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Exotic matter driving wormholes

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Summary. — We develop an iterative approach to span the whole set of exotic matter models able to drive a traversable wormhole. The method reduces the Einstein equation to an infinite set of algebraic conditions and easily allows the implementation of further conditions linking the stress-energy tensor components among each other, like symmetry conditions or equations of state.

1. – Introduction

Wormhole solutions to Einstein equations can be seen as handles connecting two asymptotically flat regions of the space-time manifold, *i.e.* a short-cut or a bridge linking together two distant portions of the universe. An example of wormhole is given by the Einstein-Rosen bridge [1]. However, such a bridge is not traversable because it is part of the Schwartzschild vacuum solution [2]. In 1988, Thorne and Morris [3] introduced the concept of *traversable* wormhole and found the condition the bridge must satisfy in order to be safely crossed by hypothetical travellers. In ref. [4] we considered a static spherically symmetric traversable wormhole. Following refs. [2, 3, 5-7] the most general metric without horizons in the Schwarzschild coordinate system (t, r, θ, ϕ) is $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -e^{2\Phi(r)}\mathrm{d}t^2 + \left(1 - \frac{b(r)}{r}\right)^{-1} \mathrm{d}r^2 + r^2\,(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$, where $b(r)$ and $\Phi(r)$ are the so-called *shape* and *red-shift* functions, respectively. A traversable wormhole has to satisfy the *flaring-out condition* $b'(r) < \frac{b(r)}{r}$, near the wormhole "*throat*" (the region where the radial coordinate takes its minimum value r_0 [2, 3]). Furthermore, r and $b(r)$ must have the same value at the throat, namely $b(r_0) = r_0$ [3]. Hence at r_0 one gets $b'(r_0) < 1$. According to ref. [3], Birkhoff's theorem forbids a traversable wormhole with the desired symmetries to occur in vacuum. Thus, we consider Einstein equations in presence of some matter with non-trivial stress-energy tensor [2, 3]. We adopt the *proper reference frame* $(\hat{t}, \hat{r}, \hat{\theta}, \hat{\phi})$ introduced in refs. [3,8]. The metric $g_{\mu\nu}$ locally reduces to the Minkowski's one and the Einstein tensor $G_{\mu\nu}$ is diagonal. The Einstein equations imply that the stress-energy tensor $T_{\mu\nu}$ is diagonal as well and its non-vanishing components are $T_{\hat{t}\hat{t}} \equiv \rho(r)$, $T_{\hat{r}\hat{r}} \equiv -\tau(r)$, $T_{\hat{\theta}\hat{\theta}} = T_{\hat{\phi}\hat{\phi}} \equiv p(r)$. The flaring-out condition implies that $\tau(r) > \rho(r)$ so the considered material violates the Null Energy Condition (NEC) [2,3,7,9].

2. – Exotic matter in a neighbourhood of the wormhole throat

In the chosen frame $G_{\hat{\mu}\hat{\nu}}$ and $T_{\hat{\mu}\hat{\nu}}$ are both diagonal and Einstein equations read $G_{\hat{\mu}\hat{\nu}} = \frac{1}{m_P^2} T_{\hat{\mu}\hat{\nu}}$, where m_P is the Planck mass. We recast all quantities involved in terms of dimensionless ones, namely $\bar{r} \equiv m_P\, r$, $\bar{b} \equiv m_P\, b$, $\bar{\Phi} = \Phi$, $\bar{\rho} \equiv \rho/m_P^4$, $\bar{\tau} \equiv \tau/m_P^4$ and $\bar{p} \equiv p/m_P^4$. We aim to characterise the whole set of exotic matter models able to drive a traversable wormhole. To achieve such a goal we expand around the throat all quantities entering in equations. A set of algebraic conditions is obtained order by order in the relative distance from the throat $(\bar{r} - \bar{r}_0)/\bar{r}_0$. Due to flaring-out, one has to fix $\bar{b}_0 = 1$ and $\bar{b}_1 < 1$. The zero-th order equations are $\{\bar{b}_1 = \bar{\rho}_0 < 1,\ \bar{\tau}_0 = 1,\ \bar{\Phi}_1 = \frac{2\bar{p}_0}{1-\bar{\rho}_0} - 1\}$, and first order conditions are $\{\bar{b}_2 = 2\bar{\rho}_0 + \bar{\rho}_1,\ \bar{\tau}_1 = -(4\bar{p}_0 + \bar{\rho}_0 + 1),\ \bar{\Phi}_2 = \frac{6\bar{p}_0 + 2\bar{p}_1}{3(1-\bar{\rho}_0)} + \frac{2\bar{p}_0(2\bar{\rho}_0 + \bar{\rho}_1)}{3(1-\bar{\rho}_0)^2} - \frac{1}{3}(\frac{2\bar{p}_0}{1-\bar{\rho}_0} - 1)(\frac{4\bar{p}_0}{1-\bar{\rho}_0} + 1)\}$. For a more detailed list see [4]. All the above equations hold for an arbitrary throat size r_0. For each order five new coefficients enter into the game, whereas three new conditions must be satisfied, hence one can fix two of such quantities to obtain all the others. The set of equations further simplifies if one considers additional constraints $F(\bar{\rho}, \bar{p}, \bar{\tau}) = 0$ linking the components of the stress-energy among each other (for example symmetry conditions or equations of state). By expanding F we get additional conditions $\{F_0 = F(\bar{\rho}_0, \bar{p}_0, \bar{\tau}_0) = 0,\ F_1 = \frac{1}{\bar{r}_0^2}[\frac{\partial F}{\partial \bar{\rho}}|_{\bar{r}_0}\bar{\rho}_1 + \frac{\partial F}{\partial \bar{p}}|_{\bar{r}_0}\bar{p}_1 + \frac{\partial F}{\partial \bar{\tau}}|_{\bar{r}_0}\bar{\tau}_1] = 0, \ldots\}$. Far away from the wormhole throat, the metric must approach the Schwartzschild vacuum. We bound the exotic matter into a finite spherical region of radius $\bar{R}$ surrounded by a shell of different material [3, 2, 8]. However, the size of this finite region is not completely arbitrary. Indeed, we require the shape function $\bar{b}(\bar{r})$ to have only one point where $\bar{b}(\bar{r}) = \bar{r}$, which by construction is already verified for $\bar{r} = \bar{r}_0$. If a radius $\bar{r}^* > \bar{r}_0$ such that $\bar{b}(\bar{r}^*) = \bar{r}^*$ exists then the exotic material must be confined in a smaller region (this provides the upper bound $\bar{R} < \bar{r}^*$). Outside the shell $\bar{b}(\bar{r})$ is constant. We related this value with the dimensionless mass $\bar{M}$ featuring in the outer Schwartzchild solution [4]. Such a mass $\bar{M} = \int_{\bar{r}_0}^{\bar{R}} \bar{\rho}(\bar{r})\bar{r}^2 \mathrm{d}\bar{r} + \bar{r}_0$ is a function of the stress-energy and of the radii $\bar{r}_0$ and $\bar{R}$.

3. – Conclusions

By performing series expansion of functions $b(r), \phi(r), \rho(r), p(r), \tau(r)$ we reduce the Einstein equations to an infinite set of algebraic conditions. For each order we have five new coefficients and three new conditions to be satisfied. This allows one to fix arbitrarily two quantities out of five to get all the others by iterating the procedure. In order to reduce the d.o.f. of the solution, it is possible to implement an extra constraints $F(\bar{\rho}, \bar{p}, \bar{\tau}) = 0$. In ref. [4], we applied the method to some relevant examples characterised by constant mass-energy density and isotropy, or described by quintessence-like equations of state [10]. The approach provides up to a certain order in the relative distance from the throat the radial profile of all quantities of the described wormhole source.

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REFERENCES

[1] Einstein A. and Rosen N., *Phys. Rev.*, **48** (1935) 73.
[2] Visser M., *Lorentzian Wormholes. From Einstein to Hawking* (American Institute of Physics) 1996.
[3] Morris M. S. and Thorne K. S., *Am. J. Phys.*, **56** (1988) 395.
[4] Chianese M., Di Grezia E., Manfredonia M. and Miele G., *Eur. Phys. J. Plus*, **132** (2017) 164.
[5] Morris M. S., Thorne K. S. and Yurtsever U., *Phys. Rev. Lett.*, **61** (1988) 1446.
[6] Visser M., *Phys. Rev. D*, **46** (1992) 2445.
[7] Lobo F. S. N., *Int. J. Mod. Phys. D*, **25** (2016) 1630017.
[8] Misner C., Thorne K. and Wheeler J., *Gravitation* (W. H. Freeman) 1973.
[9] Visser M. and Barcelo C., *Energy conditions and their cosmological implications*, in *Proceedings, 3rd International Conference on Particle Physics and the Early Universe (COSMO 1999): Trieste, Italy, September 27-October 3, 1999* (World Scientific) 2000, pp. 98–112.
[10] Lobo F. S. N., *Phys. Rev. D*, **71** (2005) 084011.