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Compactons of nonlinear Schrödinger lattices under strong nonlinearity management

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Summary. — We review recent work on compacton matter waves in Bose-Einstein condensates (BEC) trapped in deep optical lattices in the presence of strong and rapid periodic time modulations of the atomic scattering length. In particular, we derive averaged discrete nonlinear Schrödinger equations (DNLSE) and show that compacton solutions of different types can exist as stable excitations. Stability properties are also investigated both by linear analysis and by direct numerical integrations of the DNLSE.

PACS `42.65.-k` – Nonlinear optics.
PACS `42.81.Dp` – Propagation, scattering, and losses; solitons.
PACS `03.75.Lm` – Tunneling, Josephson effect, Bose-Einstein condensates in periodic potentials, solitons, vortices, and topological excitations.

1. – Introduction

Periodic management of parameters of nonlinear wave systems is a very attractive technique for the generation of solitons with new types of properties [1]. Examples of the management technique in continuous systems are the dispersion management of solitons in optical fibers which allows to improve communication capacities [2], and the nonlinearity management of 2D and 3D Bose-Einstein condensates (BEC) or optically layered media which provides partial stabilization against collapse in the case of attractive interatomic interactions [3]. In discrete systems the diffraction management technique was used to generate spatial discrete solitons with novel properties [4, 5] which have recently been observed in experiments [5]. The suppression of the inter-well tunneling was experimentally observed also in the propagation of light in waveguide arrays [6] and in BEC's in strongly driven optical lattices [7]. In these cases, however, the system is typically subjected either to resonant modulations of the dispersion (coupling between waveguides for the case of light propagation) or to external linear forces (shaking of the optical lattices for the case of BEC). The inhibition of the inter-well tunneling, however, may become possible also in the presence of fast periodic time variations of the nonlinearity, the so-called *strong nonlinearity management* (SNLM). In this case new phenomena, such as strong localization and formation of (discrete counterparts of) the so called *compactons* [8] *e.g.* localized nonlinear waves with compact support, can arise.

The aim of the present work is to demonstrate the existence of compacton solutions in the prototypical dynamical lattice of the discrete nonlinear Schrödinger (DNLS) form [9] subjected to SNLM. It is shown that, contrary to ordinary solitons, the amplitude of a compacton reduces exactly to zero outside the localizing domain, this implying the total suppression of the inter-well tunneling at the compacton edges. For this, we derive averaged discrete nonlinear Schrödinger equations (DNLSE) and show that compacton solutions of different types can exist as stable excitations [10]. Stability properties are studied by linear analysis and by direct numerical integrations of the DNLS system. Similar compact excitations can exist also in binary BEC mixtures modeled by a vector DNLSE [11] and in multidimensional contexts [12].

2. – The model

The dynamical lattice considered here is the well known DNLS equation [9]

$$i\dot{u}_n + \kappa(u_{n+1} + u_{n-1}) + (\gamma_0 + \gamma(t))|u_n|^2 u_n = 0, \tag{1}$$

which serves as a prototypical model both for matter waves in BEC arrays and for light propagation in arrays of optical waveguides. In the BEC context κ quantifies the coupling (tunneling of matter) between adjacent wells of the optical lattice, t represents the time and γ_0 and $\gamma(t)$ represent the constant and the modulated part of the interatomic interaction (nonlinearity), respectively. In this case the management corresponds to periodically varying in time the atomic scattering length which can be achieved by the Feshbach resonance technique. In the optical context the time t should be replaced by the propagation distance z, κ quantifies the coupling between adjacent waveguides, and the nonlinear management consists in periodically varying in space the Kerr nonlinearity around a constant value γ_0. In the following we shall refer to the BEC context and assume $\gamma(t)$ to be a periodic, $\gamma(t) = \gamma(t+T)$, and rapidly varying function of time of the form $\gamma(t) = \frac{\gamma_1}{\varepsilon}\cos(\Omega\frac{t}{\varepsilon})$, with $\varepsilon \ll 1$ and $T = 2\pi/\Omega$ the period.

To investigate the existence of discrete compacton solitons in this model, we shall derive averaged equations over rapid modulations, using the method developed in [13]. Following this approach, we introduce the new variables v_n related to the field u_n as

$$u_n(t) = v_n(t)e^{i\Gamma(t)|v_n|^2(t)}, \quad \Gamma(t) = \frac{1}{\epsilon}\int_0^t \mathrm{d}t\gamma_1\left(\frac{t}{\epsilon}\right). \tag{2}$$

Substituting this expression into eq. (1) and averaging the resulting equation over the period of the rapid modulation, we obtain

$$\begin{aligned} iv_{n,t} = &-\alpha\kappa v_n[(v_{n+1}v_n^* + v_{n+1}^* v_n)J_1(\alpha\theta_+) + (v_{n-1}v_n^* + v_{n-1}^* v_n)J_1(\alpha\theta_-)] \\ &- \kappa[v_{n+1}J_0(\alpha\theta_+) + v_{n-1}J_0(\alpha\theta_-)] - \gamma_0|v_n|^2 v_n, \end{aligned} \tag{3}$$

where $J_n(x)$ is the Bessel function of order n, $\theta_\pm = |v_{n\pm1}|^2 - |v_n|^2$ and $\alpha = \gamma_1/K$. This modified DNLS equation has the essentially nonlinear neighbor-neighbor interactions and can be put in Hamiltonian form $i\dot{v}_n = \frac{\delta H}{\delta v_n^*}$, with Hamiltonian

$$H_{av} = -\sum_n \left\{ \kappa J_0(\alpha\theta_+) \left[v_{n+1} v_n^* + v_{n+1}^* v_n \right] + \frac{\gamma_0}{2} |v_n|^4 \right\}.$$

For small $\alpha\theta_+$ the function J_0 can be expanded in series giving the same averaged Hamiltonian of the DNLS equation obtained in [14] in the limit of *weak* nonlinearity management.

3. – Compactly supported localized modes

In this section we demonstrate the existence of exact compactons in this averaged system. We remark that compacton solutions were initially reported as a "mathematical curiosity" of somewhat artificial variants of the DNLS where linear dispersion is absent [15], by analogy to their continuum siblings [8]. The present setting, however, is in some sense unique, in that linear dispersion is not, generally speaking, absent. In fact, there is a linear spectrum of the background state in the linearization of even these compact solutions and can be analytically shown to extend from $[-\mu - 2\kappa, -\mu + 2\kappa]$ and from $[\mu - 2\kappa, \mu + 2\kappa]$. Yet, there exist particular values of the amplitude which essentially completely *inhibit* the inter-well tunneling to the nearest neighbors and hence enable the formation of such compact structures.

To this regard we seek for stationary solutions of the form $v_n = A_n e^{-i\mu t}$ for which eq. (4) becomes

$$\begin{aligned} \mu A_n + \gamma_0 A_n^3 + \kappa(A_{n+1} J_0(\alpha\theta_+) + A_{n-1} J_0(\alpha\theta_-)) \\ + 2\alpha\kappa A_n [A_{n+1} A_n J_1(\alpha\theta_+) + A_{n-1} A_n J_1(\alpha\theta_-)] = 0, \end{aligned} \tag{4}$$

and look for conditions of tunneling suppression at the last site of vanishing amplitude (edge of the compacton) denoted as n_0 below. In the setting of eq. (4), this directly establishes that

$$J_0(\alpha |u_{n_0+1}|^2) = 0 \Rightarrow |u_{n_0+1}|^2 = 2.4048/\alpha, \tag{5}$$

which yields the solution (based on the first zero of the Bessel function) for the "boundary" of the compactly supported site. Then, for $\mu = -\gamma_0 |u_{n_0+1}|^2$, both the condition for compact support at $n_0 \pm 1$, and the equation for $n = n_0$ are satisfied. Hence eq. (5) yields a single-site discrete compacton, which linearization illustrates to be stable (both the solution and its typical linearization are shown in fig. 1.

One can then generalize this type of consideration to two-sites, which are either in phase (2nd column of fig. 1) or out-of-phase, these last also called twisted modes (1st column of fig. 2). The only thing that changes here is that in order to satisfy the equation at the non-vanishing sites,

$$\mu = -\kappa - \gamma_0 |u_{n_0+1}|^2, \qquad \mu = \kappa - \gamma_0 |u_{n_0+1}|^2, \tag{6}$$

respectively, for the in-phase and out-of-phase two-site modes. We stress that these are *exact solutions* of the reduced system. Susprisingly, and completely contrary to what is the case for DNLS, *both* of these solutions are spectrally stable, as shown in fig. 2

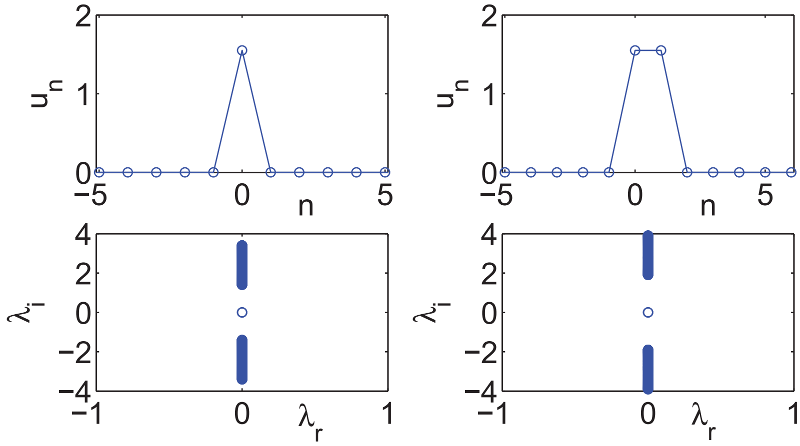

Fig. 1. – Typical examples for $\kappa = 0.5$, $\alpha = 1$ of compact localized mode solutions of eq. (4) (top panels) and of the spectral plane (λ_r, λ_i) of their linearization eigenvalues $\lambda = \lambda_r + i\lambda_i$. 1st column: on-site, 2nd column: inter-site in-phase compacton. Remarkably, all solutions are spectrally stable.

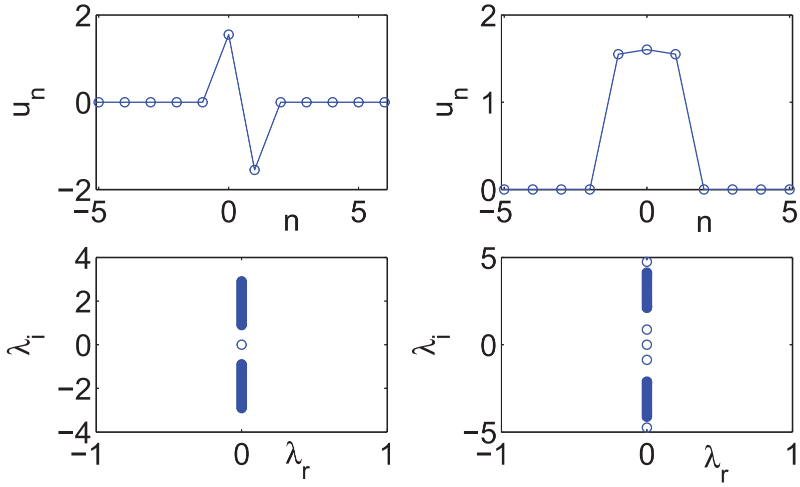

Fig. 2. – Compacton solutions of eq. (4) 1st column: inter-site out-of-phase mode, 2nd column: symmetric 3-site compacton.

Moreover, one can generalize these considerations to an arbitrary number of sites. As a typical example, a 3 site mode with amplitudes $(\ldots, 0, A_1, A_2, A_1, 0, \ldots)$ will satisfy in addition to the "no tunneling condition" $J_0(\alpha A_1) = 0$, the constraints

$$\mu A_1 + \kappa A_2 J_0(\alpha(A_2^2 - A_1^2)) + 2\alpha\kappa A_1^2 A_2 J_1(\alpha(A_2^2 - A_1^2)) + \gamma_0 A_1^3 = 0, \tag{7}$$

$$\mu A_2 + \kappa A_1 J_0(\alpha(A_1^2 - A_2^2)) + 4\alpha\kappa A_2^2 A_1 J_1(\alpha(A_1^2 - A_2^2)) + \gamma_0 A_2^3 = 0, \tag{8}$$

which can be easily solved to yield a solution as the one shown in the 2nd column of fig. 2. Even such more complex solutions which would be highly unstable in DNLS are dynamically robust in the present setting. To examine the full nonlinear dynamical

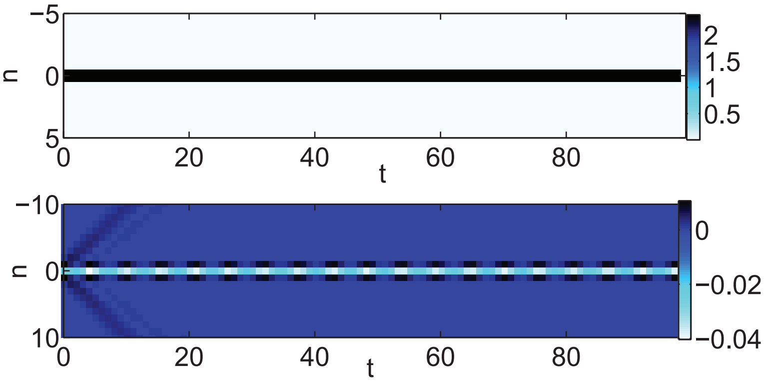

Fig. 3. – Propagation of single-site compacton solution. The top panel shows the evolution obtained from the averaged equation in eq. (4), while the bottom panel refers to the numerical integration of the original DNLS system in eq. (1).

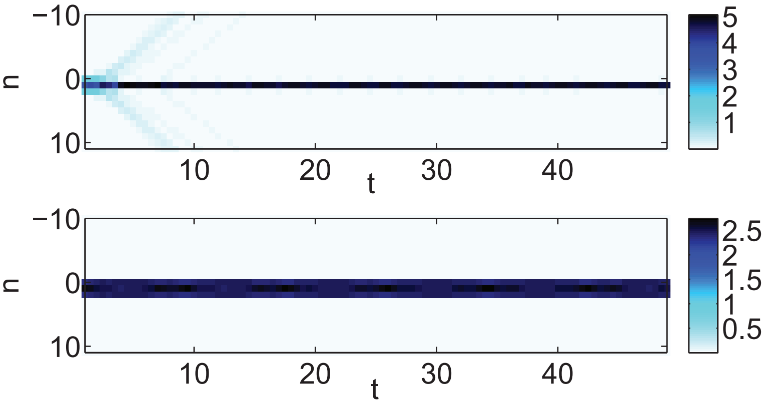

Fig. 4. – Propagation of a perturbed 3-site compacton for $\kappa = 1$ decaying into a single-site compacton for $\epsilon = 0.1$ (top panel) and remaining stable for $\epsilon = 0.025$ (bottom panel).

stability of these solutions, we considered them as initial conditions both in the averaged eq. (4), as well as in the full eq. (1). The results are shown in fig. 3 for the case of a single-site compaction (similar findings were obtained for other modes). The top panel (of large colorbar amplitude) shows the space-time contour map of the solution modulus, while the bottom panel (of small colorbar amplitude) illustrates the deviation from the original solution. To further ensure robustness, a uniformly distributed random perturbation of small amplitude was added to the original solution. It can clearly be seen that in *all* cases, both in the averaged equation and in the original system of eq. (1), the relevant perturbation stays uniformly bounded and never exceeds 2% of the solution amplitude. The waveforms remain remarkably localized in their compact shape (after a transient stage of shedding off small amplitude wavepackets). Notice that for eq. (1), $\gamma(t) = 1 + \frac{1}{\epsilon}\cos(t/\epsilon)$, with $\epsilon = 0.1$ was used.

It should, however, be noted that if one departs from the regime of validity of the averaging, interesting deviationsm from the above behavior (and stability) may ensue. An example of this is shown in the panels of fig. 4. In this case, the 3-site solution was initialized in eq. (1) with $\epsilon = 0.1$ in the top panel, while $\epsilon = 0.025$ in the bottom one. While in the latter case, the above argued robustness of the averaged modes was observed, in the former one, the apparent lack thereof was clearly due to the use of an ϵ outside of the regime of applicability of the averaging approximation. Nevertheless, the resulting evolution confirms the general preference of the system towards settling in compact modes, since despite the large coupling $\kappa = 1$ used in this case, the evolution asymptotes to an essentially single-site solution.

4. – Conclusions

In conclusion, we have shown that in the SNLM limit stable discrete compactons can exist both in one-dimensional BEC trapped in deep optical lattices and in optical waveguides arrays with Kerr nonlinearity periodically varied along the propagation distance. In this last case, the absence of interacting tails in compactons should permit a maximal rate of information transfer without disturbing interferences. The setting considered herein could therefore lead to experimental observation of discrete compactons both in BEC arrays and in nonlinear optical waveguides under SNLM.

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