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Two-loop corrections to electron-muon scattering at NNLO in QED

A. Primo(*)

Department of Physics, University of Zürich - CH-8057 Zürich, Switzerland

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Summary. — In this paper, we discuss the evaluation of the two-loop virtual corrections to the electron-muon scattering at next-to-next-to-leading order (NNLO) in QED. These radiative corrections are relevant for the analysis of the MUonE experiment, recently proposed at CERN. MUonE aims at the high-precision determination of the QED running coupling constant in the space-like region from the measurement of the differential cross section of the elastic scattering of high-energy muons on atomic electrons. The precise theoretical knowledge of QED corrections to the process will allow to extract from the experimental data the full hadronic contribution to the running coupling constant. This will provide a new and independent determination of the leading-order hadronic correction to the muon $g - 2$. As an essential step towards the full theoretical prediction, we present the decomposition of the NNLO virtual amplitude in terms of basic integrals and the analytical evaluation of the latter by means of differential equations and the Magnus exponential method. We work in the massless electron approximation, while we retain full dependence on the muon mass. The presented results are also relevant for crossing-related processes, such as di-muon production at e^+e^- colliders, as well as for the QCD corrections to top-pair production at hadron colliders.

1. – Introduction

In this contribution, we report on the progress in the study of the next-to-next-to-leading-order (NNLO) virtual corrections to the elastic scattering of muons and electrons in Quantum Electrodynamics (QED), which we have presented in [1, 2]. In particular, we discuss the analytic evaluation of the complete set of —planar and non-planar— two-loop four-point master integrals (MIs) that arise from the relevant Feynman diagrams

(*) Based on work in collaboration with S. di Vita, S. Laporta, P. Mastrolia, M. Passera, J. Ronca, U. Schubert and W. J. Torres Bobadilla.

of order α^3 [3]. The present results make the analytic evaluation of the NNLO virtual amplitude for μe scattering within reach, for instance, in the framework of the adaptive integrand decomposition [4], as discussed in [5].

Together with the full NLO corrections [6] and the estimation of the NNLO hadronic effects [7,8], the NNLO QED contribution to $\mu e \to \mu e$ will play a crucial role in the interpretation of future high-precision experiments, like MUonE, recently proposed at CERN, which aims at measuring the cross section of the elastic scattering of high-energy muons on atomic electrons as a function of the negative squared momentum transfer [9-11] with statistical and systematic uncertainties of the order of 10 ppm. This measurement will provide the running of the effective electromagnetic coupling in the space-like region and, as a result, an independent determination of the leading hadronic contribution to the muon $g-2$, which is expected to be competitive with the precision of the traditional dispersive calculations (see [12] for a review). For a detailed description of the experiment we refer to [13].

Given the hierarchy between the electron mass m_e and the muon mass m, $m_e/m \sim 5 \cdot 10^{-3}$, we work in the approximation $m_e = 0$. We use integration-by-parts identities (IBPs) [14-16] in order to identify a set MIs, which we analytically computed by means of the differential equations (DEQs) method [17-19]. The solution of the system of DEQs for small values of the dimensional regulator $\epsilon = (4-d)/2$ in terms of generalised polylogarithms (GPLs) [20, 21] is facilitated by the identification of a canonical basis of MIs, in the sense of [22]. Such basis is determined through a well-consolidated procedure, based on the Magnus exponential [23, 24], which has been successfully applied in the context of multi-loop integrals involving several kinematic scales [25-28].

The same set of MIs required for μe scattering will allow the determination of the NNLO QED corrections to the crossing-related process $e^+e^- \to \mu^+\mu^-$. The latter will be relevant for some of the high-precision studies planned at upcoming low-energy e^+e^- experiments, like Belle-II and VEPP-2000, which will target the forward-backward asymmetry [29] in muon pair production and the determination of the $R(s)$-ratio [30, 31]. In addition, due to the $m_e = 0$ approximation, the hereby computed MIs constitute a subset of those needed for the complete QCD corrections to the $t\bar{t}$-pair production at hadron colliders [32-34] and, together with the recent result of [28, 35], allow the fully analytic evaluation of the two-loop amplitude in the light-quark-annihilation channel. The remaining part of the presentation is organised as follows: in sect. **2** we classify the relevant four-point integral families, in sect. **3** we discuss the solution of the DEQs for the associated MIs and in sect. **4** we address the numerical checks and the analytic continuation of the result. We present our conclusions in sect. **5**.

2. – Integral families

We study the box-type two-loop corrections to the scattering process $\mu^+(p_1) + e^-(p_2) \to e^-(p_3) + \mu^+(p_4)$, with kinematics $p_1^2 = p_4^2 = m^2$, $p_2^2 = p_3^2 = 0$. The representative Feynman diagrams of the 10 relevant four-point topologies T_i are shown in fig. 1. All these topologies can be organised into 3 distinct integral families F_i, $i = 1, 2, 3$, of the type

$$I^{[d]}(a_1, \ldots, a_9) \equiv \frac{1}{\Gamma_\epsilon^2} \left(\frac{m^2}{\mu^2}\right)^\epsilon \int \frac{\mathrm{d}^d k_1}{i\pi^{d/2}} \frac{\mathrm{d}^d k_2}{i\pi^{d/2}} \frac{1}{D_1^{a_1} \ldots D_9^{a_9}}, \quad a_i \in \mathbb{Z}, \tag{1}$$

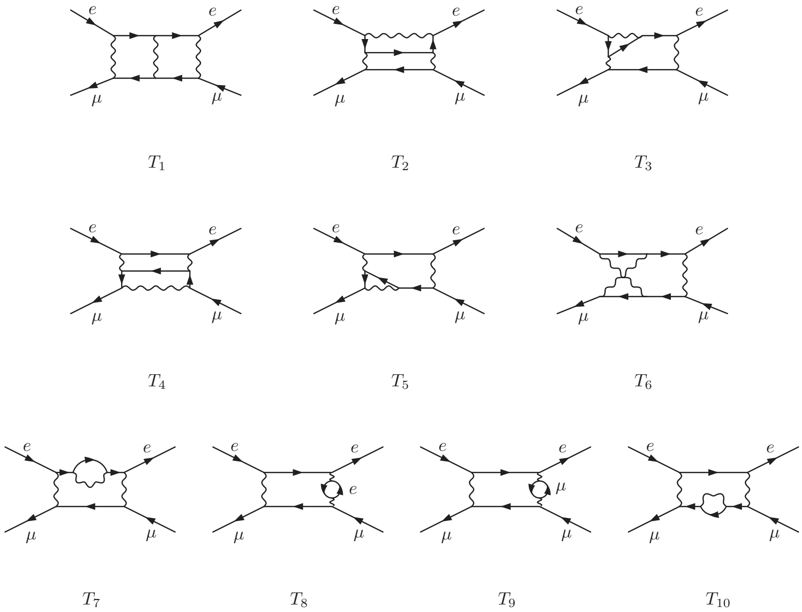

Fig. 1. – Two-loop integral families for $\mu e \to \mu e$.

where k_1 and k_2 are the loop momenta and $D_i = q_i^2 - m_i^2$ correspond to inverse scalar propagators. We use IBPs to reduce all the integrals belonging to each of the families F_i to a finite set of MIs:

1) the first (planar) integral family F_1, which includes the topologies T_1, T_2, T_3, T_7 and T_8, is reduced to 34 MIs;

2) the second (planar) integral family F_2, which includes the topologies T_4, T_5, T_9 and T_{10}, is reduced to 42 MIs;

3) the third integral family F_3, which corresponds to the single non-planar topology T_6, is reduced to 44 MIs;

We refer to [1,2] for the explicit definition of the integral families F_i and the corresponding bases of MIs chosen for the computation.

3. – Differential equations

In order to determine the analytic expression of MIs identified through IBPs reduction, we solve their DEQs in the independent kinematic invariants $s = (p_1+p_2)^2$, $t = (p_2-p_3)^2$ and m^2. These three dimensionful parameters can be combined into two independent dimensionless variables x_1 and x_2 that parametrise (up to a trivial scaling factor) the full dependence of the MIs on the kinematics. A suitable choice of the differentiation variables x_i can greatly simplify the determination of MIs, as it can be used to remove non-rational terms in the DEQs. For the integrals under study, we obtain a completely rational system of DEQs by choosing $t = -m^2(1-x_2)^2/x_2$ and by parametrising s as $s = -m^2 x_1$ for F_1, F_2 and as $(m^2-s-t)/(s-m^2) = -x_1^2/x_2$ for as F_3.

3˙1. *Canonical systems of differential equations.* – For each integral family, we derived canonical systems of DEQs according to the Magnus algorithm described in [23, 24]. It is worth noticing that, for the case of canonical DEQs, the basis change obtained through the Magnus exponential is equivalent to the Wronskian matrix (formed by the solutions of the associated homogenous equations), which has been shown to allow extensions to systems of DEQs involving elliptic solutions [36, 37]. Once combined into a single total differential, the canonical DEQs in x_1 and x_2 read

$$\mathrm{d}\mathbf{I}(\epsilon, x_1, x_2) = \epsilon \mathrm{d}A(x_1, x_2)\mathbf{I}(\epsilon, x_1, x_2), \tag{2}$$

where $\mathbf{I}$ is a vector that collects the MIs of a given integral family and

$$\mathrm{d}A(x_1, x_2) = \sum_i M_i \mathrm{dlog}(\eta_i(x_1, x_2)), \tag{3}$$

with M_i being rational constant matrices. The arguments η_i of the dlog-form define the so-called *alphabet* of the DEQs. All MIs are normalised in such a way that they are finite in the $\epsilon \to 0$ limit, so that $\mathbf{I}(x_1, x_2)$ admits a Taylor expansion in ϵ,

$$\mathbf{I}(\epsilon, x_1, x_2) = \sum_{n\geq 0} \epsilon^n \left(\sum_{i=0}^{n} \Delta^{(n-i)}(x_1, x_2; x_{1,0}, x_{2,0})\mathbf{I}^{(i)}(x_{1,0}, x_{2,0}) \right), \tag{4}$$

where $\mathbf{I}^{(i)}(x_{1,0}, x_{2,0})$ is a vector of boundary constants and $\Delta^{(k)}$ the weight-k operator

$$\Delta^{(k)}(x_1, x_2; x_{1,0}, x_{2,0}) = \int_\gamma \underbrace{\mathrm{d}A \ldots \mathrm{d}A}_{\text{k times}}, \qquad \Delta^{(0)}(x_1, x_2; x_{1,0}, x_{2,0}) = 1, \tag{5}$$

which iterates k-ordered integrations of the matrix-valued 1-form $\mathrm{d}A$ along a path γ in the x_1x_2-plane. In the presence of a rational alphabet with algebraic roots, the iterated integrals (5) can be directly expressed in terms of GPLs. For all the three integral families, the solutions of the system of DEQs is derived in the unphysical region $s < 0 \wedge t < 0$, where potential imaginary parts of the solution can originate from the integration constants only.

3˙2. *Boundary constants.* – The iterative integration of eq. (2) leads to a general solution of the DEQs in terms of GPLs that depends on arbitrary integration constants. The latter are determined by imposing a suitable set of boundary conditions. On a limited number of cases, it was possible to fix the integration constants by exploiting the knowledge of the analytic expression of the MIs in special kinematic configurations, derived from either direct computation or from the solution of auxiliary, simpler, system of DEQs. For the majority of the integrals, however, it was sufficient to impose the regularity of the solution at pseudo-thresholds of the DEQs in order to completely determine the boundary constants, at every order in ϵ, as a transcendentally uniform combination of the constants π, ζ_k and $\log 2$.

4. – Numerical evaluation and analytic continuation

As a validation of our result, we numerically evaluated the analytic expression of all MIs by means of the `GiNac` library [38] and checked them against independent numerical calculations. The check was performed in the Euclidean region $s < 0 \wedge t < 0$ where all planar integrals are real valued. All the MIs, with the only exception of the four-point non-planar integrals that belong to F_3, have been cross-checked against the results provided by the code `SecDec` [39]. For the challenging numerical evaluation of the non-planar integrals we resorted on a different strategy: after identifying an alternative set of quasi finite MIs in $d = 6$, we evaluated their Feynman parametric representation by carrying out as many analytic integrations as possible and by numerically evaluating the leftover integrals by means of Gauss quadrature. Dimension-shifting identities and IBPs establish analytical relations between this set of integrals and the original MIs computed around $d = 4$ and allow the cross-checks of the numerical result of the chosen basis of integrals. Once the analytic expression of the MIs in the Euclidean region $s < 0 \wedge t < 0$ is established, all physically relevant kinematic regions can be reached through a consistent analytic continuation procedure. We addressed the problem in [28], where we provided a practical prescription for the analytic continuation of the result not only to the kinematic region relevant for μe-scattering but also to the $t\bar{t}$-production kinematics.

5. – Conclusions

In this paper, we reported on the analytic evaluation of the two-loop master integrals needed for the NNLO virtual corrections to μe elastic scattering in QED that has been presented in [1, 2]. These results pave the way to the evaluation of the NNLO virtual amplitude, which is currently under investigation [3,5]. The two-loop amplitude will constitute an essential part of the theoretical input required by the ambitious experimental goal of the MUonE project, which will determine the leading hadronic contribution to the muon $g - 2$ by measuring the scattering of high-energy muons on atomic electrons.

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