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Stellar structures in Extended Gravity

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Summary. — Stellar structures are investigated by considering the modified Lané-Emden equation coming out from Extended Gravity. In particular, this equation is obtained in the Newtonian limit of $f(R)$-gravity by introducing a polytropic relation between the pressure and the density into the modified Poisson equation. The result is an integro-differential equation, which, in the limit $f(R) \to R$, becomes the standard Lané-Emden equation usually adopted in the stellar theory. We find the radial profiles of gravitational potential by solving for some values of the polytropic index. The solutions are compatible with those coming from General Relativity and could be physically relevant in order to address peculiar and extremely massive objects.

PACS `04.25.Nx` – Post-Newtonian approximation; perturbation theory; related approximations.
PACS `04.50.Kd` – Modified theories of gravity.
PACS `04.40.Nr` – Einstein-Maxwell spacetimes, spacetimes with fluids, radiation or classical fields.
PACS `97.10.Cv` – Stellar structure, interiors, evolution, nucleosynthesis, ages.

1. – Introduction

Extended Theories of Gravity (ETG) [1] are a new paradigm of modern physics aimed to address several shortcomings coming out in the study of gravitational interaction at ultra-violet and infra-red scales. In particular, instead of introducing unknown fluids, the approach consists in extending General Relativity (GR) by taking into account generic functions of curvature invariants. These functions can be physically motivated and capable of addressing phenomenology at galactic, extragalactic, and cosmological scales [2].

This viewpoint does not require to find out candidates for dark energy and dark matter at fundamental level (not detected up to now), but takes into account only the observed ingredients (*i.e.* gravity, radiation and baryonic matter), changing the l.h.s. of the field equations. Despite this modification, it is in agreement with the spirit of GR since the only request is that the Hilbert-Einstein action should be generalized asking for a gravitational interaction acting, in principle, in different ways at different scales but preserving the robust results of GR at local and Solar System scales (see [1] for a detailed discussion). This is the case of $f(R)$-gravity which reduces to GR as soon as $f(R) \to R$.

Other issues as, for example, the observed Pioneer anomaly problem [3] can be framed into the same approach [4] and then, apart the cosmological dynamics, a systematic analysis of such theories urges at short scales and in the low energy limit.

On the other hand, the strong gravity regime [5] is another way to check the viability of these theories. In general the formation and the evolution of stars can be considered suitable test-beds for Alternative Theories of Gravity. Considering the case of $f(R)$-gravity, divergences stemming from the functional form of $f(R)$ may prevent the existence of relativistic stars in these theories [6], but thanks to the chameleon mechanism, introduced by Khoury and Weltman [7], the possible problems jeopardizing the existence of these objects may be avoided [8]. Furthermore, there are also numerical solutions corresponding to static star configurations with strong gravitational fields [9] where the choice of the equation of state is crucial for the existence of solutions.

Furthermore some observed stellar systems are incompatible with the standard models of stellar structure. We refer to anomalous neutron stars, the so-called "magnetars" [10] with masses larger than their expected Volkoff mass. It seems that, on particular length scales, the gravitational force is larger or smaller than the corresponding GR value. For example, a modification of the Hilbert-Einstein Lagrangian, consisting of R^2 terms, enables a major attraction while a $R_{\alpha\beta}R^{\alpha\beta}$ term gives a repulsive contribution [11]. Understanding on which scales the modifications to GR are working or what is the weight of corrections to gravitational potential is a crucial point that could confirm or rule out these extended approaches to gravitational interaction.

The plan of paper is the following: In sect. **2**, we review briefly the classical hydrostatic problem for stellar structures. In sect. **3** we derive the Newtonian limit of $f(R)$-gravity obtaining the modified Poisson equation. The modified Lané-Emden equation is obtained in sect. **4** and its structure is compared with respect to the standard one. In sect. **5**, we show the analytical solutions of standard Lané-Emden equation and compare them with those obtained perturbatively from $f(R)$-gravity. With help of plot we can compare between them all results. Discussion and conclusions are drawn in sect. **6**.

2. – Hydrostatic equilibrium of stellar structures

The condition of hydrostatic equilibrium for stellar structures in Newtonian dynamics is achieved by considering the equation

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{\mathrm{d}\Phi}{\mathrm{d}r}\rho, \tag{1}$$

where p is the pressure, $-\Phi$ is the gravitational potential, and ρ is the density [12]. Together with the above equation, the Poisson equation

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = -4\pi G\rho, \tag{2}$$

gives the gravitational potential as solution for a given matter density ρ. Since we are taking into account only static and stationary situations, here we consider only time-independent solutions([1]). In general, the temperature τ appears in eqs. (1) and (2) the density satisfies an equation of state of the form $\rho = \rho(p, \tau)$. In any case, we assume that there exists a polytropic relation between p and ρ of the form

$$p = K\rho^{\gamma}, \tag{3}$$

where K and γ are constant. Note that $\Phi > 0$ in the interior of the model since we define the gravitational potential as $-\Phi$. The polytropic constant K is fixed and can be obtained as a combination of fundamental constants. However there are several realistic cases where K is not fixed and another equation for its evolution is needed. The constant γ is the *polytropic exponent*. Inserting the polytropic equation of state into eq. (1), we obtain

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = \gamma K \rho^{\gamma-2} \frac{\mathrm{d}\rho}{\mathrm{d}r}. \tag{4}$$

For $\gamma \neq 1$, the above equation can be integrated giving

$$\frac{\gamma K}{\gamma - 1}\rho^{\gamma-1} = \Phi \rightarrow \rho = \left[\frac{\gamma - 1}{\gamma K}\right]^{\frac{1}{\gamma-1}} \Phi^{\frac{1}{\gamma-1}} \doteq A_n \Phi^n \tag{5}$$

where we have chosen the integration constant to give $\Phi = 0$ at surface ($\rho = 0$). The constant n is called the *polytropic index* and is defined as $n = \frac{1}{\gamma-1}$. Inserting the relation (5) into the Poisson equation, we obtain a differential equation for the gravitational potential

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -4\pi G A_n \Phi^n. \tag{6}$$

Let us define now the dimensionless variables

$$\begin{cases} z = |\mathbf{x}|\sqrt{\dfrac{\mathcal{X} A_n \Phi_c^{n-1}}{2}}, \\ w(z) = \dfrac{\Phi}{\Phi_c} = \left(\dfrac{\rho}{\rho_c}\right)^{\frac{1}{n}}, \end{cases} \tag{7}$$

where the subscript c refers to the center of the star and the relation between ρ and Φ is given by eq. (5). At the center ($r = 0$), we have $z = 0$, $\Phi = \Phi_c$, $\rho = \rho_c$ and therefore $w = 1$. Then eq. (6) can be written

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \frac{2}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + w^n = 0. \tag{8}$$

([1]) The radius r is assumed as the spatial coordinate. It varies from $r = 0$ at the center to $r = \xi$ at the surface of the star.

This is the standard *Lane-Embden equation* describing the hydrostatic equilibrium of stellar structures in the Newtonian theory [12].

3. – The Newtonian limit of $f(R)$-gravity

Let us start with a general class of Extended Theories of Gravity given by the action

$$\mathcal{A} = \int \mathrm{d}^4x\sqrt{-g}[f(R) + \mathcal{X}\mathcal{L}_m], \tag{9}$$

where $f(R)$ is an analytic function of the curvature invariant R. $\mathcal{L}_m$ is the minimally coupled ordinary matter Lagrangian density. In the metric approach, the field equations are obtained by varying the action (9) with respect to $g_{\mu\nu}$. We get

$$\begin{cases} f'R_{\mu\nu} - \dfrac{f}{2}g_{\mu\nu} - f_{;\mu\nu} + g_{\mu\nu}\Box f' = \mathcal{X}\,T_{\mu\nu}, \\ 3\Box f' + f'R - 2f \ = \ \mathcal{X}\,T, \end{cases} \tag{10}$$

where the second equation is the trace of the field equations.

Here, $T_{\mu\nu} = \frac{-1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$ is the the energy-momentum tensor of matter; $T = T^{\sigma}{}_{\sigma}$ is the trace; $f' = \frac{\mathrm{d}f(R)}{\mathrm{d}R}$, $\Box = {}_{;\sigma}{}^{;\sigma}$ the d'Alembert operator and $\mathcal{X} = 8\pi G$. We assume $c = 1$ is adopted. The conventions for Ricci's tensor are $R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu}$; the Riemann tensor is $R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} + \ldots$. The affine connections are the Christoffel's symbols of the metric $\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\sigma}(g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma})$. The signature is $(+ - - -)$.

In order to achieve the Newtonian limit of the theory the metric tensor $g_{\mu\nu}$ have to be approximated as follows

$$g_{\mu\nu} \sim \begin{pmatrix} 1 - 2\,\Phi(t,\mathbf{x}) + \mathcal{O}(4) & \mathcal{O}(3) \\ \mathcal{O}(3) & -\delta_{ij} + \mathcal{O}(2) \end{pmatrix}, \tag{11}$$

where $\mathcal{O}(n)$ (with $n =$ integer) denotes the order of the expansion (see [13] for details). The set of coordinates([2]) adopted is $x^{\mu} = (t, x^1, x^2, x^3)$. The Ricci scalar formally becomes

$$R \sim R^{(2)}(t,\mathbf{x}) + \mathcal{O}(4). \tag{12}$$

The n-th derivative of Ricci function can be developed as

$$f^n(R) \sim f^n(R^{(2)} + \mathcal{O}(4)) \sim f^n(0) + f^{n+1}(0)R^{(2)} + \mathcal{O}(4), \tag{13}$$

here $R^{(n)}$ denotes a quantity of order $\mathcal{O}(n)$. From lowest-order of field eqs. (10), we have $f(0) = 0$ which trivially follows from the above assumption (11) for the metric. This means that the space-time is asymptotically Minkowskian and we are discarding a

([2]) The Greek index runs between 0 and 3; the Latin index between 1 and 3.

cosmological constant term in this analysis ([3]). Equations (10) at $\mathcal{O}(2)$-order, that is at Newtonian level, are

$$\begin{cases} R_{tt}^{(2)} - \dfrac{R^{(2)}}{2} - f''(0)\triangle R^{(2)} = \mathcal{X}\, T_{tt}^{(0)}, \\ -3f''(0)\triangle R^{(2)} - R^{(2)} = \mathcal{X}\, T^{(0)}, \end{cases} \tag{14}$$

where $\triangle$ is the Laplacian in the flat space, $R_{tt}^{(2)} = -\triangle\Phi(t, \mathbf{x})$ and, for the sake of simplicity, we set $f'(0) = 1$. We recall that the energy-momentum tensor for a perfect fluid is

$$T_{\mu\nu} = (\epsilon + p)\, u_\mu u_\nu - p\, g_{\mu\nu}, \tag{15}$$

where p is the pressure and ϵ is the energy density. Being the pressure contribution negligible in the field equations in Newtonian approximation, we have

$$\begin{cases} \triangle\Phi + \dfrac{R^{(2)}}{2} + f''(0)\triangle R^{(2)} = -\mathcal{X}\rho, \\ 3f''(0)\triangle R^{(2)} + R^{(2)} = -\mathcal{X}\rho, \end{cases} \tag{16}$$

where ρ is now the mass density([4]). We note that for $f''(0) = 0$ we have the standard Poisson equation: $\triangle\Phi = -4\pi G\rho$. This means that as soon as the second derivative of $f(R)$ is different from zero, deviations from the Newtonian limit of GR emerge.

The gravitational potential $-\Phi$, solution of eqs. (16), has in general a Yukawa-like behavior depending on a characteristic length on which it evolves [13]. Then as it is evident the Gauss theorem is not valid ([5]) since the force law is not $\propto |\mathbf{x}|^{-2}$. The equivalence between a spherically symmetric distribution and point-like distribution is not valid and how the matter is distributed in the space is very important [13-15].

Besides the Birkhoff theorem results modified at Newtonian level: the solution can be only factorized by a space-depending function and an arbitrary time-depending function [13]. Furthermore the correction to the gravitational potential is depending on the only first two derivatives of $f(R)$ in $R = 0$. This means that different analytical theories, from the third derivative perturbation terms on, admit the same Newtonian limit [13,14].

Equations (16) can be considered the *modified Poisson equation* for $f(R)$-gravity. They do not depend on gauge condition choice [15].

4. – Stellar hydrostatic equilibrium in $f(R)$-gravity

From the Bianchi identity, satisfied by the field eqs. (10), we have

$$T^{\mu\nu}{}_{;\mu} = 0 \rightarrow \frac{\partial p}{\partial x^k} = -\frac{1}{2}(p + \epsilon)\frac{\partial \ln g_{tt}}{\partial x^k}. \tag{17}$$

([3]) This assumption is quite natural since the contribution of a cosmological constant term is irrelevant at stellar level.
([4]) Generally it is $\epsilon = \rho\, c^2$.
([5]) It is worth noticing that also if the Gauss theorem does not hold, the Bianchi identities are always valid so the conservation laws are guaranteed.

If the dependence on the temperature τ is negligible, *i.e.* $\rho = \rho(p)$, this relation can be introduced into eqs. (16), which become a system of three equations for p, Φ and $R^{(2)}$ and can be solved without the other structure equations.

Let us suppose that matter satisfies still a polytropic equation $p = K\,\rho^{\gamma}$. If we introduce eq. (5) into eqs. (16) we obtain an integro-differential equation for the gravitational potential $-\Phi$, that is

$$\triangle\Phi(\mathbf{x}) + \frac{2\mathcal{X}A_n}{3}\Phi(\mathbf{x})^n = -\frac{m^2\mathcal{X}A_n}{6}\int d^3\mathbf{x}'\mathcal{G}(\mathbf{x},\mathbf{x}')\Phi(\mathbf{x}')^n, \tag{18}$$

where $\mathcal{G}(\mathbf{x},\mathbf{x}') = -\frac{1}{4\pi}\frac{e^{-m|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}$ is the Green function of the field operator $\triangle_{\mathbf{x}} - m^2$ for systems with spherical symmetry and $m^2 = -\frac{1}{3f''(0)}$ (for details see [14,15]). The integro-differential nature of eq. (18) is the proof of the non-viability of Gauss theorem for $f(R)$-gravity. Adopting again the dimensionless variables

$$\begin{cases} z = \dfrac{|\mathbf{x}|}{\xi_0}, \\ w(z) = \dfrac{\Phi}{\Phi_c}, \end{cases} \tag{19}$$

where

$$\xi_0 \doteq \sqrt{\frac{3}{2\mathcal{X}A_n\Phi_c^{n-1}}} \tag{20}$$

is a characteristic length linked to stellar radius ξ, eq. (18) becomes

$$\begin{aligned} &\frac{d^2w(z)}{dz^2} + \frac{2}{z}\frac{dw(z)}{dz} + w(z)^n = \\ &\frac{m\xi_0}{8}\frac{1}{z}\int_0^{\xi/\xi_0} dz'\,z'\left\{e^{-m\xi_0|z-z'|} - e^{-m\xi_0|z+z'|}\right\}w(z')^n, \end{aligned} \tag{21}$$

which is the *modified Lané-Emden equation* deduced from $f(R)$-gravity. Clearly the particular $f(R)$-model is specified by the parameters m and ξ_0. If $m \to \infty$ (*i.e.* $f(R) \to R$), eq. (21) becomes eq. (8). We are only interested in solutions of eq. (21) that are finite at the center, that is for $z = 0$. Since the center must be an equilibrium point, the gravitational acceleration $|\mathbf{g}| = -d\Phi/dr \propto dw/dz$ must vanish for $w'(0) = 0$. Let us assume we have solutions $w(z)$ of eq. (21) that fulfill the boundary conditions $w(0) = 1$ and $w(\xi/\xi_0) = 0$; then according to the choice (19), the radial distribution of density is given by

$$\rho(|\mathbf{x}|) = \rho_c w^n, \qquad \rho_c = A_n{\Phi_c}^n \tag{22}$$

and the pressure by

$$p(|\mathbf{x}|) = p_c w^{n+1}, \quad p_c = K{\rho_c}^{\gamma}. \tag{23}$$

For $\gamma = 1$ (or $n = \infty$) the integro-differential eq. (21) is not correct. This means that the theory does not contain the case of isothermal sphere of ideal gas. In this case, the polytropic relation is $p = K\rho$. Putting this relation into eq. (17) we have

$$\frac{\Phi}{K} = \ln\rho - \ln\rho_c \to \rho = \rho_c\, e^{\Phi/K}, \tag{24}$$

where the constant of integration is chosen in such a way that the gravitational potential is zero at the center. If we introduce eq. (24) into eqs. (16), we have

$$\triangle\Phi(\mathbf{x}) + \frac{2\mathcal{X}\rho_c}{3}e^{\Phi(\mathbf{x})/K} = -\frac{m^2\mathcal{X}\rho_c}{6}\int \mathrm{d}^3\mathbf{x}'\mathcal{G}(\mathbf{x},\mathbf{x}')e^{\Phi(\mathbf{x}')/K}. \tag{25}$$

Assuming the dimensionless variables $z = \frac{|\mathbf{x}|}{\xi_1}$ and $w(z) = \frac{\Phi}{K}$, where $\xi_1 \doteq \sqrt{\frac{3K}{2\mathcal{X}\rho_c}}$, eq. (25) becomes

$$\begin{aligned}\frac{\mathrm{d}^2w(z)}{\mathrm{d}z^2} + \frac{2}{z}\frac{\mathrm{d}w(z)}{\mathrm{d}z} + e^{w(z)} =\\ \frac{m\xi_1}{8}\frac{1}{z}\int_0^{\xi/\xi_1}\mathrm{d}z'\,z'\left\{e^{-m\xi_1|z-z'|} - e^{-m\xi_1|z+z'|}\right\}e^{w(z')},\end{aligned} \tag{26}$$

which is the *modified "isothermal" Lané-Emden equation* derived $f(R)$-gravity.

5. – Solutions of the standard and modified Lané-Emden equations

The task is now to solve the modified Lané-Emden equation and compare its solutions to those of the standard Newtonian theory. Only for three values of n, the solutions of eq. (8) have analytical expressions [12]

$$\begin{cases} n = 0 \to w_{GR}^{(0)}(z) = 1 - \dfrac{z^2}{6}, \\ n = 1 \to w_{GR}^{(1)}(z) = \dfrac{\sin z}{z}, \\ n = 5 \to w_{GR}^{(5)}(z) = \dfrac{1}{\sqrt{1+\frac{z^2}{3}}}. \end{cases} \tag{27}$$

We label these solution with ${}_{GR}$ since they agree with the Newtonian limit of GR. The surface of the polytrope of index n is defined by the value $z = z^{(n)}$, where $\rho = 0$ and thus $w = 0$. For $n = 0$ and $n = 1$ the surface is reached for a finite value of $z^{(n)}$. The case $n = 5$ yields a model of infinite radius. It can be shown that for $n < 5$ the radius of polytropic models is finite; for $n > 5$ they have infinite radius. From eqs. (27) one finds $z_{GR}^{(0)} = \sqrt{6}$, $z_{GR}^{(1)} = \pi$ and $z_{GR}^{(5)} = \infty$. A general property of the solutions is that $z^{(n)}$ grows monotonically with the polytropic index n. In fig. 1 we show the behavior of solutions $w_{GR}^{(n)}$ for $n = 0, 1, 5$. Apart from the three cases where analytic solutions are known, the classical Lané-Emden eq. (8) has to be be solved numerically, considering

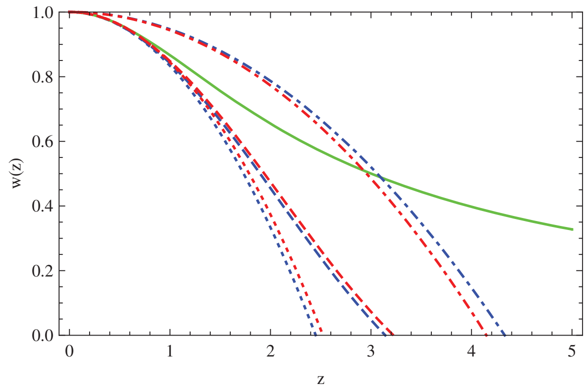

Fig. 1. – Plot of solutions (blue lines) of standard Lané-Emden eq. (8): $w^{(0)}_{GR}(z)$ (dotted line) and $w^{(1)}_{GR}(z)$ (dashed line). The green line corresponds to $w^{(5)}_{GR}(z)$. The red lines are the solutions of modified Lané-Emden eq. (21): $w^{(0)}_{f(R)}(z)$ (dotted line) and $w^{(1)}_{f(R)}(z)$ (dashed line). The blue dashed-dotted line is the potential derived from GR ($w_{GR}(z)$) and the red dashed-dotted line the potential derived from $f(R)$-gravity ($w_{f(R)}(z)$) for a uniform spherically symmetric mass distribution. The assumed values are $m\xi = 1$ and $m\xi_0 = 0.4$. From a rapid inspection of these plots, the differences between GR and $f(R)$ gravitational potentials are clear and the tendency is that at larger radius z they become more evident.

with the expression

$$w^{(n)}_{GR}(z) = \sum_{i=0}^{\infty} a^{(n)}_i z^i \tag{28}$$

for the neighborhood of the center. Inserting eq. (28) into eq. (8) and by comparing coefficients one finds, at lowest orders, a classification of solutions by the index n, that is

$$w^{(n)}_{GR}(z) = 1 - \frac{z^2}{6} + \frac{n}{120} z^4 + \dots \tag{29}$$

The case $\gamma = 5/3$ and $n = 3/2$ is the non-relativistic limit while the case $\gamma = 4/3$ and $n = 3$ is the relativistic limit of a completely degenerate gas.

Also for modified Lané-Emden eq. (21), we have an exact solution for $n = 0$. In fact, it is straightforward to find out

$$w^{(0)}_{f(R)}(z) = 1 - \frac{z^2}{8} + \frac{(1+m\xi)e^{-m\xi}}{4m^2{\xi_0}^2}\left[1 - \frac{\sinh m\xi_0 z}{m\xi_0 z}\right], \tag{30}$$

where the boundary conditions $w(0) = 1$ and $w'(0) = 0$ are satisfied. A comment on the GR limit (that is $f(R) \to R$) of solution (30) is necessary. In fact when we perform the limit $m \to \infty$, we do not recover exactly $w^{(0)}_{GR}(z)$. The difference is in the definition of quantity ξ_0. In $f(R)$-gravity we have the definition (20) while in GR it is $\xi_0 = \sqrt{\frac{2}{\mathcal{X} A_n \Phi_c^{n-1}}}$, since in the first equation of (16), when we perform $f(R) \to R$, we have to eliminate the trace equation condition. In general, this means that the Newtonian limit and the

Eddington parameterization of different relativistic theories of gravity cannot coincide with those of GR (see [16] for further details on this point).

The point $z^{(0)}_{f(R)}$ is calculated by imposing $w^{(0)}_{f(R)}(z^{(0)}_{f(R)}) = 0$ and by considering the Taylor expansion

$$\frac{\sinh m\xi_0 z}{m\xi_0 z} \sim 1 + \frac{1}{6}(m\xi_0 z)^2 + \mathcal{O}(m\xi_0 z)^4, \tag{31}$$

we obtain $z^{(0)}_{f(R)} = \frac{2\sqrt{6}}{\sqrt{3+(1+m\xi)e^{-m\xi}}}$. Since the stellar radius ξ is given by definition $\xi = \xi_0\, z^{(0)}_{f(R)}$, we obtain the constraint

$$\xi = \sqrt{\frac{3\Phi_c}{2\pi G}}\frac{1}{\sqrt{1+\frac{1+m\xi}{3}e^{-m\xi}}}. \tag{32}$$

By solving numerically the constraint ([6]) eq. (32), we find the modified expression of the radius ξ. If $m \to \infty$ we have the standard expression $\xi = \sqrt{\frac{3\Phi_c}{2\pi G}}$ valid for the Newtonian limit of GR. Besides, it is worth noticing that in the $f(R)$-gravity case, for $n = 0$, the radius is smaller than in GR. On the other hand, the gravitational potential $-\Phi$ gives rise to a deeper potential well than the corresponding Newtonian potential derived from GR [14].

In the case $n = 1$, eq. (21) can be recast as follows:

$$\frac{\mathrm{d}^2\tilde{w}(z)}{\mathrm{d}z^2} + \tilde{w}(z) = \frac{m\xi_0}{8}\int_0^{\xi/\xi_0} \mathrm{d}z' \left\{e^{-m\xi_0|z-z'|} - e^{-m\xi_0|z+z'|}\right\}\tilde{w}(z'), \tag{33}$$

where $\tilde{w} = z\,w$. If we consider the solution of (33) as a small perturbation to the one of GR, we have

$$\tilde{w}^{(1)}_{f(R)}(z) \sim \tilde{w}^{(1)}_{GR}(z) + e^{-m\xi}\Delta\tilde{w}^{(1)}_{f(R)}(z). \tag{34}$$

The coefficient $e^{-m\xi} < 1$ is the parameter with respect to which we perturb eq. (33). Besides these position ensure us that when $m \to \infty$ the solution converge to something like $\tilde{w}^{(1)}_{GR}(z)$. Substituting eq. (34) in eq. (33), we have

$$\begin{aligned}&\frac{\mathrm{d}^2\Delta\tilde{w}^{(1)}_{f(R)}(z)}{\mathrm{d}z^2} + \Delta\tilde{w}^{(1)}_{f(R)}(z) = \\ &\frac{m\xi_0\, e^{m\xi}}{8}\int_0^{\xi/\xi_0} \mathrm{d}z' \left\{e^{-m\xi_0|z-z'|} - e^{-m\xi_0|z+z'|}\right\}\tilde{w}^{(1)}_{GR}(z').\end{aligned} \tag{35}$$

([6]) In principle, there is a solution for any value of m.

and the solution is easily found

$$(36)\quad w^{(1)}_{f(R)}(z) \sim \frac{\sin z}{z}\left\{1+\frac{m^2{\xi_0}^2}{8(1+m^2{\xi_0}^2)}\left[1+\frac{2\,e^{-m\xi}}{1+m^2{\xi_0}^2}(\cos\xi/\xi_0+m\xi_0\sin\xi/\xi_0)\right]\right\}$$
$$-\frac{m^2{\xi_0}^2}{8(1+m^2{\xi_0}^2)}\left[\frac{2\,e^{-m\xi}}{1+m^2{\xi_0}^2}(\cos\xi/\xi_0+m\xi_0\sin\xi/\xi_0)\frac{\sinh m\xi_0 z}{m\xi_0 z}+\cos z\right].$$

Also in this case, for $m \to \infty$, we do not recover exactly $w^{(1)}_{GR}(z)$. The reason is the same of the previous $n = 0$ case [16]. Analytical solutions for other values of n are not available.

To conclude this section, we report the gravitational potential profile generated by a spherically symmetric source of uniform mass with radius ξ. We can impose a mass density of the form

$$(37)\qquad \rho = \frac{3M}{4\pi\xi^3}\Theta(\xi - |\mathbf{x}|),$$

where Θ is the Heaviside function and M is the mass [14, 15]. By solving field eqs. (16) *inside the star* and considering the boundary conditions $w(0) = 1$ and $w'(0) = 0$, we get

$$(38)\qquad w_{f(R)}(z) = \left[\frac{3}{2\xi}+\frac{1}{m^2\xi^3}-\frac{e^{-m\xi}(1+m\,\xi)}{m^2\xi^3}\right]^{-1}$$
$$\times\left[\frac{3}{2\xi}+\frac{1}{m^2\xi^3}-\frac{{\xi_0}^2z^2}{2\xi^3}-\frac{e^{-m\xi}(1+m\,\xi)}{m^2\xi^3}\frac{\sinh m\xi_0 z}{m\xi_0 z}\right].$$

In the limit $m \to \infty$, we recover the GR case $w_{GR}(z) = 1 - \frac{{\xi_0}^2z^2}{3\xi^2}$. In fig. 1 we show the behaviors of $w^{(0)}_{f(R)}(z)$ and $w^{(1)}_{f(R)}(z)$ with respect to the corresponding GR cases. Furthermore, we plot the potential generated by a uniform spherically symmetric mass distribution in GR and $f(R)$-gravity and the case $w^{(5)}_{GR}(z)$.

6. – Discussion and conclusions

In this paper the hydrostatic equilibrium of a stellar structure in the framework of $f(R)$-gravity has been considered. The study has been performed starting from the Newtonian limit of $f(R)$-field equations. Since the field equations satisfy in any case the Bianchi identity, we can use the conservation law of energy-momentum tensor. In particular adopting a polytropic equation of state relating the mass density to the pressure, we derive the *modified Lané-Emden equation* and its solutions for $n = 0, 1$ which can be compared to the analogous solutions coming from the Newtonian limit of GR. When we consider the limit $f(R) \to R$, we obtain the standard hydrostatic equilibrium theory coming from GR. A peculiarity of $f(R)$-gravity is the non-viability of Gauss theorem and then the *modified Lané-Emden equation* is an integro-differential equation where the mass distribution plays a crucial role. Furthermore the correlation between two points in the star is given by a Yukawa-like term of the corresponding Green function.

These solutions have been matched with those coming from GR and the corresponding density radial profiles have been derived. In the case $n = 0$, we find an exact solution, while, for $n = 1$, we used a perturbative analysis with respect to the solution coming from GR. It is possible to demonstrate that density radial profiles coming from $f(R)$-gravity analytic models and close to those coming from GR are compatible. This result rules out some wrong claims in the literature stating that $f(R)$-gravity is not compatible with self-gravitating systems. Obviously the choice of the free parameter of the theory has to be consistent with boundary conditions and then the solutions are parameterized by a suitable "wave-length" $m = \sqrt{-\frac{1}{3f''(0)}}$ that should be experimentally fixed.

The next step is to derive self-consistent numerical solutions of *modified Lané-Emden equation* and build up realistic star models where further values of the polytropic index n and other physical parameters, *e.g.* temperature, opacity, transport of energy, are considered. Interesting cases are the non-relativistic limit ($n = 3/2$) and relativistic limit ($n = 3$) of completely degenerate gas. These models are a challenging task since, up to now, there is no self-consistent, final explanation for compact objects (*e.g.* neutron stars) with masses larger than Volkoff mass, while observational evidences widely indicate these objects [10]. In fact it is plausible that the gravity manifests itself on different characteristic lengths and also other contributions in the gravitational potential should be considered for these exotic objects. As we have seen above, the gravitational potential well results modified by higher-order corrections in the curvature. In particular, it is possible to show that if we put in the action (9) other curvature invariants also repulsive contributions can emerge [11,15]. These situations have to be seriously taken into account in order to address several issues of relativistic astrophysics that seem to be out of the explanation range of the standard theory.

REFERENCES

[1] Capozziello S. and Faraoni V., *Beyond Einstein gravity: A Survey of gravitational theories for cosmology and astrophysics. Fundamental Theories of Physics*, Vol. **170** (Springer, Berlin) 2010; Capozziello S. and De Laurentis M., *Invariance Principles and Extended Gravity: Theories and Probes* (Nova Science Publishers, Inc.) 2010.

[2] Nojiri S. and Odintsov S. D., *Int. J. Geom. Meth. Mod. Phys.*, **4** (2007) 115; Capozziello S. and Francaviglia M., *Gen. Relativ. Gravit.*, **40** (2008) 357; Capozziello S. and De Laurentis M., *Phys. Rep.*, **509** (2011) 167; Sotiriou T. P. and Faraoni V., *Rev. Mod. Phys.*, **82** (2010) 451; Capozziello S., De Laurentis M. and Faraoni V., *The Open Astr. Jour*, **2** (2009) 1874.

[3] Anderson J. D. *et al.*, *Phys. Rev. Lett.*, **81** (1998) 2858; Anderson J. D. *et al.*, *Phys. Rev. D*, **65** (2002) 082004.

[4] Bertolami O., Böhmer C. G., Harko T. and Lobo F. S. N., *Phys. Rev. D*, **75** (2007) 104016.

[5] Psaltis D., *Living Rev. Relativity*, **11** (2008).

[6] Briscese F., Elizalde E., Nojiri S. and Odintsov S. D., *Phys. Lett. B*, **646** (2007) 105.

[7] Khoury J. and Weltman A., *Phys. Rev. D*, **69** (2007) 044026.

[8] Hu W. and Sawicki I., *Phys. Rev. D*, **76** (2007) 064004; Capozziello S. and Tsujikawa S., *Phys. Rev. D*, **77** (2008) 107501; Tsujikawa S., Tamaki T. and Tavakol R., *JCAP*, **0905** (2009) 020; Upadhye A. and Hu W., *Phys. Rev. D*, **80** (2009) 064002.

[9] Babichev E. and Langlois D., *Phys. Rev. D*, **80** (2009) 121501; Babichev E. and Langlois D., *Phys. Rev. D*, **81** (2010) 124051.

[10] Muno M. P. *et al.*, *Astrophys. J.*, **636** (2006) L41.

[11] Stabile A., *Phys. Rev. D*, **82** (2010) 124026.
[12] Kippenhahn R. and Weigert A., *Stellar structures and evolution* (Springer-Verlag, Berlin) 1990.
[13] Capozziello S., Stabile A. and Troisi A., *Phys. Rev. D*, **76** (2007) 104019.
[14] Stabile A., *Phys. Rev. D*, **82** (2010) 064021.
[15] Capozziello S. and Stabile A., *Class. Quantum Grav.*, **26** (2009) 085019.
[16] Capozziello S., Stabile A. and Troisi A., *Phys. Lett. B*, **686** (2010) 79.