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Dilatonic non-linear sigma models and Ricci flow extensions

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It is a pleasure to dedicate this paper to Gaetano who,
in the increasingly turbolent landscape of Theoretical Physics,
has always liked it best in Experimental Mathematics.

Summary. — We review our recent work describing, in terms of the Wasserstein geometry over the space of probability measures, the embedding of the Ricci flow in the renormalization group flow for dilatonic non-linear sigma models.

PACS `02.40.Vh` – Global analysis and analysis on manifolds.
PACS `02.40.Ky` – Riemannian geometries.
PACS `11.10.Hi` – Renormalization group evolution of parameters.
PACS `11.10.Lm` – Nonlinear or nonlocal theories and models.

1. – Introduction

Non-linear sigma models (NLSM) are quantum field theories describing, in the large deviations sense, random fluctuations of harmonic maps between a Riemann surface and a Riemannian manifold [1]. Besides their ubiquitous modeling role in theoretical physics, where they find applications ranging from condensed matter to string theory, NLSM provide a natural geometrical framework for possible generalizations of Hamilton's Ricci flow [2-5]. The rationale of this deep connection between NLSM and Ricci flow lies, as we have recently shown [6], in the metric geometry of the space of probability measures over Riemannian manifolds. This geometry, induced by a natural distance among probability measures, (the quadratic Wasserstein, or more appropriately, Kantorovich-Rubinstein distance), captures the reaction-diffusion aspects both of the Ricci and of the renormalization group flow for NLSM. In particular, it provides a natural framework for discussing in a rigorous way the embedding of Ricci flow into the renormalization group flow for non-linear sigma models, and allows for generalizations [6] of the Ricci flow that might extend significantly the theory to spaces more general than Riemannian manifolds.

2. – The geometry of dilatonic non-linear sigma models

Let $(M, g, \mathrm{d}\omega)$ be a n-dimensional compact *Riemannian metric measure space* [7,8], *i.e.* a smooth orientable manifold, without boundary, endowed with a Riemannian metric g and a positive Borel measure $\mathrm{d}\omega \ll \mathrm{d}\mu_g$, absolutely continuous with respect to the Riemannian volume element, $\mathrm{d}\mu_g$. Strictly speaking, $(M, g, \mathrm{d}\omega)$ characterizes a *weighted Riemannian manifold*, (or *Riemannian manifold with density*), the corresponding metric measure space being actually defined by $(M, d_g(\cdot,\cdot), \mathrm{d}\omega)$, where $d_g(x, y)$ denotes the Riemannian distance on (M, g). By a slight abuse of notation, we shall use $(M, g, \mathrm{d}\omega)$ and $(M, d_g(\cdot,\cdot), \mathrm{d}\omega)$ interchangeably. Also, let $\mathcal{D}iff(M)$ and $\mathcal{M}et(M)$, respectively, denote the group of smooth diffeomorphisms and the open convex cone of all smooth Riemannian metrics over M. As g varies in $\mathcal{M}et(M)$, we can characterize the set $\mathrm{Meas}(M)$ of all smooth Riemannian metric measure spaces as

$$\mathrm{Meas}(M) := \{(M, g; \mathrm{d}\omega) \mid (M, g) \in \mathcal{M}\mathrm{et}(M),\, \mathrm{d}\omega \in \mathcal{B}(M, g)\}\,, \tag{1}$$

where $\mathcal{B}(M, g)$ is the set of positive Borel measure on (M, g) with $\mathrm{d}\omega \ll \mathrm{d}\mu_g$. Since in the compact-open C^∞ topology $\mathcal{M}\mathrm{et}(M)$ is contractible, the space $\mathrm{Meas}(M)$ fibers trivially over $\mathcal{M}\mathrm{et}(M)$. In particular, the fiber $\pi^{-1}(M, g)$ can be identified with the set of all (orientation preserving) measures $\mathrm{d}\omega \ll \mathrm{d}\mu_g$ over the given (M, g),

$$\mathrm{Meas}(M, g) := \{\mathrm{d}\omega \in \mathrm{Meas}(M) :\, \mathrm{d}\omega \ll \mathrm{d}\mu_g\}\,, \tag{2}$$

endowed with the topology of weak convergence. There is a natural action of the group of diffeomorphisms $\mathcal{D}iff(M)$ on the space $\mathrm{Meas}(M)$, defined by

$$\begin{array}{ccc} \mathcal{D}iff(M) \times \mathrm{Meas}(M) & \longrightarrow & \mathrm{Meas}(M) \\ (\varphi;\ g,\, \mathrm{d}\omega) & \longmapsto & (\varphi^* g,\ \varphi^* \mathrm{d}\omega), \end{array} \tag{3}$$

where $(\varphi^* g,\ \varphi^* \mathrm{d}\omega)$ is the pull-back under $\varphi \in \mathcal{D}iff(M)$. The Radon-Nikodym derivative $\wp(g,\, \mathrm{d}\omega) := \frac{\mathrm{d}\omega}{\mathrm{d}\mu_g}$ is a local *Riemannian measure space invariant* [9] under this action, *i.e.*

$$\wp(\varphi^* g,\, \varphi^*(\mathrm{d}\omega)) = \varphi^*\, \wp(g,\, \mathrm{d}\omega) = \frac{\mathrm{d}\omega}{\mathrm{d}\mu_g} \circ \varphi\,, \quad \forall \varphi \in \mathcal{D}i\!f\!f(M), \tag{4}$$

and we can introduce the geometrical dilaton field $f : \mathrm{Meas}(M) \longrightarrow C^\infty(M, \mathbb{R})$ associated with the Riemannian metric measure space $(M, g,\, \mathrm{d}\omega)$ according to

$$(M, g,\, \mathrm{d}\omega) \longmapsto f(M, g,\, \mathrm{d}\omega) := -\ln\left(V_g(M)\, \frac{\mathrm{d}\omega}{\mathrm{d}\mu_g}\right),$$

where $V_g(M) := \int_M \mathrm{d}\mu_g$ is the Riemannian volume of $(m, g,\, \mathrm{d}\omega)$.

Two-dimensional dilatonic non-linear σ models are defined by a natural extension of the harmonic energy functional to maps

$$\phi :\, (\Sigma, \gamma) \longrightarrow (M, g,\, \mathrm{d}\omega) \tag{5}$$

between a 2-dimensional smooth orientable surface without boundary (Σ, γ), with Riemannian metric γ, and the Riemannian metric measure space $(M, g,\, \mathrm{d}\omega)$. In the metric measure space setting described above, regularity issues are rather delicate, and it can be difficult to work in local charts on the target manifold M, even at a physical level of rigor. A way out is to use the Nash embedding theorem [10, 11], according to which any compact Riemannian manifold (M, g) can be isometrically embedded into some Euclidean space $\mathbb{E}^m := (\mathbb{R}^m, \delta)$ for m sufficiently large [10]. In particular, if $J : (M, g) \hookrightarrow \mathbb{E}^m$, is any such an embedding we can define the Sobolev space of maps

$$\mathcal{H}^1_{(J)}(\Sigma, M) := \{\phi \in \mathcal{H}^1(\Sigma, \mathbb{R}^m) \mid \phi(\Sigma) \subset J(M)\}, \tag{6}$$

where $\mathcal{H}^1(\Sigma, \mathbb{R}^m)$ is the Hilbert space of square summable $\varphi : \Sigma \to \mathbb{R}^m$, with (first) distributional derivatives $\in L^2(\Sigma, \mathbb{R}^m)$, endowed with the norm

$$\| \phi \|_{\mathcal{H}^1} := \int_\Sigma \left(\phi^a(x)\, \phi^b(x)\, \delta_{ab} + \gamma^{\mu\nu}(x)\, \frac{\partial \phi^a(x)}{\partial x^\mu} \frac{\partial \phi^b(x)}{\partial x^\nu}\, \delta_{ab} \right) \mathrm{d}\mu_\gamma, \tag{7}$$

where, for $\phi(x) \in J(M) \subset \mathbb{R}^m$, $a, b = 1, \ldots, m$ label coordinates in $(\mathbb{R}^m, \delta)$, and $\mathrm{d}\mu_\gamma$ denotes the Riemannian measure on (Σ, γ). This characterization is independent of J as long as M is compact, since in that case for any two isometric embeddings J_1 and J_2, the corresponding spaces of maps $\mathcal{H}^1_{(J_1)}(\Sigma, M)$ and $\mathcal{H}^1_{(J_2)}(\Sigma, M)$ are homeomorphic [12] and one can simply write $\mathcal{H}^1(\Sigma, M)$. The space of smooth maps $C^\infty(\Sigma, M)$ is dense [13] in the Sobolev space $\mathcal{H}^1(\Sigma, M)$, however maps of class $\mathcal{H}^1(\Sigma, M)$ are not necessarily continuous, and to carry out explicit computations, one may further require that any such $\phi \in \mathcal{H}^1(\Sigma, M)$ is localizable, (cf. [14], sect. 8.4), and of bounded geometry. Explicitly, we assume that for every $x_0 \in \Sigma$ there exists a metric disk $D(x_0, \delta) := \{x \in \Sigma |\, d_\gamma(x_0, x) \leq \delta\} \subset \Sigma$, of radius $\delta > 0$ and a metric ball $B(r, p) := \{z \in M |\, d_g(p, z) \leq r\} \subset (M, g)$ centered at $p \in M$, of radius $r > 0$ such that $\phi(D(x_0, \delta)) \subset B(r, p)$, with

$$r \,<\, r_0 \,:=\, \min \left\{ \frac{1}{3}\, \mathrm{inj}\,(M),\, \frac{\pi}{6\sqrt{\kappa}} \right\}, \tag{8}$$

where $\mathrm{inj}\,(M)$ and κ, respectively, denote the injectivity radius of (M, g), and the upper bound to the sectional curvature of (M, g), (we are adopting the standard convention of defining $\pi/2\sqrt{\kappa} \doteq \infty$ when $\kappa \leq 0$). Under such assumptions, one can use local coordinates also for maps in $\mathcal{H}^1(\Sigma, M)$. In particular for any $\phi \in \mathcal{H}^1(D(x_0, \delta), M)$ we can introduce local coordinates x^α, for points in $(D(x_0, \delta), \Sigma)$, and $y^k = \phi^k(x)$, $k = 1, \ldots, n$, for the corresponding image points in $\phi(D(x_0, \delta)) \subset M$, and, by using a partition of unity, work locally in the smooth framework provided by the space of smooth maps

$$\mathrm{Map}\,(\Sigma, M) := \left\{ \phi : \Sigma \to M,\, x^\alpha \longmapsto y^k = \phi^k(x) \in C^\infty(D(x_0, \delta), M) \right\}. \tag{9}$$

Under these regularity hypotheses, we can introduce the pull-back bundle $\phi^{-1}TM$ whose sections $v \equiv \phi^{-1}V := V \circ \phi$, $V \in C^\infty(M, TM)$, are the vector fields over Σ covering the map ϕ. If $T^*\Sigma$ denotes the cotangent bundle to (Σ, γ), then the differential $d\phi = \frac{\partial \phi^i}{\partial x^\alpha} dx^\alpha \otimes \frac{\partial}{\partial \phi^i}$ can be interpreted as a section of $T^*\Sigma \otimes \phi^{-1}TM$, and its Hilbert-Schmidt norm, in the bundle metric

$$\langle \cdot, \cdot \rangle_{T^*\Sigma \otimes \phi^{-1}TM} := \gamma^{-1}(x) \otimes g(\phi(x))(\cdot, \cdot), \tag{10}$$

is provided by, (see, *e.g.*, [14]),

$$\langle \mathrm{d}\phi, \mathrm{d}\phi \rangle_{T^*\Sigma\otimes\phi^{-1}TM} = \gamma^{\mu\nu}(x)\,\frac{\partial\phi^i(x)}{\partial x^\mu}\frac{\partial\phi^j(x)}{\partial x^\nu}\,g_{ij}(\phi(x)) = \,\mathrm{tr}_{\gamma(x)}\,(\phi^*\,g). \tag{11}$$

The corresponding density

$$e(\phi)\;\mathrm{d}\mu_\gamma := \frac{1}{2}\,\langle \mathrm{d}\phi, \mathrm{d}\phi \rangle_{T^*\Sigma\otimes\phi^{-1}TM}\,\mathrm{d}\mu_\gamma \;=\; \frac{1}{2}\,\mathrm{tr}_{\gamma(x)}\,(\phi^*\,g)\,\mathrm{d}\mu_\gamma, \tag{12}$$

where $\mathrm{d}\mu_\gamma$ is the volume element of the Riemannian surface (Σ,γ), is conformally invariant under two-dimensional conformal transformations

$$(\Sigma,\gamma_{\mu\nu}) \mapsto (\Sigma, e^{-\psi}\,\gamma_{\mu\nu}), \qquad \psi\in C^\infty(\Sigma,\mathbb{R}), \tag{13}$$

and defines the harmonic map energy density associated with $\phi\in\mathrm{Map}(\Sigma,M)$. In particular, the critical points of the functional

$$E[\phi,\,g]_{(\Sigma,M)} := \int_\Sigma e(\phi)\,\mathrm{d}\mu_\gamma, \tag{14}$$

are the harmonic maps of the Riemann surface $(\Sigma,[\gamma])$ into (M,g), where $[\gamma]$ denotes the conformal class of the metric γ. In terms of the possible geometrical characterization of (Σ,γ) and (M,g), important examples of harmonic maps include harmonic functions, geodesics, isometric minimal immersions, holomorphic (and anti-holomorphic) maps of Kähler manifolds. It is worthwhile to observe that in such a rich panorama, also the seemingly trivial case of constant maps plays a basic role for the interplay between Ricci flow and (the perturbative quantization of) non-linear σ models.

With these remarks along the way, if $\phi\in\mathcal{H}^1(\Sigma,M)$ is a localizable map, then we can define the associated non-linear σ model dilatonic action, with coupling parameters $a\in\mathbb{R}_{>0}$ and $(M,g,\mathrm{d}\omega)\in\mathrm{Meas}(M,g)$, according to

$$\begin{aligned}
&(\Sigma,\gamma)\times\mathcal{H}^1(\Sigma,M)\times[\mathbb{R}_{>0}\times\ \mathrm{Meas}(M,g)]\ \longrightarrow\ \mathbb{R}\\
&(\gamma,\phi;\,a,\,(M,g,\,\mathrm{d}\omega))\qquad\longmapsto\qquad S[\gamma,\phi;\ a,\,g,\,\mathrm{d}\omega]\\
&:=\,a^{-1}\int_\Sigma\left[\mathrm{tr}_{\gamma(x)}\,(\phi^*\,g)-\,a\,\mathcal{K}_\gamma\,\ln\,\phi^*\left(\frac{\mathrm{d}\omega}{\mathrm{d}\mu_g}\,V_g(M)\right)\right]\,\mathrm{d}\mu_\gamma\\
&:=\,\frac{2}{a}\,E[\phi,\,g]_{(\Sigma,M)}\,+\int_\Sigma\mathcal{K}_\gamma\,f(\phi)\,\mathrm{d}\mu_\gamma,
\end{aligned} \tag{15}$$

where $\mathcal{K}_\gamma$ is the Gaussian curvature of the Riemannian surface (Σ,γ), and where $a>0$ is a parameter with the dimension of a length squared. This definition stresses the role of $\mathrm{Meas}(M)$ as the space of *point-dependent coupling parameters* α for dilatonic non-linear σ models

$$\alpha := (a,\,g,\,f)\,. \tag{16}$$

Indeed, the energy scale of the action $S[\gamma,\phi;\ a,\,g,\,\mathrm{d}\omega]$ is set by the *dilaton* coupling $[\,f(\phi)\,\mathcal{K}]$ associated with $(M,g,\,\mathrm{d}\omega)$ and by the length scale of the target space metric g_{ab}, *i.e.* $|Rm(g,y)|\,a$, where $|Rm(g,y)| := [R^{iklm}R_{iklm}]^{1/2}$. Roughly speaking, the parameter $a > 0$ sets the (squared) length scale at which the pair $(\phi(\Sigma),\, S)$ probes the target metric measure space $(M, g,\, d\omega)$. It is also important to recall that, in stark contrast with the harmonic map energy (14), the dilatonic term in (15),

$$-\int_\Sigma \mathcal{K}_\gamma \,\ln\, \phi^*\left(\frac{d\omega}{d\mu_g}\, V_g(M)\right)\, d\mu_\gamma = \int_\Sigma \mathcal{K}_\gamma\, f(\phi)\, d\mu_\gamma, \tag{17}$$

is not conformally invariant. As is well known, and as first stressed by Fradkin and Tseytlin [15], the role of this term is to restore the conformal invariance of $E[\phi, g]$ which is broken upon quantization.

To discuss the role that Wasserstein geometry plays in non-linear σ model theory, let us denote by $\mathrm{Prob}(M)$ denote the set of all Borel probability measure on the manifold M and restrict our attention to dilaton measures $d\omega \in \mathcal{M}$ et(M) which actually belong to the dense subspace of $\mathrm{Prob}(M)$ defined by the set of absolutely continuous probability measures $d\omega \ll d\mu_g$ on (M, g),

$$\mathcal{D}IL_{(1)}(M,g) = \mathrm{Prob}_{ac}(M,g) := \left\{ d\omega \in \mathrm{Prob}(M) |\, d\omega := e^{-f}\, \tfrac{d\mu_g}{V_g(M)},\, f \in C^\infty(M,\mathbb{R}) \right\}. \tag{18}$$

This restriction, somewhat unphysical from the point of view of non-linear σ model theory, (since it constrains *a priori* the dilaton field to be associated to a probability measure), plays a basic role in Perelman's analysis of the Ricci flow and of its interaction with the NLSM renormalization group flow.

A direct and important consequence of constraining the dilatonic measure $d\omega$ to be an absolutely continuous probability measure on (M, g) is that the space $\mathcal{D}IL_{(1)}(M, g)$ can be seen as an infinite dimensional manifold locally modelled over the Hilbert space completion of the smooth tangent space

$$T_\omega \mathrm{Prob}_{ac}(M,g) := \{h \in C^\infty(M,\mathbb{R}),\, \int_M h\, d\omega = 0\}, \tag{19}$$

with respect to the Otto inner product on $\mathrm{Prob}_{ac}(M, g)$ defined, at the given $d\omega = V_g^{-1}(M)e^{-f}d\mu_g$, by the $L^2(M, d\omega)$ Dirichlet form [16]

$$\langle \nabla\varphi, \nabla\zeta\rangle_{(g,d\omega)} \doteq \int_M \left(g^{ik}\, \nabla_k\varphi\, \nabla_i\zeta\right)\, d\omega, \tag{20}$$

for any $\varphi,\, \zeta \in C^\infty(M,\mathbb{R})/\mathbb{R}$. Hence we can set

$$T_f\mathcal{D}IL_{(1)}(M,g) = \overline{\left\{ h \in C^\infty(M,\mathbb{R}),\, \int_M h\, d\omega = 0 \right\}}^{L^2(M,d\omega)}. \tag{21}$$

Moreover, under the identification (19), one can represent infinitesimal deformations of the dilaton field $d\omega$, (thought of as vectors in $T_f\mathcal{D}IL_{(1)}(M, g)$), in terms of the mapping

$$\begin{aligned} T_f\mathcal{D}IL_{(1)}(M,g) \times \mathcal{D}IL_{(1)}(M,g) &\longrightarrow C^\infty(M,\mathbb{R})/\mathbb{R}, \\ (h,\, d\omega = V_g^{-1}(M)e^{-f}d\mu_g) &\longmapsto \quad \psi, \end{aligned} \tag{22}$$

where the function ψ associated to the given $(h, \mathrm{d}\omega)$ is formally determined on (M, g) by the elliptic partial differential equation

$$-\nabla^i \left(e^{-f}\, \nabla_i \psi\right) = h\, e^{-f}, \tag{23}$$

under the equivalence relation identifying any two such solutions differing by an additive constant. Recall that if $\mathcal{L}_V\, \mathrm{d}\omega$ denotes the Lie derivative of the volume form $\mathrm{d}\omega$ along the vector field $V \in C^\infty(M, TM)$, then the weighted divergence associated with the Riemannian measure space $(M, g,\, \mathrm{d}\omega)$ is defined by

$$\mathcal{L}_V\, \mathrm{d}\omega \;=\; (div_\omega\, V)\; \mathrm{d}\omega = \left[e^f\, \nabla_i\, \left(e^{-\,f}\, V^i\right)\right]\, \mathrm{d}\omega. \tag{24}$$

It follows that the elliptic equation (23) can be equivalently written as

$$\triangle_\omega\, \psi \;=\; -\, h, \tag{25}$$

where $\triangle_\omega$ denotes the weighted Laplacian on $(M, g,\, \mathrm{d}\omega)$ [17, 18, 7],

$$\triangle_\omega := div_\omega\, \nabla = \triangle_g - \nabla f \cdot \nabla. \tag{26}$$

If we move from infinitesimal deformations of the dilaton field $\mathrm{d}\omega$ to finite deformations then we need to compare any two distinct dilaton fields in $\mathcal{DIL}_{(1)}(M, g)$, say $(M, g, \mathrm{d}\omega_1 = e^{-f_1}\, \frac{\mathrm{d}\mu_g}{V_g(M)})$ and $(M, g, \mathrm{d}\omega_2 = e^{-f_2}\, \frac{\mathrm{d}\mu_g}{V_g(M)})$. To this end let $\mathrm{Prob}(M \times M)$ denote the set of Borel probability measures on the product space $M \times M$, and let us consider the set of measures $\mathrm{d}\sigma \in \mathrm{Prob}(M \times M)$ which reduce to $\mathrm{d}\omega_1$ when restricted to the first factor and to $\mathrm{d}\omega_2$ when restricted to the second factor, *i.e.*

$$\mathrm{Prob}_{\,\omega_1,\,\omega_2}\,(M \times M) := \left\{\mathrm{d}\sigma\, \in \mathrm{Prob}(M \times M)\; \mid\, \pi^{(1)}_\sharp\, \mathrm{d}\sigma = \mathrm{d}\omega_1,\, \pi^{(2)}_\sharp\, \mathrm{d}\sigma = \mathrm{d}\omega_2\right\}, \tag{27}$$

where $\pi^{(1)}_\sharp$ and $\pi^{(2)}_\sharp$ refer to the push-forward of $\mathrm{d}\sigma$ under the projection maps $\pi^{(i)}$ onto the factors of $M \times M$. Measures $\mathrm{d}\sigma \in \mathrm{Prob}_{\,\omega_1,\,\omega_2}\,(M \times M)$ are often referred to as *couplings* between $\mathrm{d}\omega_1$ and $\mathrm{d}\omega_2$. We shall avoid such a terminology since in our setting the term coupling has quite different a meaning. Let us recall that given a (measurable and non-negative) cost function $c\,:\, M \times M \to \mathbb{R}$, an optimal transport plan [19] $\mathrm{d}\sigma_{opt} \in \mathrm{Prob}_{\,\omega_1,\,\omega_2}(M \times M)$ between the probability measures $\mathrm{d}\omega_1$ and $\mathrm{d}\omega_2$ in $\mathrm{Prob}(M)$, (not necessarily in $\mathrm{Prob}_{ac}(M, g)$), is defined by the infimum, over all $\mathrm{d}\sigma(x, y) \in \mathrm{Prob}_{\,\omega_1,\,\omega_2}(M \times M)$, of the total cost functional

$$\int_{M\times M} c(x, y)\, \mathrm{d}\sigma(x, y). \tag{28}$$

On a Riemannian manifold (M, g), the typical cost function is provided [20-22] by the squared Riemannian distance function $d^{\,2}_g(\cdot, \cdot)$, and a major result of the theory [23, 24, 22, 25], is that for any pair $\mathrm{d}\omega_1$ and $\mathrm{d}\omega_2 \in \mathrm{Prob}(M)$, there is an optimal transport plan $\mathrm{d}\sigma_{opt}$, induced by a map $\Upsilon_{opt} : M \to M$. The resulting expression for the total cost of the plan

$$d^{\,W}_g\,(\mathrm{d}\omega_1, \mathrm{d}\omega_2) := \left(\inf_{\mathrm{d}\sigma\in\mathrm{Prob}_{\,\omega_1,\,\omega_2}(M\times M)}\, \int_{M\times M} \mathrm{d}^{\,2}_g(x, y)\, \mathrm{d}\sigma(x, y)\right)^{1/2}, \tag{29}$$

characterizes the quadratic Wasserstein, (or more appropriately, Kantorovich-Rubinstein) distance between the two probability measures $d\omega_1$ and $d\omega_2$. Note that there can be distinct optimal plans $d\sigma_{opt}$ connecting general probability measures $d\omega_1$ and $d\omega_2 \in \mathrm{Prob}(M)$, whereas on $\mathrm{Prob}_{ac}(M,g)$ the optimal transport plan is unique. The quadratic Wasserstein distance d_g^W defines a finite metric on $\mathrm{Prob}(M)$ and it can be shown that $(\mathrm{Prob}(M),\, d_g^W)$ is a geodesic space, endowed with the weak–* topology, (we refer to [26-28] for the relevant properties of Wasserstein geometry and optimal transport we freely use in the following). By an obvious dictionary, we identify the distance between the two dilaton fields f_1 and f_2 with the Wasserstein distance $d_g^W(d\omega_1, d\omega_2)$ between the corresponding probability measures. This allows to characterize $\left(\mathcal{D}IL_{(1)}(M,g),\, d_g^W\right)$ as the (dense) subset, $\left(\mathrm{Prob}_{ac}(M,g),\, d_g^W\right)$, of the (quadratic) Wasserstein space ($\mathrm{Prob}(M),\, d_g^W$).

3. – Dilaton localization and warping

To get some insight into the structure of the Wasserstein geometry of the dilatonic non-linear σ model let κ denote the upper bound to the sectional curvature of $(M, g,\, d\omega)$, and let us consider a metric ball $B(r,\, p) := \{z \in M|\, d_g(p,z) \leq r\}$, centered at $p \in M$, with radius $r < r_0$, where r_0, defined by (8) sets the length scale of the target (M,g). For $q \in \mathbb{N}$, let $\{\phi_{(k)}\}_{k=1\,\ldots,\,q} \in \mathrm{Map}\,(\Sigma, M)$ denote a collection of *reference* constant maps, (hence harmonic), taking values in the interior of $B(r,\, p)$

$$\begin{aligned} \phi_{(k)}\; : \Sigma &\longrightarrow \quad B(r,\, p) \setminus \partial B(r,\, p) \subset M \\ x &\longmapsto \phi_{(k)}(x) \,=\, y_{(k)}, \qquad \forall x \in \Sigma, \qquad k = 1,\, \ldots,\, q. \end{aligned} \tag{30}$$

We explicitly assume that $r < \frac{\pi}{6\sqrt{\kappa}}$, $\mathrm{inj}\,(y) > 3r$ for all $y \in B(r,p)$, and consider the *center of mass* [29] of the maps $\{\phi_{(k)}\}$,

$$\phi_{cm} \doteq cm\, \left\{\phi_{(1)}, \ldots, \phi_{(q)}\right\}, \tag{31}$$

characterized as the minimizer of the function

$$F(y;\, q) \doteq \frac{1}{2}\sum_{k=1}^{q} d_g^2(y, y_{(k)}), \tag{32}$$

where $d_g^2(\cdot\,,\cdot\,)$ denotes the distance in (M,g). Under the stated hypotheses the minimizer exists, is unique and $cm\,\left\{\phi_{(1)}, \ldots, \phi_{(q)}\right\} \in B(2r,p)$ [29], (see also [30] (chap. 4, p. 175)). We denote by $\{d_g(\phi_{cm}, \phi_{(k)})\}$ the distances between the maps $\{\phi_{(k)}\}$ and their center of mass ϕ_{cm}. The strategy for introducing the constant maps $\{\phi_{(k)}\}$ is to use the distances $\{d_g(\phi_{cm}, \phi_{(k)})\}$ and the dilatonic measure $d\omega$ to set the scale at which (Σ,γ) probes the geometry of (M,g). To this end, we localize $\phi \in \mathrm{Map}(\Sigma, M)$, around the center of mass of $\{\phi_{(k)}\}_{k=1}^{q}$, by choosing $d\omega$ according to

$$d\omega(z;q) := C_r^{-1}(q)\, e^{-\frac{F(z;\,q)}{2\,r^2}}\, \frac{d\mu_g(z)}{V_g(M)}\,, \qquad z \in M, \tag{33}$$

where $F(z;\, q)$ is the center of mass function (32), $V_g(M)$ is the Riemannian volume of (M,g), and $C_r(q)$ denotes a normalization constant such that $\int_M d\omega(z;q) \,=\, 1$. Since

$F(z;\, q)$ attains its minimum at ϕ_{cm}, the measure $\mathrm{d}\omega$ is concentrated around the center of mass of the $\{\phi_{(k)}\}$'s, and, as $r \searrow 0^+$, weakly converges to the Dirac measure δ_{cm} supported at ϕ_{cm}. Note that according to (32) we can factorize the density $\mathrm{d}\omega/\mathrm{d}\mu_g(z)$ as

$$\frac{\mathrm{d}\omega(z;q)}{\mathrm{d}\mu_g(z)} = V_g(M)^{-1} \prod_{k=1}^{q} e^{-\frac{d_g^2(z,\,\phi(k))}{4\,r^2} - \frac{\ln C_r(q)}{q}}. \tag{34}$$

This latter remark suggests to interpret the distances $d_g(\phi_{cm}, \phi_{(k)}), k = 1, \ldots q$ as coordinates $\{\xi_{(k)}\}$ in a q-dimensional flat torus $\mathbb{T}^q_{cm}$ of unit volume, and consider the product manifold

$$N^{n+q} := M \times_{(\omega)} \mathbb{T}^q, \tag{35}$$

endowed with the warped metric

$$h^{(q)}(y,\, \xi) := g(y) + \left(\frac{\mathrm{d}\omega(y;q)}{\mathrm{d}\mu_g(y)}\, V_g(M)\right)^{\frac{2}{q}} \sum_{i=1}^{q} \mathrm{d}\xi^2_{(i)}, \qquad \xi_{(i)} \in [0,1], \tag{36}$$

and the associated warped product measure

$$\mathrm{d}\mu_N(z,\xi) := \mathrm{d}\mu_g(z) \prod_{k=1}^{q} e^{-\frac{d_g^2(z,\,\phi(k))}{4\,r^2} - \frac{\ln C_r(q)}{q}} \mathrm{d}\xi_{(k)}\,, \qquad \{\xi_{(i)}\} \in \mathbb{T}^q. \tag{37}$$

Note that trading the Riemannian metric measure space $(M, g, \mathrm{d}\omega)$ with the warped Riemannian manifold

$$\left(N^{n+q} := M \times_{(\omega)} \mathbb{T}^q,\, h^{(q)}\right), \tag{38}$$

is a standard procedure in the Riemannian measure space setting, (cf. [9], and [31] for the application to Perelman's reduced volume). As a function of the distances from the constant maps $\{\phi_{(k)}\}$, the probability measure $\mathrm{d}\omega \in \mathrm{Prob}_{ac}(M, g) \approx \mathcal{DIL}_{(1)}(M, g)$ defined by (33) is Lipschitz on (M, g) and smooth on $M \setminus \cup_{k=1}^{q} \left\{\phi_{(k)},\, \mathrm{Cut}(\phi_{(k)})\right\}$, where $\mathrm{Cut}(\phi_{(k)})$ denotes the cut locus of each $\phi_{(k)}$. Moreover, it easily follows that the dilaton field associated to the measure $\mathrm{d}\omega$

$$f(z;q) := -\ln\left(\frac{\mathrm{d}\omega(z;q)}{\mathrm{d}\mu_g(z)}\, V_g(M)\right) = \frac{1}{4\,r^2} \sum_{k=1}^{q} d_g^2\left(z,\, \phi_{(k)}\right) + \ln C_r(q), \tag{39}$$

has a gradient ∇f which exists a.e. on (M, g) and $f \in \mathcal{H}^1(M, \mathbb{R})$. Also note that in terms of f the warped metric (36) takes the form

$$h^{(q)}(y,\, \xi) := g(y) + e^{-\frac{2f(y;q)}{q}} \sum_{i=1}^{q} d\xi^2_{(i)}, \qquad \xi_{(i)} \in [0,1], \tag{40}$$

with $\mathrm{d}\mu_{h^{(q)}}(y) = e^{-f(y;q)}\, \mathrm{d}\mu_g(y)$. It can be easily checked that the metric warping (40) on the torus fiber $\mathbb{T}^q_y$ over $y \in M$ can be compensated by the point-dependent (y)

rescaling $\xi_{(k)} \mapsto \xi_{(k)} \exp[\frac{f(y;q)}{q}]$ of the fiber itself. This suggests a natural extension of $\phi \in \mathrm{Map}(\Sigma, M)$ associated to the warping $M \times_{(f)} \mathbb{T}^q$ generated by the dilatonic measure. Explicitly, if $\phi \in \mathcal{H}^1(\Sigma, M)$ denotes a localizable map and if $f \circ \phi$ is the induced dilaton field over $\phi(\Sigma)$, then, the function $f \circ \phi$ is of class $\mathcal{H}^1(\Sigma, \mathbb{R})$ and the map $\Phi_{(q)} : (\Sigma, \gamma) \longrightarrow (N^{n+q} := M^n \times_{(f)} \mathbb{T}^q,\, h^{(q)})$ defined by

$$\Phi_{(q)}(x) := \left(\phi^i(x),\, \frac{1}{2}\, e^{\frac{f(\phi(x);q)}{q}}\, d_g(\phi_{cm}, \phi_{(1)})\,, \ldots, \frac{1}{2}\, e^{\frac{f(\phi(x);q)}{q}}\, d_g(\phi_{cm}, \phi_{(q)})\right) \tag{41}$$

is a localizable map $\in \mathcal{H}^1(\Sigma, M \times_{(f)} \mathbb{T}^q)$ describing the (fiber-wise uniform) dilatation of the torus fiber $\mathbb{T}^q_{\phi(x)}$ over $\phi(x) \in M$. As an elementary consequence of this duality between the metric warping (40) and the map warping (41) we get that the harmonic energy functional associated with the map $\Phi_{(q)}$ is provided by

$$E[\Phi_{(q)},\, h^{(q)}]_{(\Sigma, N^{n+q})} = E[\phi,\, g]_{(\Sigma, M)} + \frac{F(\phi_{cm}; q)}{2}\, \mathfrak{D}[q^{-1}\, f(\phi; q)]_{(\Sigma,\, \mathbb{R})}, \tag{42}$$

where $\mathfrak{D}[q^{-1}\, f(\phi; q)]_{(\Sigma,\, \mathbb{R})}$ is the Dirichlet energy

$$\mathfrak{D}[q^{-1}\, f(\phi; q)]_{(\Sigma,\, \mathbb{R})} := \frac{1}{2} \int_\Sigma \left| \frac{\mathrm{d} f(\phi(x);q)}{q} \right|^2_\gamma \mathrm{d}\mu_\gamma(x) \tag{43}$$

associated to the map $q^{-1}\, f \circ \phi : (\Sigma, \gamma) \longrightarrow \mathbb{R}^1$, and $F(\phi_{cm}; q)$ is the minimum of the center-of-mass function (32). This extended harmonic map set–up is interesting in many respects. In particular, if we choose the given surface (Σ, γ) to be topologically the 2-torus $\mathbb{T}^2$ endowed with a conformally flat metric associated to the dilaton field, then the functional $E\,[\Phi_{(q)},\, h^{(q)}]$ can be directly connected to the non-linear σ model dilatonic action (15). Explicitly, let $(\Sigma \simeq \mathbb{T}^2,\, \delta)$ be a flat 2-torus. For $\phi \in \mathcal{H}^1(\mathbb{T}^2, M)$ a localizable map taking values in $M \setminus \cup_{k=1}^q \mathrm{Cut}(\phi_{(k)})$, we denote by $f \circ \phi$ the induced dilaton field over $\phi(\mathbb{T}^2)$. If we endow $\mathbb{T}^2$ with the conformally flat metric $\gamma_{\mu\nu} = e^{f(\phi;q)/q}\, \delta_{\mu\nu}$ then in the resulting conformal gauge $(\mathbb{T}^2,\, \gamma)$ we can write

$$S\,[\gamma, \phi;\; F(\phi_{cm}; q),\, g,\, \mathrm{d}\omega] \underset{(f)}{=} \frac{2\, q}{F(\phi_{cm}; q)}\; E[\Phi_{(q)},\, h^{(q)}]_{(\Sigma, N^{n+q})}, \tag{44}$$

i.e., we can reduce, (*caveat*: in the given conformal gauge $\gamma = e^{f(\phi;q)/q}\, \delta\,!$), the dilatonic action $S\,[\gamma, \phi;\; F(\phi_{cm}; q),\, g,\, \mathrm{d}\omega]$ associated to the map $\phi : (\mathbb{T}^2, \gamma) \longrightarrow (M, g, \mathrm{d}\omega)$ between the Riemannian surface $(\mathbb{T}^2, \gamma)$ and the Riemannian metric measure space $(M, g, \mathrm{d}\omega)$, to the harmonic energy functional $E[\Phi_{(q)},\, h^{(q)}]$ associated to the map $\Phi_{(q)} : (\mathbb{T}^2, \gamma) \longrightarrow (N^{n+q}, h^{(q)})$ between the Riemann surface $(\mathbb{T}^2, \gamma)$ and the warped Riemannian manifold $(N^{n+q}, h^{(q)})$.

4. – A weighted heat kernel embedding

The above warping mechanism, trading the harmonic energy functional $E[\Phi_{(q)},\, h^{(q)}]$ for the dilatonic action $S\,[\gamma, \phi;\; F(\phi_{cm}; q),\, g,\, \mathrm{d}\omega]$ allows us to connect the Wasserstein geometry of the dilaton space $\mathcal{DIL}_{(1)}(M, g) = \mathrm{Prob}_{ac}(M, g)$ to the Ricci flow in a rather unexpected and deep way. As a preliminary step, we need to define the heat kernel associated the weighted Laplacian $\triangle_\omega := \triangle_g - \nabla f\cdot\nabla$ on $(M, g, d\omega)$, (cf. (24)). This is a symmetric operator with respect to the defining measure $d\omega$ and can be extended to a self-adjoint operator in $L^2(M, d\omega)$ generating the heat semigroup $e^{t\,\triangle_\omega}$, $t\in\mathbb{R}_{>0}$. The associated heat kernel $p_t^{(\omega)}(\cdot\,,z)$ is defined as the minimal positive solution of

$$\left(\frac{\partial}{\partial t}\,-\,\triangle_\omega\right)\,p_t^{(\omega)}(y,z)=0, \qquad \lim_{t\searrow 0^+}\;p_t^{(\omega)}(y,z)\,d\omega(z)=\;\delta_z, \tag{45}$$

with δ_z the Dirac measure at $z\in(M, d\omega)$. The heat kernel $p_t^{(\omega)}(y,z)$ is C^∞ on $\mathbb{R}_{>0}\times M\times M$, is symmetric, satisfies the semigroup identity $p_{t+s}^{(\omega)}(y,z)=\int_M p_t^{(\omega)}(y,x)p_s^{(\omega)}(x,z)\,d\omega(x)$, and $\int_M p_t^{(\omega)}(y,z)\,d\omega(z)=1$, (see, *e.g.*, [7]). The basic observation relating dilatonic NLSM and Wasserstein geometry is suggested by Varadhan's large deviation formula, (which holds also for the weighted heat kernel $p_t^{(\omega)}$, [7] Th.7.13, and § 7.5, Th. 7.20),

$$-\lim_{t\searrow 0^+}\,t\,\ln\left[p_t^{(\omega)}(y,z)\right]=\frac{d_g^2(y,z)}{4}, \tag{46}$$

where the convergence is uniform over all $(M,g, d\omega)$. Since the map $(M,d_g)\longrightarrow(\mathrm{Prob}(M),\,d_g^W)$ defined by $z\longmapsto\delta_z$ is an isometry, (one directly computes $d_g^W(\delta_y,\delta_z)=d_g(y,z)$, by using the obvious optimal plan $d\sigma(u,v)=\delta_y(u)\otimes\delta_z(v)$ in (29)), Varadhan's formula suggests that one may exploit the heat kernel $p_t^{(\omega)}(\cdot,z)\,d\omega$ to embed non-trivially $(M,g, d\omega)$ in $(\mathrm{Prob}(M),\,d_g^W)$. By extending to $p_t^{(\omega)}(\cdot,z)\,d\omega$ the heat kernel embedding technique introduced by Mantegazza and Gigli [32], one can prove [6] that the map

$$\begin{aligned}\Upsilon_t\,:\,(M,\,g)\;&\hookrightarrow\;\;\big(\mathrm{Prob}(M),\,d_g^W\big)\\ z\;\;\;&\longmapsto\;\Upsilon_t(z)\,:=\,p_t^{(\omega)}(\cdot\,,z)\,d\omega(\cdot),\end{aligned} \tag{47}$$

is, for any $t\geq 0$, injective. Hence we can indeed exploit $p_t^{(\omega)}(\cdot\,,z)\,d\omega(\cdot)$ to embed $(M,g, d\omega)$ in the corresponding Wasserstein space of probability measures $\big(\mathrm{Prob}(M),\,d_g^W\big)$. Notice that this result also implies that if $U\in C^\infty(M,TM$ is a vector field on M, and $t\longmapsto p_t^{(\omega)}(\cdot\,,z)\,d\omega(\cdot), t\in(0,\infty)$, denotes the flow of probability measure in $\mathrm{Prob}(M,g)$ defined by the weighted heat kernel $p_t^{(\omega)}(\cdot\,,z)$, then, the map $TM\times(0,\infty)\longrightarrow C^\infty(M,\mathbb{R})$

$$(z,U(z);\,t)\;\longmapsto\;U^i(z)\,\nabla_i^{(z)}\,\ln p_t^{(\omega)}(\cdot\,,z), \tag{48}$$

defines, for each $t\in(0,\infty)$, a tangent vector in $T_{p_t(d\omega)}\,\mathrm{Prob}(M,g)$, (the exponent ${}^{(z)}$ in (48) signifies that the differentiation is applied to the indicated variable). We can go a step further and exploit Otto's parametrization (cf. (25)), to represent $U^i(z)\,\nabla_i^{(z)}\,p_t^{(\omega)}(\cdot\,,z)$ as (the gradient of) a scalar potential, $\widehat{\psi}_{(t,z,U)}\in C^\infty(M,\mathbb{R})$. This is a basic result that follows by a rather natural extension of [32]. In particular, we have that for each fixed $t > 0$, and for any $U \in C^{\infty}(M, TM)$, the elliptic partial differential equation,

$$\mathrm{div}^{(y)}_{\omega}\left(p^{(\omega)}_t(y,z)\,\nabla^{(y)}\,\widehat{\psi}_{(t,z,U)}(y)\right) \;=\; -\,U(z)\cdot\,\nabla^{(z)}\,p^{(\omega)}_t(y,z), \tag{49}$$

admits a unique solution $\widehat{\psi}_{(t,z,U)} \in C^{\infty}(M,\mathbb{R})$, with $\int_M \widehat{\psi}_{(t,z,U)}\,\mathrm{d}\omega \;=0$, smoothly depending on the data t, z, U, and such that $\nabla^{(y)}\,\widehat{\psi}_{(t,z,U)}(y) \;\not\equiv\; 0$ if $U \;\neq\; 0$. Hence, if we denote by $\mathcal{H}_{t,\,z}(TM)$ the Hilbert space of gradient vector fields obtained by completion with respect to the $L^2(p_t(\mathrm{d}\omega,z))$ norm then we have that the map $T_z\,M \longrightarrow T_{p_t(\mathrm{d}\omega)}\mathrm{Prob}(M,g) \xrightarrow{\simeq} \mathcal{H}_{t,\,z}(TM)$

$$U(z)\;\longmapsto\;U(z)\;\cdot\nabla^{(z)}\,\ln\,p^{(\omega)}_t(y,z)\;\longmapsto\;\nabla\widehat{\psi}_{(t,z,U)}, \tag{50}$$

is, for any $t \in (0,\infty)$, an injection. According to (50), and in the spirit of Otto's formal Riemmanian calculus [33], we can interpret $\nabla\widehat{\psi}_{(t,z,U)}$ as the push-forward of $U \in T_z\,M$ to the tangent space $T_{p_t(\mathrm{d}\omega)}\,\mathrm{Prob}(M,\,g)$, under the heat kernel embedding map (47). This remark motivated a basic observation by Gigli and Mantegazza (see Def. 3.2 and Prop. 3.4 of [32]) which we extended to the weighted heat kernel so as to prove [6] that for any $t > 0$, $z \in M$, and $U, W \in\, T_z\,M$, the symmetric bilinear form defined by

$$g^{(\omega)}_t\;(U(z),W(z))\;:=\;\int_M g_{ik}(y)\,\nabla^i_{(y)}\,\widehat{\psi}_{(t,z,U)}\,\nabla^k_{(y)}\,\widehat{\psi}_{(t,z,W)}\,p^{(\omega)}_t(y,z)\,\mathrm{d}\omega(y),$$

is a scale-dependent (t) metric tensor on M, varying smoothly in $0 \;<\; t \;<\; \infty$, and reducing to the original metric g on M in the singular limit $t \searrow 0^+$. This construction can be easily extended to the warped manifold $M \times \mathbb{T}^q$ by considering the solution $(t,f) \longmapsto \exp[-\,\frac{2f^{(\omega)}_t}{q}]$ of the heat equation associated to the warping factor $e^{-\,\frac{2f}{q}}$ in the metric (40), *viz.*

$$\left(\tfrac{\partial}{\partial t}-\triangle^{(z)}_{\omega}\right)\,e^{-\,\frac{2f^{(\omega)}_t(z)}{q}}\;=0\;, \qquad t\in(0,\infty), \tag{51}$$

with $\lim_{t\searrow 0^+}\,f^{(\omega)}_t \;=\; f$. In this way we can associate to the embedding (47) the t-dependent metric tensor on $N^{n+q} := M \times \mathbb{T}^q$ defined by

$$h_t(z) := g^{(\omega)}_t(z)\;+\;e^{-\,\frac{2}{q}\,f^{(\omega)}_t(z)}\,\delta, \tag{52}$$

varying smoothly with $t \in (0,\infty)$. It follows that the heat kernel embedding (47) induces a t-dependent deformation of the harmonic energy functional (42) on the warped manifold

$M \times \mathbb{T}^q$ according to [6]

$$
\begin{aligned}
(53)\qquad E[\Phi_{(q)},\, h_t^{(q)}]_{(\Sigma, N^{n+q})} &:= \frac{1}{2}\int_\Sigma \gamma^{\mu\nu}\,\frac{\partial\Phi^a_{(q)}(x,\xi)}{\partial x^\mu}\frac{\partial\Phi^b_{(q)}(x,\xi)}{\partial x^\nu}\,(h_t)_{ab}(\phi)\,\mathrm{d}\mu_\gamma \\
&= \frac{1}{2}\int_\Sigma \gamma^{\mu\nu}\,\frac{\partial\phi^i}{\partial x^\mu}\frac{\partial\phi^j}{\partial x^\nu}\,(g_t^{(\omega)})_{ij}(\phi)\,\mathrm{d}\mu_\gamma \\
&+ \frac{1}{8}\sum_{k=1}^{q} d_g^2(\phi_{cm},\phi_{(k)})\int_\Sigma \left|\mathrm{d}f_t^{(\omega)}(\phi(x);q)\right|_\gamma^2 \mathrm{d}\mu_\gamma,
\end{aligned}
$$

where $(t,h) \longmapsto h_t,\ \ t \in (0,\infty)$, is the flow of metrics defined by (52). This directly implies that if $t \mapsto (\gamma_t) = e^{f_t^{(\omega)}(\phi)}\,\delta,\ t \in (0,\infty)$, denotes the family of conformally flat metrics on $\Sigma \simeq \mathbb{T}^2$ associated with $(t,f) \mapsto f_t^{(\omega)}$, then in the conformal gauge (Σ,γ_t) the harmonic energy functional (53) provides a scale-dependent family of dilatonic actions

$$
(54)\qquad S_M\left[\gamma_t,\phi;\ F(\phi_{cm};q),\ f_t^{(\omega)},\ g_t^{(\omega)}\right] \underset{(f_t)}{:=} \frac{2}{F(\phi_{cm};q)}\,E[\Phi_{(q)},\, h_t^{(q)}]_{(\Sigma,N^{n+q})},
$$

such that in the singular limit $t \searrow 0^+$ we have

$$
(55)\qquad \lim_{t\searrow 0} S_M\left[\gamma_t,\phi;\ F(\phi_{cm};q),\ f_t^{(\omega)},\ g_t^{(\omega)}\right] = S_M\left[\gamma,\phi;\ F(\phi_{cm};q),\ f,\ g\right].
$$

5. – Beta functions and an extension of the Hamilton-Perelman Ricci flow

The above results imply that along the heat kernel embedding Υ_t we get the induced flow

$$
(56)\qquad [0,\infty) \ni t \longmapsto S_M\left[\gamma_t,\phi;\ F(\phi_{cm};q),\ f_t^{(\omega)},\ g_t^{(\omega)}\right],
$$

deforming the dilatonic action $S_M\left[\gamma,\phi;\ F(\phi_{cm};q),\ f,\ g\right]$ in the direction of the non-trivial geometric rescaling $(t,\ g,\ f) \longmapsto g_t^{(\omega)},\ f_t^{(\omega)}$ of the *couplings* defined by the given Riemannian metric measure space $(M,g,\,\mathrm{d}\omega)$. This strongly suggests a connection between heat kernel embedding and the circle of ideas and techniques related to renormalization group. The strategy of the renormalization group analysis of the non-linear σ model [34-37], is to discuss the scaling behavior of the (quantum) fluctuations of the maps $\phi : \Sigma \to M$ around the background average field ϕ_{cm}, defined by the distribution of the center of mass of a large ($q \to \infty$) number of randomly distributed independent copies $\{\phi_{(\mathrm{j})}\}_{\mathrm{j}=1}^q$ of ϕ itself. This is the *background field technique* which allows to check perturbatively if the theory is renormalizable by a renormalization of the couplings (g,f) associated with the Riemannian metric measure space $(M,g,\,\mathrm{d}\omega)$. The constant maps localization described above can be seen as a rigorous formulation of this technique if we consider the heat kernel embedding as a *toy model* for the full renormalization group (RG) flow. As in the standard renormalization group analysis, this toy model is characterized by the running metric $t \mapsto h_t^{(q)}$, *i.e.* by the coupling between the running metric $g_t^{(\omega)}$ and the running dilaton $f_t^{(\omega)}$ describing the behavior of the warped harmonic energy functional

$E[\Phi_{(q)},\, h_t^{(q)}]$ as the distances in $(M^q \times \times_{(f)} \mathbb{T}^q)$ are rescaled by the heat kernel embedding. The associated tangent vector, the so-called *beta*-function,

$$\beta\left(h_t^{(q)}\right) := \frac{\mathrm{d}}{\mathrm{d}t}\, h_t^{(q)}, \tag{57}$$

plays a major role in renormalization group theory. Typically, in quantum field theory one is able to compute the beta function only up to a few leading order terms in the perturbative expansion of the effective action. Notwithstanding this limitation, the resulting truncated flow can be exploited to study both the validity of the perturbative expansion and the nature of the possible fixed points of the renormalization group action [37]. In our case, we have the exact expression for the *effective action* $E[\Phi_{(q)},\, h_t^{(q)}]$, and the beta-function (57) computed along the heat kernel embedding turn out to be remarkably similar to the ones obtained, to leading order, in the standard perturbative analysis of the RG flow for the non-linear sigma model. To wit, by considering the beta-function associated to the metric flow $t \longmapsto g_t^{(\omega)}$, we get that along the weighted heat kernel embedding $(0,\infty)\times M \ni (t,z) \longmapsto p_t^{(\omega)}(\cdot,z)\mathrm{d}\omega(\cdot) \in \mathrm{Prob}_{ac}(M,g)$, the beta-function $\beta(g_t^{(\omega)})$ associated to the scale dependent metric $(M,\, g_t^{(\omega)})$ is provided by [6]

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\, g_t^{(\omega)}(U,W) \;=\; & -2\widetilde{\mathrm{Ric}}^{(t)}(U,W) \\
& -2\int_M \left(\nabla\psi_{(t,U)}\cdot\nabla\nabla f\cdot\nabla\psi_{(t,W)}\right)\, p_t^{(\omega)}(y,z)\mathrm{d}\omega(y) \\
& -2\int_M \left(\mathrm{Hess}\,\widehat{\psi}_{(t,U)}\cdot \mathrm{Hess}\,\widehat{\psi}_{(t,W)}\right)\, p_t^{(\omega)}(y,z)\,\mathrm{d}\omega(y),
\end{aligned} \tag{58}$$

where $\widetilde{\mathrm{Ric}}^{(t)}$ denotes the Ricci curvature of the evolving metric $(M,\, g_t^{(\omega)})$. This result indicates a striking connection between the beta function for the scale dependent flow $[t,(M,g)] \mapsto (M,\, g_t^{(\omega)})$, $t>0$, and (a generalized version of) the DeTurck-Hamilton version of the Ricci flow. This is further supported by the behavior of (58) in the singular limits $t \searrow 0$ and $q \nearrow \infty$. The former controls how the curve of heat kernel embeddings $(0,\infty)\times M \ni (t,z) \longmapsto p_t^{(\omega)}(\cdot,z)\mathrm{d}\omega(\cdot) \in \mathrm{Prob}_{ac}(M,g)$ approaches the isometric embedding of (M,g) in the non-smooth $\mathrm{Prob}(M) \supset \mathrm{Prob}_{ac}(M,g)$. The latter is related to our choice (33) of the dilatonic measure $\mathrm{d}\omega(q)$ localizing the NLσM maps $\phi:\Sigma \longrightarrow (M,g,\mathrm{d}\omega)$ around the center of mass of the constant maps $\{\phi_{(k)}\}_{k=1}^q$. By a non-trivial extension of a deep result in [32] we get that if $[0,1]\ni s \mapsto \gamma_s$, $\gamma(0)\equiv z$ denotes a geodesic in (M,g) with tangent vector $\dot{\gamma}_s$, then the beta function (58) associated with the weighted heat kernel embedding $(M,g,\,\mathrm{d}\omega) \longrightarrow \left(\mathrm{Prob}_{ac}(M,g),\, d_g^W\right)$ is tangent for $t=0$ and $q \nearrow \infty$ to the perturbative beta-functions for the dilatonic non-linear σ model

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\; g_t^{(\omega)}(\dot{\gamma}_s,\dot{\gamma}_s)\Big|_{t=0}\Big|_{q\to\infty} &= -2\left[\mathrm{Ric}_g(\dot{\gamma}_s,\dot{\gamma}_s) + \mathrm{Hess}\, f(\dot{\gamma}_s,\dot{\gamma}_s)\right], \\
\frac{\mathrm{d}}{\mathrm{d}t}\; f_t^{(\omega)}\Big|_{t=0}\Big|_{q\to\infty} &= \triangle_g f - |\nabla f|_g^2,
\end{aligned}$$

where the equality holds for almost every $s\in[0,1]$. Moreover, under the constraint $\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{d}\mu_{h_t^{(q)}}|_{t=0} = 0$, we get that the beta function (58) is tangent, for $t=0$ and $q \nearrow \infty$, to the generators of the Hamilton-Perelman Ricci flow according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\; g_t^{(\omega)}(\dot\gamma_s,\dot\gamma_s)\Big|_{t=0}\Big|_{q\to\infty} = -2\left[\mathrm{Ric}_g(\dot\gamma_s,\dot\gamma_s) + \mathrm{Hess}\, f(\dot\gamma_s,\dot\gamma_s)\right],$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\; f_t^{(\omega)}\Big|_{t=0}\Big|_{q\to\infty} = -\triangle_g f - R^{(g)} ,$$

where, again, the equality holds for almost every $s \in [0,1]$. Hence, we can connect the singular limit $t \searrow 0$, $q \nearrow \infty$ of the beta function (58) to the Hamilton-Perelman Ricci flow.

The strong similarity between (58) and the (DeTurck version [38] of the) Ricci flow, and the tangency conditions described above, may suggest that (58) is indeed an extension of the Ricci flow. In the case of the standard heat kernel embedding [32], the induced flow on the distance function, tangential to the Ricci flow for $t = 0$, is well defined for any $t \geq 0$, and with strong control on the topology of M and good continuity properties with respect to (measured) Gromov-Hausdorff convergence. As argued in [32], these properties strongly contrast with the typical behavior of the Ricci flow, characterized by the development of curvature singularities and by a poor control on Gromov-Hausdorff limits of sequences of Ricci evolved manifolds. Conversely, the explicit expression (58) for the heat kernel induced flow $(t,g) \longmapsto g_t^{(\omega)}$, and in particular the presence of the norm–contracting term $\int_M |\mathrm{Hess}\,\widehat{\psi}_{(t,U)}|_g^2\, p_t^{(\omega)}(y,z)\,\mathrm{d}\omega(y)$, suggests that along the flow $(t,g) \longmapsto g_t^{(\omega)}$ there is a strong control of the metric geometry of $g_t^{(\omega)}$ a property, this latter, that indicates that (58) provides indeed a non-trivial extension of the Ricci flow. The reader, hopefully stimulated by these remarks, can find an introduction to the geometry of the renormalization group in [34, 39] and a detailed analysis with the relevant technical proofs of the results described here in [6].

* * *

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