IL NUOVO CIMENTO **38 C** (2015) 155
DOI 10.1393/ncc/i2015-15155-5

Colloquia: VILASIFEST

Exact results in the Skyrme model in $(3+1)$ dimensions via the generalized hedgehog ansatz

Fabrizio Canfora

Centro de Estudios Científicos (CECS) - Arturo Prat 514, Valdivia, Chile

received 11 January 2016

Summary. — We present exact results in the $(3+1)$-dimensional Skyrme model. First of all, it will be shown that, in the Pionic sector, a quite remarkable phenomenon for a non-integrable $(3+1)$-dimensional field theory appears: a non-linear superposition law is available allowing the composition of solutions in order to generate new solutions of the full field equations keeping alive, at the same time, the interactions terms in the energy-density. Secondly, it will be shown that the generalized hedgehog ansatz can be extended to suitable curved backgrounds. Interestingly, one can choose the background metric in such a way to describe finite-volume effects and, at the same time, to simplify the Skyrme field equations. In this way, it is possible to construct the first exact multi-Skyrmionic configurations of the $(3+1)$-dimensional Skyrme model with arbitrary high winding number and living at finite volume. Last but not least, a novel BPS bound (which is sharper than the usual one in term of the winding number) will be derived which can be saturated and reduces the field equations to a first-order equation for the profile.

1. – Introduction

One of the main teaching of Prof. Vilasi which has been extremely important through all my research career is the search of beauty in theoretical physics: one should only publish papers in which nice mathematical patterns are disclosed. Especially when only few experiments are available, this is the most reasonable guide in our field and often such nice patterns have a deep physical meaning.

One of the model in which the search for mathematical beauty has produced some of the most remarkable physical results is the one introduced by Skyrme long ago [1]. Indeed, the Skyrme theory is one of the most important model of theoretical physics due to its wide range of applications. Skyrme [1] introduced his famous term to allow the existence of static soliton solutions with finite energy representing Fermionic degrees of freedom (despite the fact that the basic field of the Skyrme theory are scalars). Moreover, these Fermionic topological solitons called *Skyrmions* turn out to be suitable to describe nucleons (see [2-8] and references therein).

It is worth to emphasize that the topic of bosonization in which one is able to make up Fermions from Bosonic degrees of freedom became very popular in the '80 (see, for two detailed reviews [9, 10]). However, most of the examples of that period were in low dimensions. In fact, Skyrme already in the early sixties had this idea but in $(3+1)$ dimensions.

A characteristic of the Skyrme model which, until very recently, basically prevented the construction of non-trivial analytical configurations in which the non-linear effects of the Skyrme term were manifest is the fact that, unlike what happens for instance in the case of monopoles and instantons in Yang-Mills-Higgs theory (see, for instance, [2,3,11]), the Skyrme-BPS bound on the energy in term of the winding number cannot be saturated for non-trivial spherically symmetric Skyrmions. In other words, it seems to be impossible to reduce the Skyrme field equations to a first-order equation keeping alive, at the same time, non-trivial topological charges.

A fundamental theoretical challenge is to construct exact configurations of the Skyrme model in which the non-linear effects are explicitly present. This would shed considerable light on the peculiar interactions of the Skyrme model. In particular, a very interesting issue is to explain how, in a non-linear and non-integrable theory such as the Skyrme model in four dimensions, nice crystal-like structures (see [13] and references therein) are able to appear. These beautiful configurations, which have been constructed numerically, look almost like non-trivial super-positions of "elementary solutions" but, at a first glance, it appears to be impossible to accommodate both non-linear effects and (any sort of) superposition law.

On the other hand, in the mathematical literature, the concept of *non-linear superposition law* is already known since Lie (which to the best of the author's knowledge was the first to formulate this idea). Unfortunately, all the examples constructed so far (for a nice review see [12]) correspond to systems of ordinary differential equations which are cooked up in such a way to possess such remarkable property (which will be defined more precisely in the next sections). Examples of systems of partial differential equations (PDEs henceforth) arising from realistic non-linear field theory models are still missing.

In the present paper, it will be shown that using the generalized hedgehog ansatz introduced in [14-17] for $SU(2)$-valued scalar fields (which recently has also been extended to the $SU(N)$ case in [18]) one can show that a remarkable phenomenon takes place: a non-linear superposition law appears which allows to combine two or more "elementary" configurations (whose profile depends in a non-trivial way on all the space-like coordinates) into a new exact composite configuration representing an interacting cloud of Pions. Despite the explicit presence of non-linear effects, the interaction energy between the moduli of elementary configuration *can be computed exactly.* Moreover the same formalism allows [17] the construction of multi-Skyrmions at finite volume: namely, exact solutions of the Skyrme model representing interacting elementary Skyrmions with a non-trivial winding number, in which finite-volume effects can be explicitly taken into account. The way to do this is to write the system in a modified "cylinder-like" metric whose curvature is parametrized by a length R_0. The ground state of such multi-Skyrmions has the remarkable property that although the BPS bound in term of the winding cannot be saturated, a new topological charge exists leading to a different BPS bound which instead can be saturated allowing to reduce the second-order field equations to a first-order equation in a genuine BPS-style.

This paper is organized as follows: in the second section, the generalized hedgehog ansatz as well as the non-linear superposition law will be presented. In the third section, how the generalized hedgehog leads to the first multi-Skyrmionic configurations living at finite volume will be described. In the fourth sections some conclusions as well as interesting directions for future investigation will be discussed.

2. – Generalized hedgehog and non-linear superposition law

The action of the $SU(2)$ Skyrme system in four-dimensional space-times is

$$S_{\text{Skyrme}} = \frac{K}{2}\int d^4x\sqrt{-g}\,\mathrm{Tr}\left(\frac{1}{2}R^\mu R_\mu + \frac{\lambda}{16}F_{\mu\nu}F^{\mu\nu}\right), \qquad K>0, \qquad \lambda>0, \tag{1}$$
$$R_\mu := U^{-1}\nabla_\mu U = R^j_\mu t_j,\ F_{\mu\nu} := [R_\mu, R_\nu], \qquad \hbar = 1, \qquad c = 1,$$

where the Planck constant and the speed of light have been set to 1, g is the determinant of the metric, the coupling constants K and λ are fixed by comparison with experimental data (see for instance [7]) and the t^j are the basis of the $SU(2)$ generators

$$t_j = -i\frac{\sigma_j}{2}$$

(where the Latin index j corresponds to the group index and σ_j are the Pauli matrices).

The Skyrme field equations are

$$\nabla^\mu R_\mu + \frac{\lambda}{4}\nabla^\mu[R^\nu, F_{\mu\nu}] = 0, \tag{2}$$

where ∇^μ corresponds to the derivative operator. In this section the following flat background metric will be considered:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

The standard parametrization of the $SU(2)$-valued scalar $U(x^\mu)$ will be adopted

$$U^{\pm 1}(x^\mu) = Y^0(x^\mu)\mathbf{1} \pm Y^i(x^\mu)t_i, \qquad \left(Y^0\right)^2 + Y^iY_i = 1, \tag{3}$$
$$Y^0 = \cos\alpha, \qquad Y^i = \widehat{n}^i \sin\alpha, \tag{4}$$
$$\widehat{n}^1 = \sin\Theta\cos\Phi, \qquad \widehat{n}^2 = \sin\Theta\sin\Phi, \qquad \widehat{n}^3 = \cos\Theta, \tag{5}$$

where α, Φ and Θ can depend, in principle on all the four space-time coordinates([1])

$$\alpha = \alpha(x^\mu), \quad \Theta = \Theta(x^\mu), \quad \Phi = \Phi(x^\mu). \tag{6}$$

Since the parametrization in eqs. (3), (4) and (5) corresponds to the most general element of $SU(2)$, by replacing it into the action in eq. (1) as well as into the equations of motion in eq. (2) the $SU(2)$ Skyrme theory can be interpreted as a theory of three interacting scalar fields α, Φ and Θ. Usually, the scalar field α is called profile while Φ and Θ describe the orientation within isospin space.

([1]) For instance, on flat spaces in spherical coordinates, when $\alpha = \alpha(r)$, $\Theta = \theta$ and $\Phi = \varphi$ one gets back the usual hedgehog ansatz for the spherycally symmetric Skyrmion.

One of the main features of the well known hedgehog ansatz introduced by Skyrme to describe his topological solitons (which however is rarely emphasized in the textbooks and papers on the subject) is that, with a suitable ansatz for Φ and Θ, it reduces a coupled system of PDEs for α, Φ and Θ to just one non-linear equation for the profile α (see, for instance, [2]). This is a quite remarkable property since usually Φ and Θ are chosen *a priori* to have non-trivial topological charge and, at the same time, a spherically symmetric energy-density. In this way one is left with just one free degree of freedom (namely, the profile α) but still three field equations to solve and yet the system is consistent. The above property define the generalized hedgehog ansatz [14-16]: namely under which conditions on α, Φ and Θ the full system of field equations of the Skyrme theory reduces to just one equation for the profile α. Interestingly enough, such formalism can also be applied to the coupled Einstein-Skyrme system: interesting cosmological applications have been investigated in [19, 20].

In the Pionic sector [14-16] such conditions([2]) read

$$Y^0 = \cos\alpha, \quad Y^i = \widehat{n}^i \sin\alpha, \tag{7}$$

$$\widehat{n}^1 = \cos\Phi, \quad \widehat{n}^2 = \sin\Phi, \quad \widehat{n}^3 = 0, \ (\nabla_\mu \alpha)(\nabla^\mu \Phi) = 0, \tag{8}$$

with Φ a linear function of the coordinates. This is called Pionic sector since the winding number vanishes identically.

As it has been discussed in [16] a very convenient choice is to take Φ in such a way that

$$(\nabla_\mu \Phi)(\nabla^\mu \Phi) = (\nabla\Phi)^2 = \text{const} \neq 0, \tag{9}$$

since the system becomes rather trivial when $(\nabla\Phi)^2$ vanishes (in which case the field equation for α linearizes and, mainly, the energy-momentum tensor becomes quadratic in α so that the non-linear effects disappear). Thus when eq. (9) is satisfied, as one can check directly, the full Skyrme field equations eq. (2) (which are a coupled system of non-linear partial differential equations for α, Φ and Θ) reduce to the following single scalar non-linear partial differential equation for the Skyrmion profile α:

$$\begin{aligned} 0 &= \left(1 + \lambda (\nabla\Phi)^2 \sin^2\alpha\right) \Box\alpha + \frac{\lambda (\nabla\Phi)^2 (\nabla\alpha)^2}{2} \sin(2\alpha) - \frac{(\nabla\Phi)^2}{2} \sin(2\alpha) = 0, \\ \Box &= \nabla_\mu \nabla^\mu, \end{aligned} \tag{10}$$

while the *t-t* component of the energy-momentum (which represents the energy-density) reads

$$\begin{aligned} T_{tt} = K \Big\{ & (\nabla_t \alpha)^2 + \sin^2\alpha (\nabla_t \Phi)^2 - \frac{g_{tt}}{2} \left[(\nabla\alpha)^2 + \sin^2\alpha (\nabla\Phi)^2 \right] \\ & + \lambda \sin^2\alpha \left((\nabla\Phi)^2 (\nabla_t \alpha)^2 + (\nabla\alpha)^2 (\nabla_t \Phi)^2 \right) - \frac{\lambda g_{tt}}{2} \sin^2\alpha \left((\nabla\Phi)^2 (\nabla\alpha)^2 \right) \Big\}. \end{aligned} \tag{11}$$

([2]) There are also two further technical conditions which become trivial on flat spaces [15].

The class of exact time-periodic solutions (denoted as "periodic Skyrmions") corresponds to the following choices of the profile of the hedgehog α and of the function Φ:

$$\alpha = \alpha(x,y,z), \quad (\nabla\alpha)^2 = (\partial_x\alpha)^2 + (\partial_y\alpha)^2 + (\partial_z\alpha)^2, \tag{12}$$

$$\Phi = \omega t, \quad \omega \in \mathbb{R}, \quad (\nabla\Phi)^2 = -\omega^2. \tag{13}$$

The ansatz in eq. (13) describes Skyrmions with a profile α which depends on all the space-like coordinates. On the other hand, the internal vector $\widehat{n}^i$ which describes the orientation of the Skyrmion in the internal $SU(2)$ space oscillates in time with frequency ω between the first and the second generators of the $SU(2)$ algebra: the description of this dynamical situation would be impossible with the usual spherical hedgehog ansatz.

With the above choice of α and Φ, the full Skyrme field equations eq. (2) reduce consistently to (see [14-16]) the following scalar elliptic non-linear partial differential equation for the Skyrmion profile

$$0 = \left(1 - \lambda\omega^2\sin^2\alpha\right)\triangle\alpha + \frac{\omega^2}{2}\sin(2\alpha) - \frac{\lambda\omega^2\sin(2\alpha)}{2}(\nabla\alpha)^2, \tag{14}$$

where $\triangle$ is the flat three-dimensional Laplacian.

It is possible to define a convenient change of variable in eq. (14) which discloses one of the very intriguing effects mentioned in the introduction. In terms of the function $H(\alpha)$ of the profile α

$$\begin{aligned} H(\alpha) = \int^{\alpha} ds\sqrt{1-\lambda\omega^2\sin^2 s} \;\Rightarrow\; & \frac{\triangle H}{\sqrt{1-\lambda\omega^2\sin^2\alpha}} \\ & = \left[\triangle\alpha - \frac{\lambda\omega^2}{2}\frac{\sin(2\alpha)(\nabla\alpha)^2}{1-\lambda\omega^2\sin^2\alpha}\right], \end{aligned} \tag{15}$$

eq. (14) can be written as

$$\triangle H + \frac{\omega^2\sin(2\alpha)}{2\sqrt{1-\lambda\omega^2\sin^2\alpha}} = 0, \tag{16}$$

where one should express α in terms of H inverting the elliptic integral in eq. (15). A very surprising phenomenon is now apparent: if

$$\omega = \omega^* = \pm\frac{1}{\sqrt{\lambda}}, \tag{17}$$

then eq. (14) with the change of variable in eq. (15) reduces to the following (*linear!*) Helmholtz equation

$$\triangle H + \frac{1}{\lambda}H = 0, \qquad \alpha = \arcsin H. \tag{18}$$

which, as it will be discussed below, allows to define an exact *non-linear superposition law.* The energy-density (defined in eq. (11)) in terms of H becomes

$$T_{tt} = K\left\{\frac{1}{2\lambda}H^2 + \frac{1}{2}\left(\frac{1+H^2}{1-H^2}\right)(\nabla H)^2\right\}. \tag{19}$$

The explicit presence of non-linear effects in the energy-momentum tensor in eq. (19) despite the fact that H satisfies the linear Helmholtz equation is related to the breaking of the homogeneous scaling symmetry. Unlike what happens in free field theories, the energy-density does not scale homogeneously under the rescaling

$$H \to \rho H, \qquad \rho \in \mathbb{R}.$$

In particular, by definition (see eqs. (18) and (7)), $|H|$ cannot be larger than 1 and, moreover, from the energetic point of view, it may be very "expensive" for $|H|$ to get close to 1 as it is clear from eq. (19). Therefore, one can multiply a given solution $H_{(0)}$ for a constant ρ *only provided* the (absolute value of the) new solution $\rho H_{(0)}$ of the Helmholtz equation does not exceed 1.

The non-analytic dependence of the energy density in eq. (19) on the Skyrme coupling λ clearly shows both the non-perturbative nature of the present effect and the fact that it is closely related to the Skyrme term. Here only the cases in which eq. (18) has a unique solution up to integration constants (which play the role of moduli of the Skyrmions) will be considered since they allow a more transparent physical interpretation (but more general situations can be analyzed as well). The periodic Skyrmions corresponding such unique solution will be denoted as *elementary Skyrmions*.

Let us consider the case in which the soliton profiles depend on three space-like coordinates x, y and z. Periodic boundary conditions (with periods $2\pi L_i$) in the spatial directions will be considered. Let

$$H_i = H\left(\overrightarrow{x} - \overrightarrow{x}_i\right) = A_i\left(\sin\mu_1\left(x - x_i\right)\sin\mu_2\left(y - y_i\right)\sin\mu_3\left(z - z_i\right)\right), \tag{20}$$

$$\frac{1}{\lambda} = \sum_{i=1}^{3}\mu_i^2, \quad \mu_i = \frac{1}{L_i}, \quad \overrightarrow{x}_i = \left(x_i, y_i, z_i\right), \quad \alpha_i = \arcsin H_i, \tag{21}$$

where $H\left(\overrightarrow{x} - \overrightarrow{x}_i\right)$ is the solution of eq. (18). As one can check in eq. (19), the positions of the peaks in the energy density in eq. (19) corresponding to the elementary Skyrmions $\alpha_i = \arcsin H_i$ are determined by $\overrightarrow{x}_i$ which therefore plays the role of the moduli of α_i since $\overrightarrow{x}_i$ identifies the "position" of the elementary Skyrmion. On the other hand, the overall constant A_i strictly speaking does not represent a moduli of the elementary Skyrmion since, when one replaces the expression in eq. (20) into eq. (19), one can see that the total energy depends on A_i while it does not depend on $\overrightarrow{x}_i$.

The most natural way to define a continuous composition of N elementary Skyrmions $\alpha_i = \arcsin H_i$ with moduli $\overrightarrow{x}_i$ (defined in eqs. (20) and (21)) with the property that, when all the H_i are small, the profile of the sum reduces to the sum of the profiles is

$$\alpha_{1+2+\ldots+N} = \arcsin\left(H_1 + H_2 + \ldots + H_N\right), \qquad \text{if} \quad \left|H_1 + H_2 \ldots + H_N\right| \leq 1, \tag{22}$$

$$\alpha_{1+2+\ldots+N} = \frac{\pi}{2}, \qquad \text{if} \quad H_1 + H_2 \ldots + H_N > 1, \tag{23}$$

$$\alpha_{1+2+\ldots+N} = -\frac{\pi}{2}, \qquad \text{if} \quad H_1 + H_2 \ldots + H_N < -1, \tag{24}$$

where one must take into account that $\arcsin x$ is only defined when $|x| \leq 1$. At a first glance, in the cases in which $|H_1 + H_2 + \ldots| > 1$, discontinuities in the first derivatives of the composite Skyrmion can appear. In fact, it is unlikely that such non-smooth solutions can survive since they are very expensive energetically (since the corresponding gradient would be unbounded). Hence, the non-linear superposition of elementary Skyrmions is only allowed when they satisfy $|H_1 + H_2 + \ldots| < 1$ otherwise it is not energetically convenient to combine the elementary Skyrmions into the composite Skyrmion.

As far as the appearance of ordered patterns is concerned, from eqs. (11) and (19) one can see that the energy-density of the composition $\alpha_{1+2+\ldots+N}$ of N elementary Skyrmions (whose integral represents the interaction energy between the N elementary Skyrmions and can be computed in principle for any N) depends in a complicated way on the moduli $\overrightarrow{x}_i$. However, one can observe in eqs. (11) and (19) that, in the expression of the energy-density of the composition $\alpha_{1+2+\ldots+N}$, in order to find configurations of the $\overrightarrow{x}_i$ which are favorable energetically, one should minimize with respect to the moduli $\overrightarrow{x}_i$ quadratic sums of the following type:

$$\Psi = \left(\sum_{i=1}^{N} H\left(\overrightarrow{x} - \overrightarrow{x}_i\right) \right)^2 .$$

In all the cases in which $H\left(\overrightarrow{x}\right)$ involves trigonometric functions, the theory of interference in optics([3]) can be applied to minimize sums of the type appearing in the above equation since one can interpret Ψ as the interference of many elementary waves. Hence, placements in which the $\overrightarrow{x}_i$ follow patterns of negative interference are always favorable energetically (although other local minima of the total energy appear as well). In the cases in which $H\left(\overrightarrow{x}\right)$ involves a different basis of functions (such as the Bessel functions which naturally appear when analyzing the Helmholtz equation in unbounded domains) the known results in optics cannot be applied directly, but it is reasonable to expect that also in those cases the $\overrightarrow{x}_i$ follow patterns associated to "negative interference of Bessel functions".

It is worth emphasizing that the present remarkable non-linear composition phenomenon in a non-integrable theory such as the four-dimensional Skyrme model depends crucially on the (square of the gradient of) the field Φ (as it is clear from eq. (13)) as in eq. (17). It is natural to wonder whether the corresponding energy scale defined in eq. (17) has actually a deeper physical meaning or the appearance of such phenomenon is just a "coincidence". This very interesting question is under investigation.

Thus, from the above analysis, one can conclude that it is not true (as one could naively think) that the Skyrme term always makes the field equations more complicated than the one of the non-linear sigma model (namely, its $\lambda = 0$ limit). It is precisely *because of the presence of the Skyrme term* that the such non-linear composition law appears. Besides its relations with the appearance of ordered patterns, this shows that the Skyrme term is also very relevant in the Pionic sector where the winding number vanishes.

3. – Exact multi-Skyrmions at finite volume

In this section, the construction of analytic multi-Skyrmionic configurations at finite volume in the $SU(2)$ case [17] will be described.

([3]) Namely, if one considers sums such as $\sum_j^{N-1} A \exp(i\xi_j)$ one can show that it vanishes when $\xi_j - \xi_{j-1}$ is equal to $2\pi/N$ for any j. In the present case, the difference between the arguments in the summands is related to the distance between peaks of neighboring Skyrmions.

Firstly however, it is necessary to emphasize why it is important to analyze finite-volume effects in the Skyrme model. One of the most important open problems in Skyrme theory is related with the compression modulus (see for instance [21]) which is an experimental measurable quantity related to the second derivative of the total energy with respect to the volume of a system possessing many Skyrmions (therefore, one needs to put Skyrmions within a finite volume in order to analyze this issue). It has been argued (see for instance, [22]) that the Skyrme model is unable to produce a theoretical value for the compression modulus close to the experimental one. On the other hand, such argument was proposed in a situation in which there were still no analytic configurations of the original Skyrme model with non-vanishing winding number. Moreover, it appeared that to find analytic solutions with high winding numbers living at finite volume was even more difficult than in the usual case. In fact, it will be now shown how one can construct analytic multi-Skyrmionic configurations living at finite volumes with a suitable choice of the background metric.

As in the previous section, the parametrization of the $SU(2)$-valued scalar $U(x^\mu)$ in eqs. (3), (4) and (5) will be adopted. In order to describe a spherically symmetric field configuration we use the hedgehog ansatz:

$$Y^0 = \cos\alpha, \qquad Y^i = \widehat{n}^i \sin\alpha, \qquad \alpha = \alpha(x,t), \tag{25}$$

$$\widehat{n}^1 = \sin\theta\cos\varphi, \qquad \widehat{n}^2 = \sin\theta\sin\varphi, \qquad \widehat{n}^3 = \cos\theta. \tag{26}$$

The key idea of [17] for the $SU(2)$ case (which has been recently extended to the $SU(N)$ case in [18]) is that in order to mimic finite-volume effects without loosing the nice symmetries of the hedgehog ansatz one can analyze the Skyrme theory on a suitable curved background of finite volume. The best choice is the following curved background

$$ds^2 = -dt^2 + dx^2 + R_0^2(d\theta^2 + (\sin\theta)^2 d\varphi^2), \qquad -\frac{L}{2} \le x \le \frac{L}{2}, \tag{27}$$

$$0 \le \theta \le \pi, \qquad 0 \le \varphi \le 2\pi, \tag{28}$$

where L is the length of the x-interval. The total volume of space is $V = 4\pi R_0^2 L$. As one can check directly, the above ansatz for the Skyrme configuration in eqs. (25) and (26) satisfies the generalized hedgehog property (defined in the previous section). Namely, with the above ansatz for the Skyrme field, the full Skyrme field equations on the metric in eq. (27) reduce consistently to just one equation for the profile α. This geometry describes three-dimensional cylinders whose sections are S^2 spheres, so that parameter R_0 plays the role of the (finite) diameter of the transverse sections of the tube. The fact that this parameter replaces the radial variable r in the metric also leads, as we will see, to considerable simplification of the equations of motion, even allowing to find exact solutions in the $SU(2)$ case. The main reason behind the simplifications is that, unlike what happens in the case of the flat metric in polar coordinates, the determinant of the metric in eq. (27) does not depend on x.

Moreover, the curvature of this metric is proportional to $(R_0)^{-2}$. The explicit presence of this parameter in eq. (27) allows to define a smooth flat limit in which $R_0 \to \infty$ and so all the effects of the curvature disappear (however, the global topology of the space remains cylindrical even in the flat limit and so it differs from the trivial S^3 topology of flat static *unbounded* Skyrmions). In the present context, "flat limit" really means

$$R_0 \gg 1 \text{ fm}, \tag{29}$$

so that, from the practical point of view, already when R_0 is around 100 fm all the effects of the curvature are negligible compared, for instance, with QCD corrections. Consequently, even in the flat limit in eq. (29) finite-volume effects will not disappear. It is also worth to emphasize that the well known result that elementary Skyrmions should be quantized as Fermions (which originally was derived on flat spaces) has been extended to space-times with compact orientable three-dimensional spatial sections in [23] (and the metric in eq. (27) belongs to this class).

One may wonder whether it would be possible to start from the very beginning with a flat metric. In fact, as it has been shown in [17], the background metric in eq. (27) is a very suitable tool to take into account finite-volume effects (since the total spatial volume is finite) without breaking relevant symmetries of the hedgehog ansatz, with the additional advantage of simplifying the field equations. Therefore, it is much more convenient to analyze the Skyrme theory first within the background metric in eq. (27), and take the flat limit only later.

With the above ansatz the Skyrme field equations reduce in the static case to the following scalar differential equation for the Skyrmion profile α [17]:

$$\left(1+\frac{2\lambda}{R_0^2}\sin^2\alpha\right)\ddot{\alpha}-\frac{\sin(2\alpha)}{R_0^2}\left(1-\lambda\left[\dot{\alpha}^2-\frac{\sin^2\alpha}{R_0^2}\right]\right)=0, \tag{30}$$

$$\frac{\mathrm{d}\alpha}{\mathrm{d}x}=\dot{\alpha}.$$

Here the advantage of the choice of the background metric in eq. (27) is apparent: although the above equation has a very similar structure to the usual one (see, for instance, [2]) it is an autonomous equation for α (namely, no explicit dependence on the independent variable x appears). As it will be shown in a moment, this allows to solve the above equation by quadratures.

The winding number W for such a configuration reads:

$$W=-\frac{1}{24\pi^2}\int\epsilon^{ijk}\mathrm{Tr}\left(U^{-1}\partial_iU\right)\left(U^{-1}\partial_jU\right)\left(U^{-1}\partial_kU\right)=-\frac{2}{\pi}\int\left(\dot{\alpha}\sin^2\alpha\right)\mathrm{d}x. \tag{31}$$

In the present case, the natural boundary conditions correspond to the choice:

$$\alpha\left(\frac{L}{2}\right)-\alpha\left(-\frac{L}{2}\right)=n\pi,\qquad n\in\mathbb{Z}. \tag{32}$$

and with these boundary conditions the winding number takes the integer value n. These boundary conditions are unique in that they ensure $U(-\frac{L}{2})=(-1)^nU(\frac{L}{2})$, which correspond to Bosonic and Fermionic states for even and odd n, respectively.

It will be now shown that smooth solutions exist for any n satisfying the above boundary conditions for a finite range $(-L/2, L/2)$. In particular multi-soliton solutions exist, which represent Skyrmions with winding number n living in a finite spatial volume $V=4\pi R_0^2L$. It is worth to remark that the large n limit in the present context is quite natural since we want to consider thermodynamical properties of the multi-Skyrmions system and n is the baryon number: obviously, a thermodynamical analysis only makes sense in the cases in which the particles number is very large.

The first step to achieve this goal is to observe that the energy density (which is the 0-0 component of the $T_{\mu\nu}$) derived from the Skyrme action reads

$$T_{00} = \frac{KF(\alpha)}{2}\left[\dot{\alpha} \mp \left(\frac{2G(\alpha)}{F(\alpha)}\right)^{1/2}\right]^2 \pm \sqrt{2}K\left[F(\alpha)G(\alpha)\right]^{1/2}\dot{\alpha}. \tag{33}$$

In the above equations, F and G are defined as

$$F(\alpha) = \left(1 + \frac{2\lambda}{R_0^2}\sin^2\alpha\right), \quad G(\alpha) = \frac{\sin^2\alpha}{R_0^2}\left(1 + \frac{\lambda}{2R_0^2}\sin^2\alpha\right). \tag{34}$$

It is worth to note that the second term on the right hand side of eq. (33) is a total derivative.

The second step is to observe that, by multiplying eq. (30) by $\dot{\alpha}$, one can easily see that it can be reduced to a quadrature:

$$\left(\dot{\alpha}\right)^2 = \frac{I + 2G(\alpha)}{F(\alpha)}, \tag{35}$$

where I is an integration constant (of dimensions *lenght*$^{-2}$). Equation (35) is suitable to discuss the boundary conditions in eq. (32) are realized. In particular, one can find a closed equation which determines the dependence of the integration constant I in terms of the parameters of the model:

$$\frac{L}{n} = \pm\int_0^\pi \mathrm{d}z \left[\frac{1 + \frac{2\lambda}{R_0^2}\sin^2 z}{I + 2\frac{\sin^2 z}{R_0^2}\left(1 + \frac{\lambda}{2R_0^2}\sin^2 z\right)}\right]^{1/2}, \tag{36}$$

where n is the winding number. From the above relation one can easily see that, in order to have n Skyrmions at finite volume I must be positive. Moreover, for fixed volume (namely, fixed L and R_0) and very large n one gets that I is proportional to n^2. The other interesting limit is L and n both large but L/n fixed (which is the relevant region of parameters for the compression modulus): in this case I only depends on the ratio L/n (as well as R_0). In particular, within this region one can find a range of values for the integration constant I which makes the theoretical compression modulus close to the experimental one [17].

Moreover, from the explicit expression of the energy density, one can derive a novel BPS bound for the total Energy E_{tot}:

$$E_{tot} = 4\pi R_0^2 K \int T_{00}\mathrm{d}x \geq |Q|, \tag{37}$$

$$Q = \sqrt{2}4\pi R_0^2 K \int \left(\left[F(\alpha)G(\alpha)\right]^{1/2}\frac{\mathrm{d}\alpha}{\mathrm{d}x}\right)\mathrm{d}x. \tag{38}$$

Clearly, the term Q in eq. (38) is a boundary term and is therefore invariant under continuous deformations of the fields in the bulk, as suits a topological invariant. The above BPS bound can be saturated when α satisfies the following first-order differential equation:

$$\alpha' = \pm \left(\frac{2G(\alpha)}{F(\alpha)} \right)^{1/2} . \tag{39}$$

It is worth to note that the above BPS equation does imply the field equation as it can be explicitly seen from eq. (35). If one chooses $I = 0$ in eq. (35) one obtains exactly the BPS equation (39). Hence, the present Skyrmions have two type of topological charges. The first one is the winding and the second one is the Q charge defined in eq. (38). It can be seen that the topological charge Q is bounded from below, $|Q| > |W|$. The configurations which saturate the bound in eq. (37) necessarily have winding number $n = \pm 1$ (consequently, they correspond to Fermions) and live in an unbounded domain: namely, $L \to \infty$ in eq. (27). To the best of author's knowledge, this is the first non-trivial example in which the full Skyrme field equations can be reduced to first order in terms of the novel topological charge defined in eq. (38).

Hence, this analysis shows that although the BPS bound in terms of the winding number cannot be saturated different BPS bounds can exist which allow to reduce the field equations to first order in genuine BPS style. It is also worth emphasizing that this strategy to adapt the geometry to the field equation can be very effective. In particular, it is quite surprising that it is not the flat metric the one to simplify the most the Skyrme field equations.

4. – Conclusions and perspectives

In the present paper, the generalized hedgehog ansatz for the $SU(2)$ Skyrme model in $(3+1)$ dimensions has been described through two of its most interesting applications.

First of all, it has been shown that even in the Pionic sector with vanishing winding number, the Skyrme term plays a fundamental role. In particular, it gives rise to a quite remarkable phenomenon for a non-integrable four-dimensional field theory: a non-linear superposition law appears which allows composing solutions in order to generate new solutions of the full field equations keeping alive, at the same time, the interactions terms in the energy-density. This intriguing phenomenon is closely related to the appearance of ordered patterns in the Skyrme theory.

Secondly, it has been shown that the generalized hedgehog ansatz can be extended to suitable curved backgrounds. In particular, one can choose the background metric in such a way to describe finite-volume effects and, at the same time, to simplify the Skyrme field equations. In this way, it is possible to construct the first exact multi-Skyrmionic configurations of the $(3+1)$-dimensional Skyrme model with arbitrary high winding number and living at finite volume. Moreover, a novel BPS bound (which is sharper than the usual one in terms of the winding number) has been derived which can be saturated and reduces the field equations to a first-order equation for the profile. Such multi-Skyrmionic configurations allow to derive an analytic expression for the compression modulus in good agreement with experiments.

It is worth emphasizing that, from the point of view of the theory of PDEs, these multi-Skyrmionic configurations made of n elementary Skyrmions are very different from the solutions arising in the Pionic sector in which the non-linear composition law appears. In the Fermionic case in which the winding number n is non-vanishing, no matter how large n is, the multi-Skyrmionic configurations defined implicitly in eq. (35) only have one non-trivial integration constant (namely I). In the Pionic sector corresponding to the non-linear superposition law, by composing many elementary solutions one can get a new solution characterized by many integration constants. Thus, in a sense, the Fermionic sector is more rigid.

There are many interesting ways to extend the above results. First of all, one can try to extend such formalism to the $SU(N)$ case. The results obtained recently in [18] are very promising in this respect. It is also very interesting to try to extend the above results in the presence of a non-vanishing chemical potential. In [24, 25] it has been shown that one can introduce an isospin chemical potential in the Skyrme model by simply replacing the ordinary derivative with a suitable covariant derivative. In [26, 27] this formalism has been successfully applied to the usual Skyrme solitons. It would be very interesting to see whether the above results obtained with the generalized hedgehog ansatz also hold in the presence of a non-vanishing chemical potential using [24-27]. One could try to see if the energy scale at which the non-linear superposition law appears has some deeper physical meaning related with Pions physics. Last but not least, it is very natural to wonder wether the present strategy to search for exact non-trivial configurations of the Skyrme model in (3+1) dimensions can be extended to the Einstein-Skyrme system. This issue is very interesting as the present analysis clearly shows the prominent role of the geometry within the present framework. Moreover, it is an extremely important issue in itself to analyze self-gravitating topological solitons since, until very recently, there were no analytic examples of self-gravitating objects possessing a discrete topological charge similar to the Skyrmions. At a first glance, such a goal could appear over-ambitious as already the Skyrme field equations are very hard to solve by themselves. Hence, one could think that to solve the coupled Einstein-Skyrme system is much more difficult. In fact, in [28] the first analytic family of globally regular self-gravitating Skyrmions in $(3+1)$ dimensions have been constructed. These configurations offer the unique opportunity to study in details the physical consequences of a discrete conserved charge in a realistic self-gravitating system together with the fact that, already at a classical level, the Skyrmions have a characteristic size.

Therefore, one can conclude that the search for mathematical beauty pays off with the appearance of patterns rich of physical implications.

* * *

This work has been funded by the Fondecyt grants 1120352. The Centro de Estudios Científicos (CECs) is funded by the Chilean Government through the Centers of Excellence Base Financing Program of Conicyt.

REFERENCES

[1] Skyrme T., *Proc. R. Soc. London A*, **260** (1961) 127; **262** (1961) 237; *Nucl. Phys.*, **31** (1962) 556.
[2] Manton N. and Sutcliffe P., *Topological Solitons* (Cambridge University Press, Cambridge) 2007.
[3] Olive David I. and West Peter C. (Editor(s)), *Duality and Supersymmetric Theories* (Cambridge University Press, Cambridge) 1999.
[4] Balachandran A. P., Gomm H. and Sorkin R. D., *Nucl. Phys. B*, **281** (1987) 573.
[5] Balachandran A. P., Barducci A., Lizzi F., Rodgers V. G. J. and Stern A., *Phys. Rev. Lett.*, **52** (1984) 887.

[6] Balachandran A. P., Lizzi F., Rodgers V. G. J. and Stern A., *Nucl. Phys. B*, **256** (1985) 525.
[7] Adkins G. S., Nappi C. R. and Witten E., *Nucl. Phys. B*, **228** (1983) 552.
[8] Guadagnini E., *Nucl. Phys. B*, **236** (1984) 35.
[9] Schonhammer K. and Meden V., *Am. J. Phys.*, **64** (1996) 1168.
[10] Frishman Y. and Sonnenschein J., *Phys. Rep.*, **223** (1993) 309.
[11] Manton N. S., *Phys. Lett.*, **110B** (1982) 54.
[12] Carinena J., Grabowski J. and Marmo G., *Rep. Math. Phys.*, **60** (2007) 237.
[13] Battye R. A., Sutcliffe P. M., *Phys. Rev. Lett.*, **79** (1997) 363; *Phys. Lett. B*, **391** (1997) 150.
[14] Canfora F. and Salgado-Rebolledo P., *Phys. Rev. D*, **87** (2013) 045023.
[15] Canfora F. and Maeda H., *Phys. Rev. D*, **87** (2013) 084049.
[16] Canfora F., *Phys. Rev. D*, **88** (2013) 065028.
[17] Canfora F., Correa F. and Zanelli J., *Phys. Rev. D*, **90** (2014) 085002.
[18] Canfora F., Kurkov M., Di Mauro M. and Naddeo A., *SU(N) Multi-Skyrmions at Finite Volume*, arXiv:1508.07055, to appear on *Eur. Phys. J.* **C**.
[19] Canfora F., Giacomini A. and Pavluchenko S. A., *Phys. Rev. D*, **90** (2014) 043516.
[20] Parisi L., Radicella N. and Vilasi G., *Phys. Rev. D*, **91** (2015) 063533.
[21] Blaizot J. P., *Phys. Rep.*, **64** (1980) 171; Jennings B. K. and Jackson A. D., *Phys. Rep.*, **66** (1980) 141.
[22] Adam C., Naya C., Sanchez-Guillen J., Speight J. M. and Wereszczynski A., *Phys. Rev. D*, **90** (2014) 045003.
[23] Auckly D. and Speight M., *Commun. Math. Phys.*, **263** (2006) 173; Krusch S. and Speight M., *Commun. Math. Phys.*, **264** (2006) 391.
[24] Weldon H. A., *Phys. Rev. D*, **26** (1982) 1394.
[25] Actor A., *Phys. Lett. B*, **157** (1985) 53.
[26] Loewe M., Mendizabal S. and Rojas J. C., *Phys. Lett. B*, **609** (2005) 437.
[27] Loewe M., Mendizabal S. and Rojas J. C., *Phys. Lett. B*, **632** (2006) 512.
[28] Ayon-Beato E., Canfora F. and Zanelli J., *Analytic self-gravitating Skyrmions, cosmological bounces and wormholes*, arXiv:1509.02659.