contributions have been considered in [5, 6].

(²) A different approach, which aims at a dispersive description of the muon vertex function instead of the HLbL tensor, has been presented in [12].

(³) In 4 space-time dimensions, there are two more linear relations, hence a basis consists of 41 elements [18].

of kinematic singularities and has to be supplemented by additional structures. We find a redundant generating set of dimension 54:

$$\Pi^{\mu\nu\lambda\sigma}(q_1,q_2,q_3)=\textstyle\sum_{i=1}^{54}T_i^{\mu\nu\lambda\sigma}\Pi_i(s,t,u), \tag{3}$$

where the scalar functions Π_i are free of kinematic singularities and zeros. The Mandelstam variables are defined by $s=(q_1+q_2)^2$, $t=(q_1+q_3)^2$, $u=(q_2+q_3)^2$. Both crossing symmetry and gauge invariance are implemented in a manifest way in the set $\{T_i^{\mu\nu\lambda\sigma}\}$: on the one hand, crossing results just in permutations of the 54 structures, on the other hand each structure fulfils the WT identities. Since the scalar functions Π_i are free of kinematics, they are the well-suited quantities for a dispersive description.

3. – HLbL contribution to the $(g-2)_\mu$

The extraction of the HLbL contribution to $a_\mu=(g-2)_\mu/2$ with the help of Dirac projector techniques is well-known [19]. With our decomposition of the HLbL tensor in 54 structures, this amounts to the calculation of the following two-loop integral:

$$\begin{aligned}a_\mu^{\mathrm{HLbL}}=&-\frac{e^6}{48m_\mu}\int\frac{\mathrm{d}^4q_1}{(2\pi)^4}\frac{\mathrm{d}^4q_2}{(2\pi)^4}\frac{1}{q_1^2q_2^2(q_1+q_2)^2}\frac{1}{(p+q_1)^2-m_\mu^2}\frac{1}{(p-q_2)^2-m_\mu^2}\\&\times\mathrm{Tr}\Big((\not{p}+m_\mu)[\gamma_\rho,\gamma_\sigma](\not{p}+m_\mu)\gamma_\mu(\not{p}+\not{q}_1+m_\mu)\gamma_\lambda(\not{p}-\not{q}_2+m_\mu)\gamma_\nu\Big)\\&\times\sum_{i=1}^{54}\left(\frac{\partial}{\partial q_{4\rho}}T_i^{\mu\nu\lambda\sigma}(q_1,q_2,q_4-q_1-q_2)\right)\bigg|_{q_4=0}\Pi_i(q_1,q_2,-q_1-q_2).\end{aligned} \tag{4}$$

After a Wick rotation of the momenta, five of the eight loop integrals can be carried out with the technique of Gegenbauer polynomials [20]. In analogy to the pion-pole contribution [21], a Master formula for the full HLbL contribution to the $(g-2)_\mu$ can be worked out:

$$a_\mu^{\mathrm{HLbL}}=\frac{2\alpha^3}{3\pi^2}\int_0^\infty\mathrm{d}Q_1\int_0^\infty\mathrm{d}Q_2\int_{-1}^1\mathrm{d}\tau\sqrt{1-\tau^2}Q_1^3Q_2^3\sum_{i=1}^{12}T_i(Q_1,Q_2,\tau)\bar{\Pi}_i(Q_1,Q_2,\tau), \tag{5}$$

where $\alpha=e^2/(4\pi)$ and the T_i are integration kernels. Only twelve independent linear combinations of the hadronic scalar functions Π_i contribute, denoted by $\bar{\Pi}_i$. They have to be evaluated for the reduced kinematics

$$\begin{aligned}s&=-Q_3^2,\quad t=-Q_2^2,\quad u=-Q_1^2,\\q_1^2&=-Q_1^2,\quad q_2^2=-Q_2^2,\quad q_3^2=-Q_3^2=-Q_1^2-2Q_1Q_2\tau-Q_2^2,\quad q_4^2=0.\end{aligned} \tag{6}$$

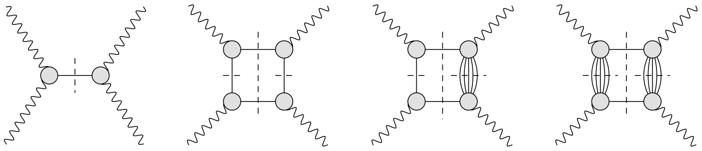

Fig. 1. – Unitarity diagrams according to the Mandelstam representation. Crossed diagrams are omitted.

4. – Mandelstam representation

Gauge invariance, encoded in the decomposition (3), leads to Lorentz structures $T_i^{\mu\nu\lambda\sigma}$ of mass dimension 4, 6, and 8. Hence, we expect the scalar functions Π_i to be rather strongly suppressed at high energies. This allows us to write down unsubtracted double-spectral (Mandelstam) representations for the Π_i [22], *i.e.* parameter-free dispersion relations. The input to the dispersion relation are the residues at poles (due to single-particle intermediate states) and the discontinuities along branch cuts (due to two-particle intermediate states). Both are defined by the unitarity relation, in which the intermediate states are always on-shell. We neglect contributions from intermediate states consisting of more than two particles in the primary cut.

In the Mandelstam representation, the sum over intermediate states in the unitarity relations (for the primary and secondary cuts) translates into a splitting of the HLbL tensor into several topologies, shown in fig. 1. The first topology consists of the pion pole, *i.e.* the terms arising from a single pion intermediate state. This contribution is well-known [21] and given by

$$\begin{aligned}\bar{\Pi}_1^{\pi^0\text{-pole}} &= -\frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}\left(-Q_1^2,-Q_2^2\right)\mathcal{F}_{\pi^0\gamma^*\gamma^*}\left(-Q_3^2,0\right)}{Q_3^2+M_\pi^2}, \\ \bar{\Pi}_2^{\pi^0\text{-pole}} &= -\frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}\left(-Q_1^2,-Q_3^2\right)\mathcal{F}_{\pi^0\gamma^*\gamma^*}\left(-Q_2^2,0\right)}{Q_2^2+M_\pi^2},\end{aligned} \tag{7}$$

where $\mathcal{F}_{\pi^0\gamma^*\gamma^*}$ denotes the pion transition form factor (for off-shell photons but an on-shell pion).

The other topologies are obtained by selecting two-pion intermediate states in the primary cut. The sub-process $\gamma^*\gamma^* \to \pi\pi$ is again cut in the crossed channel. If we single out the pion-pole contribution in both of the sub-processes, we obtain the box topologies for HLbL. For higher intermediate states in the crossed channel of $\gamma^*\gamma^* \to \pi\pi$, we obtain boxes with multi-particle cuts instead of poles in the sub-processes.

By explicitly constructing the Mandelstam representation, we have shown that the box topologies in the sense of unitarity have the same analytic structure as the sQED loop contribution, multiplied with pion electromagnetic form factors $F_\pi^V(q_i^2)$ for each of the off-shell photons (FsQED). The dispersion relation defines unambiguously this particular q_i^2 dependence. With the construction of the Mandelstam representation, we prove that FsQED and box topologies are the same. Note that the sQED loop contribution in terms of Feynman diagrams consists of boxes, triangles, and bulbs, but that the corresponding unitarity diagrams are just box topologies.

We treat the contribution from topologies with higher intermediate states in a partial-wave picture. This means that the multi-particle cut is approximated by a polynomial,

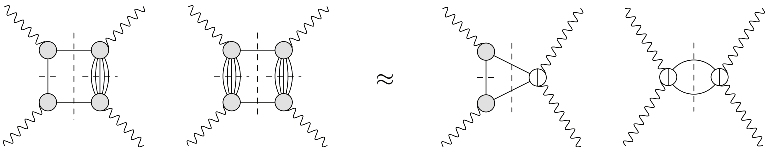

Fig. 2. – Partial-wave approximation of multi-particle intermediate states in the secondary cut.

as illustrated in fig. 2. The dispersive formulation allows us to describe here the effect of two-pion rescattering in the primary channel. In [10], we already discussed the S-wave contribution. The new Lorentz decomposition allows us to include also higher partial waves. The contribution of these topologies is given by dispersion integrals over products of $\gamma^*\gamma^{(*)} \to \pi\pi$ helicity partial waves. The Born terms of the sub-process have to be properly subtracted to avoid double-counting with the box topologies. The imaginary parts in the integrand of the dispersion integrals are simply obtained by projecting the partial-wave unitarity relation onto the scalar functions Π_i.

5. – Conclusion and outlook

Using the Mandelstam representation for the hadronic scalar functions Π_i, we have split a_μ^{HLbL} into three contributions: pion-pole contributions, box topologies, and $\pi\pi$-rescattering contributions:

$$a_\mu^{\mathrm{HLbL}} = a_\mu^{\pi^0\text{-pole}} + a_\mu^{\mathrm{box}} + a_\mu^{\pi\pi} + \dots, \tag{8}$$

where the dots denote neglected higher intermediate states in the primary cut. The input quantities in this dispersive description are the pion transition form factor $\mathcal{F}_{\pi^0\gamma^*\gamma^*}(q_i^2, q_j^2)$, the pion electromagnetic form factor $F_\pi^V(q_i^2)$, and the $\gamma^*\gamma^* \to \pi\pi$ helicity partial waves. In the absence of experimental data on the doubly-virtual processes, these quantities will be reconstructed again dispersively [10, 23-29].

We stress that the dispersive formalism defines unambiguously both the pion-pole and pion-box contribution. They are treated without any approximation. For the two-pion rescattering contribution a partial-wave expansion is employed.

We have limited the discussion to pions although the formalism can be extended to higher pseudoscalar poles (η, η') or KK intermediate states.

The presented dispersive approach provides a first model-independent description of HLbL scattering and shows a path towards a more data-driven evaluation of the HLbL contribution to the $(g-2)_\mu$. A careful numerical study is currently under way to identify the experimental input with the largest impact on the theory uncertainty.

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