IL NUOVO CIMENTO **38 C** (2015) 173
DOI 10.1393/ncc/i2015-15173-3

Colloquia: VILASIFEST

Star product and contact Weyl manifold

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received 11 January 2016

Dedicated to Professor Gaetano Vilasi
on the occasion of his 70th birthday

Summary. — Contact algebra is introduced, which is a Lie algebra given as a one-dimesional exrention of a Weyl algebra. A contact Lie algebra bundle called a contact Weyl manifold is considered over a symplectic manifold which contains a Weyl manifold as a subbundle. A relationship is discussed between deformation quantization on s symplectic manifold and a Weyl manifold over the symplectic manifold. The contact Weyl manifold has a canonical connection which gives rise the relation, and is regarded as an extension of Fedosov connection.

1. – Introduction

The Moyal product [1] is used to quantize a classical mechanical system in $\mathbb{R}^{2m}$ with the canonical Poisson bracket, so-called *Moyal quantization*, which can be extended to $\mathbb{R}^n$ or $\mathbb{C}^n$ by replacing the Poisson bracket with some slighty wider class of biderivations in a convergent way (cf. [2]). The products can be considered on any manifold to give rise to a concept of deformation quantization [3], and any Poisson manifold has deformation quantizations in a formal sense [4].

The formal deformation quantization is given a geometric picture on a symplectic manifold by using Weyl manifold, a Weyl algebra bundle over the symplectic manifold [5]. Roughly speaking, a Weyl manifold is a quantized symplectic manifold in a sense that, while a symplectic manifold is regarded as a patch work of canonical coordinates by means of canonical transformations, the Weyl manifold is given as a patch work of *quantized canonical coordinates* by means of *quantized canonical transformations*. From a Weyl manifold over a symplectic manifold, one can obtain conversely, form a deformation quantization on a symplectic manifold, one can obtain a Weyl manifold over the symplectic manifold. On the other hand, Fedosov considered a connection on a Weyl

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algebra bundle over a symplectic manifold to investigate index theory on symplectic manifolds, and he applied the connection to deformation quantization theory to obtain a simple construction of deformation quantization [6].

The contact Weyl algebra is introduced in [5] and is used as a tool to make a patch work of quantized canonical coordinates by means quantized canonical transformations called Weyl diffeomorpisms. The contact Weyl algebra is a Lie algebra containing the Weyl algebra as a subalgebra. The Weyl algebra is regarded as a quantization of the canonical symplectic structure in terms of the Moyal product, and the contact Weyl algebra is regarded as a contactification [7] of quantized canonical symplectic structure in this sense.

In [8], using the contact Weyl algebra as a fiber, Yoshioka defined an algebra bundle over a symplectic manifold called a *contact Weyl manifold*, and showed its existence for any symplectic manifold. A contact Weyl manifold contains a Weyl manifold as a subbundle, which is considered as a contactification of quantized symplectic manifold. Then the existence theorem means that every quantized symplectic manifold has a quantized contactification.

A contact Weyl manifold has a canonical connection [8] and the connection characterizes the set of Weyl functions which is the structue of Weyl manifold, and it was shown that when the connection is restricted to the subbundle, that is the Weyl manifold, it gives the so-called Fedosov connection. Then we have seen that the bundle that Fedosov used in his theory [6] coincides with the Weyl manifold [5] and the canonical connection on the contact Weyl manifold is an extenstion of Fedosov connection.

The curvature of the canonical connection on a contact Weyl manifold is a formal power series of closed 2-forms on the base symplectic manifold whose lowest order term is the symplectic 2-form. Then the curvature form determines a formal power series of the second cohomology class of the base manifold. Since the connection is equal to Fedosov connection on the Weyl manifold, the cohomology class given by the canonical connection is equal to the characteristic class which induces a moduli space of deformation quantization on the sympectic manifold. In [9], in the process of gluing quantized canonical coordinates, namely the local Weyl functions, naturally emerges a formal power series of the second Čech cohomology class called a *Poincaré-Cartan class*, and the class also characterizes the equivalence classes of Weyl manifolds and then those of deformation quantizations on the symplectic manifold. It is proved in [8] that the cohomology class by Čech is equal to that given by the curvature form of the canonical connection.

In this paper, we give a brief review on deformation quantization and Weyl manifolds and also on contact Weyl manifolds.

2. – Star product, deformation quantization

In this section, we discuss deformation quantiations, or star products.

2˙1. *Example: Moyal product.* – We start by the well-known example of star product, the Moyal product. Let M be a $2n$-dimensional Euclidean space $\mathbb{R}^{2n}$. We write the coordinates as $(x_1, \dots, x_n, y_1, \dots, y_n) \in \mathbb{R}^{2n}$, and the canocnical symplectic structure as $\omega = \sum_{k=1}^{n} dy_k \wedge dx_n$, respectively. The Poisson bracket $\{f, g\} = \sum_{k=1}^{n} (\partial_{x_k} f \, \partial_{y_k} g -$

$\partial_{y_k} f\, \partial_{x_k} g)$ is written as the following biderivation such that

$$\begin{aligned}\{f,g\} &= \sum_{k=1}^{n} (\partial_{x_k} f\, \partial_{y_k} g - \partial_{y_k} f\, \partial_{x_k} g) = \sum_{k=1}^{n} (f \overleftarrow{\partial}_{x_k} \overrightarrow{\partial}_{y_k} g - f \overleftarrow{\partial}_{y_k} \overrightarrow{\partial}_{x_k} g) \\ &= f \overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_y g - f \overleftarrow{\partial}_y \cdot \overrightarrow{\partial}_x g = f(\overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_y - \overleftarrow{\partial}_y \cdot \overrightarrow{\partial}_x) g = f \overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y g\end{aligned}$$

for smooth functions f, g on $\mathbb{R}^{2n}$. The l th porwer of the biderivation $\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y$ is calculated by means of the binomal theorem such as

$$\left(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right)^l = \sum_{k=0}^{l} \begin{pmatrix} l \\ k \end{pmatrix} (-1)^k (\overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_y)^{l-k} (\overleftarrow{\partial}_y \cdot \overrightarrow{\partial}_x)^k$$

and defines a bidifferential operator on $\mathbb{R}^{2n}$.

The Moyal product $*_0$ is then given by a formal power seris in ν of the biderivation of the exponential type such that

$$\begin{aligned} f *_0 g &= fg + (\tfrac{\nu}{2}) f(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y) g + \cdots + (\tfrac{\nu}{2})^l \tfrac{1}{l!} f(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y)^l g + \cdots \\ &= f \exp\left(\tfrac{\nu}{2} \overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right) g \end{aligned}$$

for any $f, g \in C^\infty(\mathbb{R}^{2n})$. The Moyal product is extended naturally to the formal power series such as $f = \sum_{l \geq 0} f_l\, \nu^l, g = \sum_{l \geq 0} g_l\, \nu^l \in C^\infty(\mathbb{R}^{2n})[[\nu]]$, and then the Moyal product is an associative product on $C^\infty(\mathbb{R}^{2n})[[\nu]]$.

We sometimes write the Moyal product in general form such as

$$f *_0 g = fg + \nu C_1(f,g) + \nu^2 C_2(f,g) + \cdots + \nu^l C_l(f,g) + \cdots,$$

where $C_l(f,g) = \frac{1}{l!}(\frac{1}{2})^l (\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y)^l$, $l = 1, 2, \ldots$.

2˙2. *Star product*. – The definition of star product is direct from the Moyal product. For a manifold M, we consider a binary product on the space of formal power series $C^\infty(M)[[\nu]]$ such that

$$f * g = fg + \nu C_1(f,g) + \nu^2 C_2(f,g) + \cdots + \nu^l C_l(f,g) + \cdots,$$

where $C_l(\cdot,\cdot)$ is a bilinear map from $C^\infty(M) \times C^\infty(M)$ to $C^\infty(M)$. In what follows, we set $\mathcal{A}_\nu(M) = C^\infty(M)[[\nu]]$ for simplicity.

Definition 1. A product $f * g$, $f, g \in \mathcal{A}_\nu(M)$ is called a star product when it is associative.

Then for a star product $*$ on M, we have an associative algebra $(\mathcal{A}_\nu(M), *)$, called a *star product algebra*. We have the following.

Poisson structure. We see that a star product naturally induces a Poisson structure on the manifold M.

Consider skewsymmetric part C_1^- of C_1, nemely, $C_1^-(f,g) = \frac{1}{2}(C_1(f,g) - C_1(g,f))$, $\forall f, g \in C^\infty(M)$. Then, the associative product satisfies

1. $[f, g * h]_* = [f,g]_* * h + g * [f,h]_*$,
2. $[f, [g,h]_*]_* +$ (cyclic with respect to $f, g, h) = 0$,

where $[f,g]_* = f*g - g*f$ is the commutator of the product, and the lowest order terms of the expansion of the above give

Proposition 2. The skew symmetric part of C_1 is a Poisson bracket on M.

Suppose we have given a Poisson structure $\{\cdot,\cdot\}$ on M.

Definition 3. A star product $*$ on M is called a deformation quantization of the Poisson manifold $(M, \{\cdot,\cdot\})$ when the skew symmetric part of C_1 is equal to $\{\cdot,\cdot\}$.

Equivalence. Suppose we have star products $*$, $*'$ on a manifold M, and then we have star product algebras $(\mathcal{A}_\nu(M), *)$, $(\mathcal{A}_\nu(M), *')$

Definition 4. The star products $*$, $*'$ are equivalent if there exists an algebra isomorphism $T : (\mathcal{A}_\nu(M), *) \to (\mathcal{A}_\nu(M), *')$ of the form

$$T(f) = f + \nu T_1(f) + \nu^2 T_2(f) + \cdots + \nu^l T_l(f) + \cdots$$

where T_l is a linear map of $C^\infty(M)$, $l = 1, 2, \ldots$,

We have the followng (see [10]).

Proposition 5. For every equivalence class of star product on M, there is a representative $f * g = fg + \nu C_1(f,g) + \cdots + \nu^l C_l(f,g) + \cdots$, $\forall f, g \in \mathcal{A}_\nu(M)$ such that C_1 is a Poisson bracket, namely, its symmetric part is zero. Moreover, we can take every C_l ($l = 1, 2, \ldots$) is local, namely, a differential operator on M.

Back ground. Star products are already treated by Weyl, Wigner, Moyal, Groenewold. These can be regarded as a deformation of the usual multiplication of functions. For these, Bayen-Flato-Fronsdal-Lichnerowicz-Sternheimer proposed a concept of deformation quantization on a manifold.

By many people's efforts, the existence and classification problem for formal deformation quantization became clear and was established (*e.g.*, see [11]). Kontsevich proved that there is a deformation quantiaztion on every Poisson manifold in a formal semse.

3. – Weyl manifold

When M is a symplectic manifold, a star product has a geometric picture which we call a Weyl manifold. From a Weyl manifold over a symplectic manifold, we can obtain a deformation quantization of symplectic manifold.

In what follows, we will explain the construction of Weyl manifold over arbitrary symplectic manifold and also explain how we obtain a deformation quantization from a Weyl manifold.

This section is based on the joint work with Omori, Maeda [5].

Let (M, ω) be a $2n$-dimensional symplectic manifold. A Weyl manifold W_M is a Weyl algebra bundle over (M, ω) with certain properties. Weyl manifold has a deep relationship with deformation quantization of symplectic manifold. Namely, from W_M we can obtain a deformation quantization of (M, ω) and conversely from a deformation quantization of (M, ω) we obtain W_M, and in this sense W_M is regarded as a geometric picture of deformation quantization of (M, ω).

3˙1. *Idea.* – The basic idea of the construction of Weyl manifold is to embed local functions on a Darboux chart into Weyl algebra whose embedded image is called Weyl functions.

Quantized Daruboux chart. For any point $p \in M$, by Darboux theorem there exists a coordinate neighborhood $(U, (x_1, \ldots, x_n, y_1, \ldots, y_n))$ such that $\omega = \sum_{j=1}^{n} dy_j \wedge dx_j$, which is called canonical coordinates or Darboux chart. With respect to this chart the Poisson bracket of (M, ω) is written as $\{\cdot, \cdot\} = \overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y$, then we have the Moyal product on U;

$$\begin{aligned} f *_0 g &= fg + (\tfrac{\nu}{2}) f(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y) g + \cdots + (\tfrac{\nu}{2})^l \tfrac{1}{l!} f(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y)^l g + \cdots \\ &= f \exp\left(\tfrac{\nu}{2} \overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right) g, \quad f, g \in \mathcal{A}_\nu(U) = C^\infty(U)[[\nu]]. \end{aligned}$$

The triplet $(U, (x, y), *_0)$ is regarded as a quantized Darboux chart.

Quantized Darboux theorem. Suppose we have a deformation quantization $*$ of the symplectic manifold (M, ω):

$$f * g = fg + \nu C_1(f, g) + \nu^2 C_2(f, g) + \cdots + \nu^l C_l(f, g) + \cdots .$$

By the previous Proposition, we can assume $C_1 = \frac{1}{2}\{\cdot, \cdot\}$ and C_l $(l = 1, 2, \ldots)$ are bidifferential operators. Then the star product $*$ is local and we can restrict on U to have local star product algebra $(\mathcal{A}_\nu(U), *)$ with the coordinates with product $(U, (x, y), *)$. Then we have a "Quantized Darboux theorem" as follows.

Proposition 6. Suppose we have a star product $*$ on M. Then on every U, the local star product algebra $(\mathcal{A}_\nu(U), *)$ with local coodinates with product $(U, (x, y), *)$ is equivalent to the local Moyal product algebra $(\mathcal{A}_\nu(U), *_0)$ with $(U, (x, y), *_0)$. Hence, the star product $*$ has a local coordinate expression of quantized Darboux chart $(U, (x, y), *_0)$ for every point.

In what follows, we write the canonical coordinates as $(x_1, \cdots, x_n, y_1, \cdots, y_n) = (z_1, \cdots, z_{2n}) = z$ for simplicity.

Quantized symplectic atlas. Suppose we have a symplectic atlas $\{(U_\alpha, z_\alpha)\}_{\alpha \in \Lambda}$, where (U_α, z_α) is a Darboux chart for each α. Then quantized Darboux theorem shows that the star product $*$ has a quantized symplectic atlas $\{(U_\alpha, z_\alpha, *_0)\}_{\alpha \in \Lambda}$, and yields local star product algebras $\{(\mathcal{A}_\nu(U_\alpha), *_0)\}_{\alpha \in \Lambda}$ glued together by algebra isomorphisms:

$$T_{\beta\alpha} : (\mathcal{A}_\nu(U_\alpha), *_0)|_{U_\alpha \cap U_\beta} \to (\mathcal{A}_\nu(U_\beta), *_0)|_{U_\beta \cap U_\alpha},$$

These isomorphisms obviously satisfy

Lemma 7. *i)* $T_{\alpha\gamma}T_{\gamma\beta}T_{\beta\alpha} = 1$ *for* $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$.

ii) $T_{\beta\alpha}^{-1} = T_{\alpha\beta}$ *for* $U_\alpha \cap U_\beta \neq \emptyset$.

3˙2. *Weyl manifold.* – According to the previous argument, conversely, for any symplectic manifod (M, ω) it is natural to consider to construct a deformation quantization $*$ of (M, ω) by gluing local Moyal algebras, or Quantized Darboux charts by algebra isomorphisms. For this purpose, we first construct a Weyl algebra bundle over (M, ω) called Weyl manifold from which we can obtain a deformation quantization.

In a word, Weyl manifold is a locally trivial Weyl algebra bundle over a symplectic manifold whose gluing maps are Weyl diffeomorphisms of local trival bundles. Here Weyl diffeomorphism is the bundle isomorphism which preserves Weyl functions. The Weyl functions are the key concept, or the geometric structure, of quantized symplectic manifold or Weyl manifold.

Formal Weyl algebra. A formal Weyl algebra W is an associative algebra, with the multiplication denoted by $\hat{*}$, formally generated over $\mathbb{R}$ or $\mathbb{C}$ by elements $\nu, X_1, \ldots, X_n, Y_1, \ldots, Y_n$ where ν commutes with any elements and satisfy the canonical commutation relation

$$[X_j, Y_k]_* = \nu\delta_{jk}, \quad [X_j, X_k]_* = [Y_j, Y_k]_* = 0, \quad j, k = 1, 2, \ldots, n.$$

Here the bracket $[\cdot,\cdot]_*$ is the commutator of W; $[F, G]_* = F\hat{*}G - G\hat{*}F$, $F, G \in W$.

For simplicity, instead of $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ we sometimes use a notation

$$(X_1, \ldots, X_n, Y_1, \ldots, Y_n) = (Z_1, \ldots, Z_{2n}).$$

Weyl ordered expression and Moyal product formula. In W we consider the completely symmetric polynomials such as $X_1\hat{*}X_2 + X_2\hat{*}X_1/2$, which we denote by X_1X_2, etc.

It is easy to see the set of all symmetric polynomials forms a linear basis of W. Using this basis, the formal Weyl algebra W is expressed as the formal power series of the generators $\nu, X_1, \ldots, X_n, Y_1, \ldots, Y_n$ with the Moyal product formula. Namley, we have a linear isomorphism

$$\sigma : W \to \mathbb{C}[[\nu, X_1, \ldots, X_n, Y_1, \ldots, Y_n]].$$

And with this identification the multiplication $\hat{*}$ is given in $\mathbb{C}[[\nu, X_1, \ldots, X_n, Y_1, \ldots, Y_n]]$ as the Moyal product that is, any elements are expressed as a formal power serires, $F = \sum_{l\alpha} a_{l\alpha}\nu^l Z^\alpha$, $G = \sum_{m\beta} b_{m\beta}\nu^m Z^\beta$, and we have

Lemma 8.

$$\begin{aligned} F\hat{*}G &= F \exp\left(\tfrac{\nu}{2}\overleftarrow{\partial}_X \wedge \overrightarrow{\partial}_Y\right) G \\ &= FG + (\tfrac{\nu}{2})F(\overleftarrow{\partial}_X \wedge \overrightarrow{\partial}_Y)G + \cdots + (\tfrac{\nu}{2})^l \tfrac{1}{l!} F(\overleftarrow{\partial}_X \wedge \overrightarrow{\partial}_Y)^l G + \cdots \end{aligned}$$

Weyl function. Let U be an open subset of $\mathbb{R}^{2n}$. We consider to embed a function f on U into a formal Weyl algebra W. The embedding is given as a similar fashion as Taylor expnasion and is called a Weyl continuation of function denoted by $f^{\#}$ such that

$$f^{\#}(z) = \sum_{\alpha} \tfrac{1}{\alpha!}\, \partial_z^{\alpha} f(z)\, Z^{\alpha}, \quad z \in U.$$

The Weyl continuation is obviously extended to the formal power series $\mathcal{A}_\nu(U) = C^\infty(U)[[\nu]]$, and give a section of the trivial Weyl algebra bundle $U \times W = W_U$, namely, $f^{\#} \in \Gamma(W_U)$. We denote the image of $\#$ by $\mathcal{F}(W_U) = \mathcal{A}_\nu(U)^{\#} \subset \Gamma(W_U)$.

It is direct to see that the Moyal products $*_0$, $\hat{*}$ and the Weyl continuation $\#$ commute, namely we have

Proposition 9.

$$(f *_0 g)^{\#} = f^{\#} \hat{*} g^{\#}, \quad \forall f, g \in \mathcal{A}_\nu(U).$$

Then we have

Corollary 10. *i) The space of the Weyl functions is an associative algebra under the multiplication $\hat{*}$, and $(\mathcal{F}(W_U), \hat{*})$ is an associative algebra.*

ii) The Weyl continuation is an algebra isomorphism

$$\# : (\mathcal{A}_\nu(U), *_0) \to (\mathcal{F}(W_U), \hat{*}).$$

3·3. *Weyl diffeomorphism.* – Instead of gluing local Moyal algebras $(\mathcal{A}_\nu(U), *_0)$, we glue the algebras $(\mathcal{F}(W_U), \hat{*})$ of Weyl functions. Since $\mathcal{F}(W_U)$ is a space of sections of the trivial bundle W_U, we gain a bundle picture for the star product algebra $(\mathcal{A}_\nu(U), *_0)$.

Consider trivial bundles $W_U = U \times W$, and $W_{U'}$ for open subsets $U, U' \subset \mathbb{R}^{2n}$.

Definition. A bundle isomorphism $\Phi : W_U \to W_{U'}$ with induced map $\phi : U \to U'$ is called a *Weyl diffeomorphism* when

i) $\Phi(\nu) = \nu$.

ii) $\Phi^*(\mathcal{F}(W_{U'}) = \mathcal{F}(W_U)$.

iii) $\Phi^* f^{\#} = (\phi^* f)^{\#} + O(\nu^2), \quad f \in \mathcal{A}_\nu(U')$.

Remark 11. *1. The condition i) is natural which means that Φ is $\mathbb{C}[[\nu]]$-linear.*

2. The condition ii) is essential to our theory. We regard the Weyl functions $\mathcal{F}(W_U)$ as the geometric structure of Weyl manifold W_M, or quantized symplectic manifold.

3. The conditions iii) is optional. The condition iii) corresponds that the symmetric part C_1^+ vanishes.

A bundle map naturally induces a map between the base space $\phi : U \to U'$. As to Weyl diffeomorpism we have the following (see [5] Lemma 3.3).

Lemma 12. The induced map $\phi : U \to U'$ of a Weyl differomorphism $\Phi : W_U \to W_{U'}$ is a symplectic diffeomorphism.

On the other hand, we have (see [5] Theorem 3.7)

Theorem 13. For a symplectic diffeomorphism $\phi : U \to U'$, there exists a Weyl diffeomorphism $\Phi : W_U \to W_{U'}$ whose induced map is ϕ.

Contact algebra. Let $\{U_\alpha, z_\alpha)\}_{\alpha\in\Lambda}$ be a symplectic atlas of (M, ω). We consider to glue trivial bundles $\{W_{U_\alpha}\}_\alpha$ by Weyl diffeomorphisms.

We remark here the structure of a Weyl diffeomorphism $\Phi : W_U \to W_{U'}$ is roughly $\Phi = d\phi \times \exp\left(\frac{1}{\nu}\mathrm{ad}(f^\#)\right)$, where $d\phi$ is the tangent map of the induced symplectic diffeomorphism and $f^\#$ is a certain Weyl function on U. So in order to adjust Weyl diffeomorphisms to satisfy transition function rule of the bundles, we need further idea to control the center. The idea is what we call a contact Lie algebra, which can be regared as a quantized contact structure in some sense.

We introduce a degree d of the elements of the Weyl algebra W by setting

$$d(\nu) = 2, \quad d(X_j) = d(Y_j) = 1, \quad j = 1, 2, \ldots, n.$$

Then the degree is well-defined for the Weyl algebra since it is of no contradiction with the relation $[X_j, Y_k]_* = \nu\delta_{jk}$. For example we see $d(\nu X_1 \hat{*} Y_2) = 4$, etc.,

Using the degree we can introduce a derivation of the Weyl algebra $D : W \to W$ such that

$$D(\nu) = 2\nu, \quad D(X_j) = \nu X_j,\ D(Y_j) = \nu Y_j, \quad j = 1, 2, \ldots, n.$$

Notice that the center of W is equal to $\mathbb{C}[[\nu]]$ and D does not vainish on the center.

We introduce an element τ such that

$$[\tau, F] = -[F, \tau] = D(F), \quad \forall F \in W.$$

We consider a direct sum

$$\mathfrak{g} = \mathbb{C}\tau \oplus W,$$

and then we can define a Lie algebra, called a Contact Lie algebra, by putting

$$[\lambda\tau + a, \mu\tau + b] = \lambda[\tau, a] + \mu[a, \tau] + [a, b]_*, \quad \lambda, \mu \in \mathbb{C},\ a, b \in W.$$

Contact Weyl vector field. Let U be an open subset of $\mathbf{R}^{2n}$ and consider a trivial bundle $\mathfrak{g}_U = U \times \mathfrak{g}$. We denote by $\Gamma(\mathfrak{g}_U)$ the set of all smooth sections of $\mathfrak{g}_U$. Then $\Gamma(\mathfrak{g}_U)$ forms a Lie algebra by the pointwise multiplication and becomes a complete topological Lie algebra under smooth topology. We consider a section $\tau_U \in \Gamma(\mathfrak{g}_U)$ such that

$$\tau_U(z) = \tau + \sum_{i,j=1}^{2n} \omega_{ij} z^i Z^j, \quad (z \in U),$$

where ω_{ij} is the coefficient of the canonical symplectic 2-form. It is easy to see that the derivation $[\tau,\ ]$ satisfies $[\tau, F] = 2\nu^2\partial_\nu F + \nu\sum_{k=1}^{2n} Z^k\partial_{Z^k}F$ and notice $[\sum_{ij}\omega_{ij}z^iZ^j, F] =$

$\nu\sum_k z^k\partial_{Z^k}F$, $F\in W$ for each $z\in U$. Then we see easily the fiberwise derivation $[\tau_{_U},\ ]$ acts on $\Gamma(W_U)$ in the form

$$[\tau_{_U}(z),F(z)]=2\nu^2\partial_\nu F(z)+\nu\sum_{k=1}^{2n}z^{k\#}\partial_{Z^k}F(z),\quad F\in\Gamma(W_U).$$

The identity $\partial_{Z^k}f^\#=(\partial_{z^k}f)^\#$ yields

$$[\tau_{_U}(z),f^\#(z)]=2\nu^2\partial_\nu f^\#(z)+\nu\sum_{k=1}^{2n}\left(z^k\partial_{z^k}f\right)^\#(z),\quad f^\#\in\mathcal{F}(W_U).$$

Here we use the identity $z^{k\#}\left(\partial_{z^k}f\right)^\#=\left(z^k\partial_{z^k}f\right)^\#$. In fact, using the definition of $\widehat{*}$ we calculate $z^{k\#}\widehat{*}g^\#=z^{k\#}g^\#+\frac{\nu}{2}\sum_m\Lambda^{km}\left(\partial_{z^m}g\right)^\#$, where Λ^{km} is the coefficient of the canonical Poisson bracket and we see easily

$$z^{k\#}\widehat{*}g^\#=\left(z^k*_0 g\right)^\#=\left(z^kg\right)^\#+\frac{\nu}{2}\sum_m\Lambda^{km}\left(\partial_{z^m}g\right)^\#,$$

which shows $z^{k\#}g^\#=\left(z^kg\right)^\#$ for $\forall g^\#\in\mathcal{F}(W_U)$. Thus, we have

Lemma 14.

$$[\tau_{_U}(z),f^\#(z)]=2\nu^2\partial_\nu f^\#(z)+\nu\left(Ef\right)^\#(z),\quad f^\#\in\mathcal{F}(W_U),$$

where $E=\sum_{k=1}^{2n}z^k\partial_{z^k}$ is the Euler vector field.

The deviation $[\tau_{_U},\ ]$ is called a *contac Weyl vector field*.

Contact Weyl diffeomorphism. Now we extend the Weyl diffeomorphism to a contact Weyl diffeomorphism. We consider a locally trivial Lie alegebra bundle $\mathfrak{g}_{_U}=U\times\mathfrak{g}$. We define

Definition 15. A Lie algebra bundle isomorphism $\Psi:\mathfrak{g}_{_U}\to\mathfrak{g}_{_{U'}}$ is called a contac Weyl diffeomorphism when it satisfies

- *i)* $\Psi^*\tau_{_{U'}}=\tau_{_U}+f^\#,\quad\exists f\in C^\infty(U)[[\nu]]$.
- *ii) The restriction to the Weyl algebra bundle $\Psi|W_U$ induces a Weyl diffeomorphism $\Psi|W_U:W_U\to W_{U'}$.*

We have (see [5] Theorem 4.7.)

Proposition 16. *i) For a Weyl diffeomorphism $\Phi:W_U\to W_{U'}$, there exists a contact Weyl diffeomorphism $\Psi:\mathfrak{g}_{_U}\to\mathfrak{g}_{_{U'}}$ such that the restriction $\Psi|W_U$ is equal to Φ.*

ii) For contact Weyl diffeomorphisms $\Psi,\Psi':\mathfrak{g}_{_U}\to\mathfrak{g}_{_{U'}}$ having the same restriction $\Psi|W_U=\Psi'|W_U$ there extists uniquely a central element $c=c_0+c_1\nu+\cdots+c_k\nu^k+\cdots$ such that $\Psi'=\Psi\exp\left(ad(\frac{1}{\nu}c)\right)$.

Especially, a contact Weyl diffeomorphism which induces an identity Weyl diffeomorphism is uniquely written as $\Psi=\exp\left(ad(\frac{1}{\nu}c)\right)$, $c\in\mathbb{C}[[\nu]]$.

Using contact Weyl diffeomorphisms to control central elements, and we can glue a system of trivial bundles $\{\mathfrak{g}_{U_\alpha}\}_{\alpha\in\Lambda}$ by Weyl diffeomorphisms and we obtain (see [8] Theorem A)

Theorem 17. For any symplectic manifold, there exist a Weyl manifold W_M and a contact Weyl manifold $\mathfrak{g}_M$ containing W_M as a subbundle.

Remark 18. In [8]*, the parity condition or hermitian property was posed on Weyl diffeomorphisms and contact Weyl diffeomorphisms. Then we had a little stronger results in* [8] *that the central element c is in $\mathbb{C}[[\nu^2]]$ in Proposition* 16 *ii). And the moduli space of Weyl manifolds is $H_2(M)[[\nu^2]]$ which is $H_2(M)[[\nu]]$ without the parity condition. The parity condition is optional.*

4. – Deformation quantization

Using a Weyl diffeomorphism we can obtain a deformation quantization of the symplectic manifold in the following way.

By a transition functions, that is gluing Weyl diffeomorphisms, the local Weyl functions are also glued together to give a global Weyl functions. We denote this algebra by $(\mathcal{F}(W_M), \hat{*})$ called a space of Weyl functions on M.

Theorem 19 [5]. *We have a $\mathbb{C}[[\nu]]$-linear map $\sigma : C^\infty(M)[[\nu]] \to \mathcal{F}(W_M)$.*

By means of this linear isomorphism we can define an asociative product on $C^\infty(M)[[\nu]]$ by

$$f * g = \sigma^{-1}(\sigma(f)\hat{*}\sigma(g)).$$

By expanding this product in the power of ν we see that the prouct $*$ is a deformation quantization of (M, ω).

5. – Canonical connection on $\mathfrak{g}_M$

The connection ∂ is defined as a twisted exterior derivation. For this, we introduce a tensor product bundle $\Lambda_M \otimes \mathfrak{g}_M$, where Λ_M is the exterior algebra bundle over M.

Poincaré-Cartan class. In [9, 8] we proposed a Čech cohomology class $c_M \in H_2(M)[[\nu]]$ which characterize an equivalent class of Weyl manifolds over M and shown that c_M is equal to the class determined by the curvature of Fedosov connection [8]. The class c_M is called a *Poincaré-Cartan class*.

By de Rham theorem, we take a closed 2-form $\Omega_M \in \Lambda^2(M)$ on M which gives the Poincaré-Cartan class: $[\Omega_M] = c_M$.

For each coordinate neighborhood (U_α, z_α) where U_α is homeomorphic to $2n$-open disk, we take a 1-form $\xi_\alpha \in \Lambda^1(U_\alpha)[[\nu]]$ such that $d\,\xi_\alpha = \Omega_\alpha$.

Local expression. Now we consider derivations on $\Lambda_{U_\alpha} \otimes \mathfrak{g}_{U_\alpha}$. Let δ_α be a fiberwise derivation defined by

$$\delta_\alpha = \mathrm{ad}\big(\tfrac{1}{\nu}\sum_{ij} dz_\alpha^i \omega_{ij} Z^j\big) : \Lambda^p_{U_\alpha} \otimes \mathfrak{g}_{U_\alpha} \to \Lambda^{p+1}_{U_\alpha} \otimes \mathfrak{g}_{U_\alpha},$$

for each $p = 0, 1, \ldots, 2n$.

For each U_α we set $\tau_\alpha = \tau_{U_\alpha}$ for simplicity.

Now we set

$$\widehat{\xi}_\alpha = \mathrm{ad}\left(\tfrac{1}{\nu}\,\xi_\alpha\right)\tau_\alpha \in \Lambda^1_{U_\alpha}[[\nu]].$$

Then we have

Lemma 20. There exists a unique 1-form $\kappa_\alpha \in \Lambda^1_{U_\alpha}[[\nu]]$ *such that*

$$\left(d - \delta_\alpha + ad\left(\frac{1}{\nu}\kappa_\alpha\right)\right)\tau_\alpha = \widehat{\xi}_\alpha.$$

We set a derivation ∂_α acting on $\Lambda_{U_\alpha} \otimes \mathfrak{g}_{U_\alpha}$ by $\partial_\alpha = \left(d - \delta_\alpha + ad\left(\frac{1}{\nu}\kappa_\alpha\right)\right)$. Let $\{\Psi_{\alpha\beta}\}$ be a system of contact Weyl diffeomorphisms which glue the local trivializations $\{\mathfrak{g}_{U_\alpha}\}$. Then we have

Proposition 21. $\Psi^*_{\alpha\beta}\partial_\alpha = \partial_\beta$

which defines a connection ∂ on $\mathfrak{g}_M$.

The following show that the canonical connection ∂ has Ω_M as a curvature form and is an extension of Fedosov connection.

Theorem 22. *i)* $\partial^2|_{W_{U_\alpha}} = 0$.

ii) A section $F \in \Gamma(W_M)$ *satisfies* $\partial F = 0$ *if and only if* $F \in \mathcal{F}(W_M)$.

iii) $\partial^2 = ad\left(\frac{1}{\nu}\Omega_M\right)$, *i.e., the curvature form of* ∂ *is equal to* Ω_M.

As a corollary of the theorem, we have

Corollary 23. *i) The restriction* $\partial|_{W_M}$ *is a Fedosov connection.*

ii) The curvature of the connection ∂ *is given by the adjoint of a 2-form which is a curvature form of Fedosov connection, which can be obtained by* $\partial^2\tau_\alpha$.

iii) The Poincaré-Cartan class is equal to the cohomology class of Fedosov connection; $[\Omega_M] = c(W_M)$.

* * *

The author expresses his thanks to Professor Alexander Vinogradov for interesting comment. This work was supported by JSPS KAKENHI Grant number 15K04856.

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