

**A possible novel approach to the Riemann Hypothesis (RH)  
by means of Generalized Zeta Functions  
related to an infinite set of numerical sequences  
generated by the Sieve of Eratosthenes**

Bruno Aiazzi, Stefano Baronti, Leonardo Santurri, Massimo Selva<sup>(1, \*)</sup>

(<sup>1</sup>) Istituto di Fisica Applicata “Nello Carrara” del CNR (IFAC-CNR)

(\*) B.Aiazzi@ifac.cnr.it

## Contents

<b>1</b>	<b>Abstract</b>	<b>3</b>
<b>2</b>	<b>Introduction</b>	<b>3</b>
<b>3</b>	<b>The Riemann's investigation in the 1859 paper</b>	<b>5</b>
3.1	Preliminar remarks . . . . .	5
3.2	The extension of $\zeta(s)$ to the complex plane . . . . .	6
3.2.1	Noticeable values of $\zeta(s)$ in some real points . . . . .	7
3.2.2	The functional equation for the zeta function $\zeta(s)$ . . . . .	7
3.3	The function $\xi(s)$ . . . . .	8
3.3.1	The product formula for the function $\xi(s)$ . . . . .	8
3.3.2	The roots $\rho$ of the function $\xi(s)$ . . . . .	9
3.4	The Riemann hypothesis (RH) . . . . .	10
3.4.1	Computational evaluations of the RH . . . . .	10
3.4.2	The Riemann-Siegel formula . . . . .	11
3.5	The connection between $\zeta(s)$ and the primes . . . . .	12
3.5.1	The step function $J(x)$ . . . . .	12
3.5.2	Fourier inversion of $J(x)$ and the Perron formula . . . . .	13
3.5.3	The main term of $J(x)$ . . . . .	14
3.5.4	The term of $J(x)$ involving the roots $\rho$ . . . . .	14
3.5.5	The remaining terms of $J(x)$ . . . . .	15
3.5.6	The final expression for $\pi(x)$ and its connection with the non trivial zeros of $\zeta(s)$ . . . . .	16
3.5.7	The Chebyshev's weighted prime counting function . . . . .	18
3.5.8	The von Mangoldt formula . . . . .	19
<b>4</b>	<b>A possible strategy for approaching the RH</b>	<b>20</b>
<b>5</b>	<b>Conclusions and future developments</b>	<b>21</b>
	<b>References</b>	<b>21</b>

# 1 Abstract

Riemann Hypothesis (RH) is one of the most important unresolved problems in mathematics. It is based on the observation that a link exists between the positions on the critical strip  $0 < \Re(s) < 1$ ,  $s \in \mathbb{C}$ , of the non-trivial zeroes in the Riemann’s zeta function and the distribution of the primes in the succession of naturals. Riemann suggested that all of these infinitely many zeroes lie on the critical line  $\Re(s) = \frac{1}{2}$ , in such a way the distribution of the primes becomes the most regular possible. However, an important observation is that the succession of primes is the final result of an infinite number of steps of the well-known Sieve of Eratosthenes. This Report provides an overview of the Riemann’s analysis, and finally gives some hints about a possible approach to the RH, which exploits the partial numerical successions provided by the Sieve procedure steps.

# 2 Introduction

In a recent published paper [1], Aiazzi et al. showed that the Prime Characteristic Function  $\xi_p(n)$ , which is a binary sequence defined as

$$\xi_p(n) = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

can be written as the union of an infinite number of subsets  $\psi(k, n)$ , each of them ranging from  $p(k)^2$  to  $p(k+1)^2$ , where  $p(k)$  denotes the  $k$ -th prime and  $p(0) = 1$ . The Prime Characteristic Function is strictly related to the Prime Number Function  $\pi(x)$ , which denotes the quantity of prime numbers less or equal to  $x \in \mathbb{R}$ , that is, if  $p$  denotes the generic prime number, and  $N$  is the greatest integer less than  $x$ ,

$$\pi(x) = \pi(N) = \sum_{n=1}^N \xi_p(n) = \sum_{p \leq N} 1. \tag{2}$$

From ancient times, it is known that  $\pi(x)$  is unlimited. The Prime Number Theorem (P.N.T.), which was conjectured by Gauss [2] and Legendre [3] and successively demonstrated by Hadamard [4] and de la Vallée Poussin [5], gives the trend followed by  $\pi(x)$  when it approaches infinity, that is,

$$\pi(x) \sim \frac{x}{\log x} \tag{3}$$

where  $\log(x)$  denotes the natural logarithm of  $x$  and the symbol  $\sim$  indicates that the ratio between  $\pi(x)$  and  $x/\log x$  approaches 1 as  $x \rightarrow \infty$ . Such an estimation was improved by Gauss himself by considering the logarithmic integral function  $\text{Li}(x)$ , whose first term of the series expansion is just  $x/\log x$ , that is,

$$\pi(x) \sim \text{Li}(x) \tag{4}$$

with

$$\text{Li}(x) = \int_2^x \frac{dt}{\log t}. \tag{5}$$

Table 1 reports the estimate  $\text{Li}(x)$  compared with the former one  $x/\log x$ . The difference  $\text{Li}(x) - \pi(x)$  is also reported. It is apparent that  $\text{Li}(x) - \pi(x)$  is always greater than the prime number function  $\pi(x)$ . However, Littlewood demonstrated in 1914 [6] that the difference  $\text{Li}(x) - \pi(x)$ , even if it is positive up to extremely large values, changes its sign in an infinite set of integers. In the following years, the minimum value of this set has been lowered, starting from S. Skewes in 1933. For example, C. Bays and R. H. Hudson found the value of  $1.39 \cdot 10^{316}$  in 2000 [7].

From [1], the Prime Characteristic Function can be then expressed as

$$\xi_p(n) = \bigcup_{k=0}^{\infty} [\psi(k, n) \wedge \kappa(k, n)] = \begin{cases} \psi(0, n) & \text{if } p(0)^2 \leq n < p(1)^2 \\ \psi(1, n) & \text{if } p(1)^2 \leq n < p(2)^2 \\ \dots & \dots \\ \psi(k, n) & \text{if } p(k)^2 \leq n < p(k+1)^2 \\ \dots & \dots \end{cases} \tag{6}$$

where  $\psi(k, n)$  is a binary sequence obtained by setting to zero, in an initial infinite sequence of 1s, all the bits in the positions of the multiples of the primes until  $p(k)$ , the primes themselves included, and  $\kappa(k, n)$  is a binary

**Table 1:** Prime number function  $\pi(x)$  compared with the estimations  $x/\log x$  and  $\text{Li}(x)$  for  $x = 10^n$  and  $n \leq 27$ . The difference  $\text{Li}(x) - \pi(x)$  is also reported. The estimations  $x/\log x$  and  $\text{Li}(x)$  are rounded to the nearest integer.

$x = 10^n$	$\pi(x)$	$x/\log x$	$\text{Li}(x)$	$\text{Li}(x) - \pi(x)$
$10^1$	4	4	6	2
$10^2$	25	22	30	5
$10^3$	168	145	178	10
$10^4$	1 229	1 086	1 246	17
$10^5$	9 592	8 686	9 630	38
$10^6$	78 498	72 382	78 628	130
$10^7$	664 579	620 421	664 918	339
$10^8$	5 761 455	5 428 681	5 762 209	754
$10^9$	50 847 534	48 254 942	50 849 235	1 701
$10^{10}$	455 052 511	434 294 482	455 055 615	3 104
$10^{11}$	4 118 054 813	3 948 131 654	4 118 066 401	11 588
$10^{12}$	37 607 912 018	36 191 206 825	37 607 950 281	38 263
$10^{13}$	346 065 536 839	334 072 678 387	346 065 645 810	108 971
$10^{14}$	3 204 941 750 802	3 102 103 442 166	3 204 942 065 692	314 890
$10^{15}$	29 844 570 422 669	28 952 965 460 217	29 844 571 475 288	1 052 619
$10^{16}$	279 238 341 033 925	271 434 051 189 532	279 238 344 248 557	3 214 632
$10^{17}$	2 623 557 157 654 233	2 554 673 422 960 952	2 623 557 165 610 822	7 956 589
$10^{18}$	24 739 954 287 740 860	24 127 471 216 847 324	24 739 954 309 690 415	21 949 555
$10^{19}$	234 057 667 276 344 607	228 576 043 106 974 647	234 057 667 376 222 382	99 877 775
$10^{20}$	2 220 819 602 560 918 840	2 171 472 409 516 259 139	2 220 819 602 783 663 484	222 744 644
$10^{21}$	21 127 269 486 018 731 928	20 680 689 614 440 563 221	21 127 269 486 616 126 182	597 394 254
$10^{22}$	201 467 286 689 315 906 290	197 406 582 683 296 285 296	201 467 286 691 248 261 498	1 932 355 208
$10^{23}$	1 925 320 391 606 803 968 923	1 888 236 877 840 225 337 614	1 925 320 391 614 054 155 139	7 250 186 216
$10^{24}$	18 435 599 767 349 200 867 866	18 095 603 412 635 492 818 797	18 435 599 767 366 347 775 144	17 146 907 278
$10^{25}$	176 846 309 399 143 769 411 680	173 717 792 761 300 731 060 452	176 846 309 399 198 930 392 619	55 160 980 939
$10^{26}$	1 699 246 750 872 437 141 327 603	1 670 363 391 935 583 952 504 342	1 699 246 750 872 593 033 005 724	155 891 678 121
$10^{27}$	16 352 460 426 841 680 446 427 399	16 084 980 811 231 549 172 264 034	16 352 460 426 842 189 113 085 405	508 666 658 006

sequence that is 1 between  $p(k)^2$  and  $p(k + 1)^2$ , with  $p(k + 1)^2$  excluded, and zero otherwise. It follows that each  $\psi(k, n)$  is given by

$$\psi(k, n) = \begin{cases} 0 & \text{if } p(i)|n \text{ for some } i = 1, \dots, k \\ 1 & \text{otherwise} \end{cases} \tag{7}$$

being  $\psi(k, n)$  periodic with a period  $T(k)$  given by the product of all the primes until  $p(k)$ , that is,

$$\psi(k, n + T(k)) = \psi(k, n) \quad \forall n \tag{8}$$

with the period  $T(k)$  equal to

$$T(k) = \prod_{i=1}^k p(i) \tag{9}$$

and  $\psi(k, n) \wedge \kappa(k, n)$  is given by

$$\psi(k, n) \wedge \kappa(k, n) = \begin{cases} \psi(k, n) & \text{if } p(k)^2 \leq n < p(k + 1)^2 \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

**Table 2:** Periods  $T(k)$  of the sequences  $\psi(k, n)$ , for primes  $p(k) \leq p(7)$ , in comparison with the sizes  $S(k) = p(k + 1)^2 - p(k)^2$  of the intervals  $I(k)$ . The pseudo-prime  $p(0) = 1$  is put in brackets.

$k$	$p(k)$	$p(k + 1)$	$I(k)$	$S(k)$	$T(k)$	$S(k)/T(k)$
(0)	(1)	2	[1, 4)	3	1	3.000000
1	2	3	[4, 9)	5	2	2.500000
2	3	5	[9, 25)	16	6	2.666667
3	5	7	[25, 49)	24	30	0.800000
4	7	11	[49, 121)	72	210	0.342857
5	11	13	[121, 169)	48	2 310	0.020779
6	13	17	[169, 289)	120	30 030	0.003996
7	17	19	[289, 361)	72	510 510	0.000141

Table 2 reports the periods  $T(k)$  of the sequences  $\psi(k, n)$ ,  $k = 0, \dots, 7$ , in comparison with the sizes  $S(k)$  of the intervals  $I(k)$ , where subsets of each  $\psi(k, n)$  are recognizable in the Prime Characteristic Function  $\xi_p(n)$ . Noticeably, the periods  $T(k)$  grow much more faster than  $S(k)$ , as indicated by the decreasing trend of the ratios

$S(k)/T(k)$ , whose scores are approaching zero very rapidly. This explains because the regularities found in each interval  $I(k)$  are so hardly recognizable at a first sight. In summary, from Equation (6), the Prime Characteristic Function  $\xi_p(n)$  can be viewed as a limit sequence of a succession of sequences  $\xi(k, n)$ ,  $k = 1, \dots, \infty$  defined as

$$\xi(k, n) = \bigcup_{i=0}^{k-1} [\psi(i, n) \wedge \kappa(i, n)] + \psi(k, n) = \begin{cases} \psi(0, n) & \text{if } p(0)^2 \leq n < p(1)^2 \\ \psi(1, n) & \text{if } p(1)^2 \leq n < p(2)^2 \\ \dots & \\ \psi(k-1, n) & \text{if } p(k-1)^2 \leq n < p(k)^2 \\ \psi(k, n) & \text{if } n \geq p(k)^2. \end{cases} \quad (11)$$

in such a way the sequences  $\xi(k, n)$  are the partial binary sequences corresponding to run the Sieve of Eratosthenes procedure until the deletion of the multiples of the prime  $p(k)$ . By exploiting the number of 1 values in each period  $T(k)$ , it is possible to estimate the number of primes in each interval  $I(k)$ . Such an estimation is equivalent to the one obtained with the logarithmic integral function  $\text{Li}(x)$  (5).

A remarkable consequence of Equation (6) is that the trend of the primes is somewhat predictable until infinity, in the sense that their distribution is given by pieces of periodic binary sequences, so that abrupt changes seem to be unsuitable. Consequently, the regularities of the prime distribution showed in Equation (6) can be viewed as a reinforcement of the famous Riemann Hypothesis (RH), because a direct consequence of the RH is that the distribution of the primes is maximally regular. Actually, Riemann was able to find a link between the prime distribution and the non trivial zeros of his zeta function, whose positions in the complex plane is the object of the RH. In the following of this Report, the RH will be analyzed from the definition of the Riemann’s zeta function, with the end of obtaining possible links between the RH and other Generalized Zeta Functions [8], which can be defined by starting from the numerical successions described in [1]. Consequently, some hints for approaching the related unsolved problems will be proposed.

### 3 The Riemann’s investigation in the 1859 paper

#### 3.1 Preliminar remarks

In his famous eight-page paper of 1859 “On the number of primes less than a given magnitude” [9], Riemann was able to find an expression for  $\pi(x)$  as the sum of a number of terms. In the first of them, the main term is represented by the  $\text{li}(x)$  function, which is the  $\text{Li}(x)$  function computed starting from  $x = 0$ , whereas the other terms include the same  $\text{li}(x)$  computed in the square root of  $x$ , cubic root of  $x$ , and so on. Such an estimation is an improvement of the classical Gauss’ estimation given by the pure  $\text{Li}(x)$  function, and approximates very well the staircase of the primes, by means of a smoothing trend. Moreover, this estimation can be corrected until to fit the exact prime distribution, by considering a number of infinite series, where the  $\text{li}(x)$  function is computed on the infinite non trivial zeros of the Riemann’s zeta function. The convergence of such correction terms, if computed in less restrictive conditions than the ones required by the RH, is important for the demonstration of the P.N.T., which was proved only successively. In any case, the contributions of Riemann to the theory of prime numbers were manifold and highly innovative, by including the extension to the Euler’s zeta function  $\zeta$  to complex values  $s \in \mathbb{C}$ , the study of complex zeros of  $\zeta(s)$ , the development of the marvellous expression for  $\pi(x)$  through ingenious Fourier and Möbius inversions, and so on.

The Riemann’s zeta function  $\zeta(s)$ , which is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (12)$$

was initially defined by Euler only for positive integer values  $l$  of the variable  $s \in \mathbb{C}$ , that is,  $s = l + i0$ . In this case, the infinite summation converges for  $l > 1$  and diverges for  $l = 1$ . For these values, the summation belongs to the generalized harmonic series, and, if  $l = 1$ , it becomes the classical harmonic series, that is, the series of the reciprocals of the natural numbers, which is divergent with logarithmic trend, i.e.,

$$\sum_{n=1}^{\infty} \frac{1}{n} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k} = \lim_{n \rightarrow \infty} [\log n + \gamma + o(1)] = \log n + \gamma \quad (13)$$

where  $\gamma$  is the *Eulero-Mascheroni constant*, whose approximate value is  $\gamma = 0.5772$ . Euler also demonstrated the existence of a link between  $\zeta(s)$  and the succession of primes, that is,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left( \frac{1}{1 - p^{-s}} \right). \quad (14)$$

Given the divergence of (14) for  $s = 1$  (if  $s$  is real, the null imaginary part will be omitted in the following), we obtain from (14) another proof of the infinitude of the primes. A further proof can be achieved by observing that the series of the reciprocal of primes is also divergent, that is,  $\sum_{p < x} 1/p \sim \log(\log x)$  for  $x \rightarrow \infty$ .

### 3.2 The extension of $\zeta(s)$ to the complex plane

In his paper [9], Riemann started by considering the Euler product formula (14). This formula can be found by expanding each factor  $\frac{1}{(1-p^{-s})}$  as

$$\frac{1}{(1-p^{-s})} = 1 + \frac{1}{p^s} + \frac{1}{(p^2)^s} + \frac{1}{(p^3)^s} + \dots \quad (15)$$

and observing that the product of the infinite sums at the right side of (15) are terms as  $\frac{1}{(p_1^{n_1} p_2^{n_2} \dots p_r^{n_r})^s}$ , where  $p_1, p_2, \dots, p_r$  are distinct primes and  $n_1, n_2, \dots, n_r$  are natural numbers, and then using the fundamental theorem of arithmetic to obtain each term of the summation  $\sum 1/n^s$ .

The essential contribution of Riemann was the extension of the relation (14) to complex values. At a first sight, the two sides of (14) converge only for complex values  $s$  such that  $\Re(s) > 1$ , but Riemann showed that the function  $\zeta(s)$  defined by (14) could be made valid for all values of  $s$ , except for a simple pole at  $s = 1$ . In order to reach this goal, Riemann utilized the factorial function  $\Gamma(s)$  defined as

$$\Gamma(s) = \int_0^\infty e^{-x} x^s dx. \quad (16)$$

Such a function is defined for all complex values  $s$  such that  $\Re(s) > -1$  and is an extension to complex values to the factorial function, being  $\Gamma(s) = s!$  if  $s$  is a natural number. An extension of  $\Gamma(s)$  to all the complex values other than negative integers can be made by considering that  $\Gamma(s)$  can be represented by the limit

$$\Gamma(s) = \lim_{N \rightarrow \infty} \frac{N!}{(s+1)(s+2)\dots(s+N)} (N+1)^s. \quad (17)$$

Equation (17) gives a proof that  $\Gamma(s)$  is an analytical function of the complex variable  $s$  having simple poles at  $s = -1, -2, -3, \dots$  and no zeros. This example shows that a complex function defined in a limited set of complex values can be extended to all the complex plane (possibly apart from some points) by finding an alternative definition. This was the way followed by Riemann to extend the zeta function  $\zeta(s)$ .

The Riemann' scope was to extend the function  $\sum n^{-s}$  to all the complex plane. If we write Equation (16) for  $\Gamma(s-1)$  and operate the substitution  $t = nx$ , we obtain

$$\Gamma(s-1) = \int_0^\infty e^{-t} t^{s-1} dt = \int_0^\infty e^{-nx} (nx)^{s-1} d(nx) = \int_0^\infty e^{-nx} n^s x^{s-1} dx \quad (18)$$

and consequently

$$\int_0^\infty e^{-nx} x^{s-1} dx = \frac{\Gamma(s-1)}{n^s}, \quad \Re(s) > 0, \quad n = 1, 2, 3, \dots \quad (19)$$

If we sum over the index  $n$  and given  $\sum_{n=1}^\infty r^{-n} = (r-1)^{-1}$ , we obtain

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1} dx = \Gamma(s-1) \sum_{n=1}^\infty \frac{1}{n^s} = \Gamma(s-1) \zeta(s), \quad \Re(s) > 1 \quad (20)$$

and then

$$\zeta(s) = \frac{1}{\Gamma(s-1)} \cdot \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx, \quad \Re(s) > 1 \quad (21)$$

At this point, Riemann utilized the contour integral in the complex plane

$$\int_{+\infty}^{+\infty} \frac{(-x)^s}{x(e^x - 1)} dx = \int_\gamma \frac{(-x)^s}{x(e^x - 1)} dx \quad (22)$$

where the path of integration  $\gamma$  starts at  $+\infty$ , moves to the left down the positive real axis, circles the origin in the positive (counterclockwise) direction, and returns up the positive real axis to  $+\infty$ , so obtaining [10]

$$\int_{+\infty}^{+\infty} \frac{(-x)^s}{x(e^x - 1)} dx = (e^{i\pi s} - e^{-i\pi s}) \int_0^{+\infty} \frac{x^{s-1}}{e^x - 1} dx. \quad (23)$$

The integrals in the Equations (23) and (21) are the same. Therefore, we obtain

$$\int_{+\infty}^{+\infty} \frac{(-x)^s}{x(e^x - 1)} dx = (e^{i\pi s} - e^{-i\pi s}) \Gamma(s-1) \zeta(s) = 2i \sin(\pi s) \Gamma(s-1) \zeta(s) = 2i \sin(\pi s) \Gamma(s-1) \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{24}$$

and, by using the relation  $\sin(\pi s) = \frac{\pi s}{\Gamma(s)\Gamma(-s)}$ , we obtain the expression for  $\zeta(s)$  that is valid for all  $s \in \mathbb{C}$

$$\zeta(s) = \frac{\Gamma(-s)}{2\pi i} \int_{+\infty}^{+\infty} \frac{(-x)^s}{x(e^x - 1)} dx. \tag{25}$$

### 3.2.1 Noticeable values of $\zeta(s)$ in some real points

Equation (25) defines a function  $\zeta(s)$  that is the same as  $\sum n^{-s}$  for real values  $s$  with  $\Re(s) > 1$ , that is, it is the same as the function  $\zeta(s)$  defined in Equation (14). Moreover, due to the convergence of the integral for all values of  $s$  (because  $e^x$  grows much faster than  $x^s$  for  $x \rightarrow \infty$ ), Equation (25) defines a function that is analytic for all values of  $s \in \mathbb{C}$ , with the possible exceptions of  $s = 1, 2, 3, \dots$ , where  $\Gamma(-s)$  has simple poles. However, the function  $\zeta(s)$  defined in Equation (14) has no pole at  $s = 2, 3, 4, \dots$ . Consequently, the integral must have some zeros that cancel the poles of  $\Gamma(-s)$  in these points, and the function  $\zeta(s)$  of Equation (25) has only a simple pole at  $s = 1$ . This extension shows that the infinite series  $\sum n^{-s}$  defines the function  $\zeta(s)$  only for a piece of its domain, and in the rest of the domain the function  $\zeta(s)$  has to be defined otherwise through Equation (25).

When  $s = -n$ ,  $n = 0, 1, 2, \dots$ , the function  $\zeta(-n)$  can be computed by using the expansion formula

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!} \tag{26}$$

which is valid for  $|x| < 2\pi$ , because we have singularities for  $x = \pm 2\pi i$ , where  $e^{\pm 2\pi i} = 1$ , whilst  $x = 0$  is an eliminable singularity. In Equation (26),  $B_n$  are the Bernoulli numbers, which are zero for  $n = 2k + 1$ ,  $k = 1, 2, 3, \dots$ , and so on. From Equations (25) and (26), we obtain

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}, \quad n = 0, 1, 2, \dots \tag{27}$$

As a consequence, the zeta function is zero for each negative even integer, that is,  $\zeta(-2) = \zeta(-4) = \dots = 0$ . These points are called *trivial zeros* of  $\zeta(s)$ , and they can also be obtained from the nullification of the  $\sin$  function in Equation (24). Equation (27) also gives the value of  $\zeta(s)$  in the origin, that is,  $\zeta(0) = -1/2$ , and in the odd negative integers (for example,  $\zeta(-1) = -1/12$ ,  $\zeta(-3) = 1/120$ , and so on), even if their computing requires an exponential computational complexity. Concerning positive integer numbers, that is,  $s = k$ ,  $k > 1$ , the infinite summation  $\sum n^{-k} \rightarrow 1$  if  $k \rightarrow \infty$ . In the case of positive odd values, no exhaustive relation exists in order to evaluate  $\zeta(2n + 1)$ , apart from particular cases as  $\zeta(3) = 1.202$ , which is called *Apery’s constant*. Conversely, in the case of positive even values, the zeta function can be expressed in a closed form, as found by Euler [11], that is,  $\zeta(2) = \pi^2/6$ ,  $\zeta(4) = \pi^4/90$ , and, in general

$$\zeta(2n) = \frac{(2\pi)^{2n} (-1)^{n+1} B_{2n}}{2 \cdot (2n)!}, \quad n = 1, 2, \dots \tag{28}$$

### 3.2.2 The functional equation for the zeta function $\zeta(s)$

It is not easy to compute all the values of the zeta function  $\zeta(s)$  by starting from the integral formula (25). For this reason, Riemann modified Equation (25), in order to obtain a *functional equation of the zeta function*  $\zeta(s)$ . This new formulation allowed the computation of the values of  $\zeta(s)$  in the negative midplane of the complex plane by starting from the values in the positive midplane, and vice versa. Riemann found two different formulations for this functional equation. In his first derivation, by starting from Equation (25) with the integral evaluated in the reverse path of integration, he obtained the following relation

$$\int_{+\infty}^{+\infty} \frac{(-x)^s}{x(e^x - 1)} dx = 4\pi i e^{i\pi s} \sin\left(\frac{\pi s}{2}\right) (2\pi)^{s-1} \zeta(1-s) \tag{29}$$

and, consequently,

$$\zeta(s) = 2\Gamma(-s) (2\pi)^{s-1} \sin\left(\frac{\pi s}{2}\right) \zeta(1-s). \tag{30}$$

From Equation (30), the analyticity of the zeta function in all the complex plane is evidenced, being analytic all the functions in the right side of (30). If  $s$  assumes real values,  $\zeta(s)$  is real and approaches the value 1 for  $s \rightarrow +\infty$ , starting from the value  $+\infty$  for  $s = 1$ . For  $s < 1$ , Equation (30) shows that  $\zeta(s)$  increases from  $-\infty$  to  $-1/2$  at  $s = 0$ , and then it assumes alternatively positive and negative values that grow more and more in modulus, with the zeros placed in the negative even integers. The positions of these zeros is evident, by considering that the  $\sin$  function in the right side of Equation (30) becomes zero for  $s = -2n$  and all the other quantities are limited. It can be noticed that Equation (30) cannot be used for computing the zeta function on the critical line  $\Re(s) = \frac{1}{2}$ , because the relation would become an identity. In order to calculate  $\zeta\left(\frac{1}{2} + it\right)$ , which is fundamental to obtain the nontrivial zeros of  $\zeta(s)$ , we have to consider other formulas, for example the integral (22).

In the second formulation of the functional equation of his zeta function, Riemann utilized the Jacobi's function by starting from the simple pole ( $+\infty$ ) at  $s = 1$ ,

$$\theta(x) = \sum_{n=-\infty}^{+\infty} e^{-n^2\pi x} \quad (31)$$

which is absolutely convergent for each given  $x > 0$ . By considering that we have  $\theta(x) = \frac{1}{\sqrt{x}} \theta\left(\frac{1}{x}\right)$ , and  $\Gamma\left(\frac{s}{2} - 1\right) = n^s \pi^{\frac{s}{2}} \int_0^\infty x^{\frac{s}{2}-1} e^{-n^2\pi x}$ , we obtain

$$\Gamma\left(\frac{s}{2} - 1\right) \pi^{-s/2} \zeta(s) = \Gamma\left(\frac{1-s}{2} - 1\right) \pi^{-(1-s)/2} \zeta(1-s), \quad (32)$$

so that the function in the left side of (32) is unchanged by the substitution  $s = 1 - s$ .

### 3.3 The function $\xi(s)$

Once the Riemann's zeta function  $\zeta(s)$  has been introduced, we can investigate the positions of its non trivial zeros, which are the object of the RH, and the possibility to obtain a formula for the prime number function  $\pi(x)$ . To this end, Riemann defined a new function  $\xi(s)$  as

$$\xi(s) = \Gamma\left(\frac{s}{2}\right) (s-1) \pi^{-s/2} \zeta(s) \quad (33)$$

which is an analytical function that is defined for all values of  $s$  (that is, it has no poles), and such that the functional equation of the zeta function (30) simply becomes  $\xi(s) = \xi(1-s)$ . The introduction of  $\xi(s)$  was made by Riemann in order to obtain a completely analytical function whose zeros are only the nontrivial zeros of  $\zeta(s)$ . This fact is evident by Equation (33) for  $\Re(s) \geq \frac{1}{2}$ , because the zero of  $s-1$  deletes the pole of  $\zeta(s)$ , whilst the function  $\Gamma(s)$  and the term  $\pi^{-s/2}$  never become null, but it can be demonstrated also for  $\Re(s) < \frac{1}{2}$ . In fact, the left side of Equation (32) can be multiply by  $s(s-1)/2$ , in order to eliminate the poles of  $\Gamma(s)$  (in  $s = 0$ ), and of  $\zeta(s)$  (in  $s = 1$ ). The pole of  $\zeta(s)$  in  $s = 1$  is clearly shown by considering the following formulation due to Euler

$$\zeta(s) = 1 + \frac{1}{s-1} - s \int_1^\infty \frac{\{t\}}{t^{s+1}} dt, \quad \Re(s) > 0. \quad (34)$$

The functional equation  $\xi(s) = \xi(1-s)$  clearly shows the symmetry of  $\xi(s)$  with respect to the critical line  $\Re(s) = \frac{1}{2}$ . Evidently, such a symmetry even holds for the zeros of  $\xi(s)$ , that is, if  $s_0 \in \mathbb{C}$  is a zero for  $\xi(s)$ , also  $1 - s_0$  is a zero for  $\xi(s)$ . Moreover, it can be shown that  $\xi(s)$  (and  $\zeta(s)$ ) is symmetrical with respect to the real axis, that is,  $\xi(s) = \xi(\bar{s})$  [12].

#### 3.3.1 The product formula for the function $\xi(s)$

Riemann also showed that the  $\xi(s)$  function can be expressed as

$$\xi(s) = \sum_{n=0}^{\infty} \left(s - \frac{1}{2}\right) a_n \quad (35)$$

where  $a_n$  are coefficients that do not depend by  $s$ , that is,

$$a_n = 4 \int_1^\infty \frac{d(x^{3/2} \psi'(x))}{dx} x^{-1/4} \frac{\left(\frac{1}{2} \log x\right)^{2n}}{(2n)!} dx \quad (36)$$



and  $\psi(x) = \sum_{n=1}^{+\infty} e^{-n^2\pi x}$  is another Jacobi's function. Equation (35) once remarks the symmetry of  $\xi(s)$  with respect to the critical line  $\Re(s) = \frac{1}{2}$ , as well as its analyticity in all the complex plane, because it is expressed as a power series around  $s_0 = \frac{1}{2}$ . Finally, Riemann computed the function  $\xi(s)$  for  $s = \frac{1}{2} + it$ , that is,

$$\xi\left(\frac{1}{2} + it\right) = 4 \int_1^\infty \frac{d(x^{3/2}\psi'(x))}{dx} x^{-1/4} \cos\left(\frac{t}{2} \log x\right) dx. \tag{37}$$

By referring to Equations (35) and (37), Riemann asserted that the function  $\xi(s)$  can be developed as a power series that is very rapidly convergent. This result was finally demonstrated by Hadamard (1893) [13], by considering the series of logarithms

$$\sum_{\rho} \log\left(1 - \frac{s}{\rho}\right) \tag{38}$$

where  $\rho$  ranges over the roots of  $\xi(s)$ , that is, the solutions of the equation  $\xi(\rho) = 0$ , which are also the non-trivial roots of  $\zeta(s)$ . It can be demonstrated that these zeros are infinitely many. Equation (38) allows to obtain an infinite development of  $\xi(s)$  as a product over its roots, which is called product formula for  $\xi(s)$ , that is,

$$\xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right). \tag{39}$$

This is essential, because *only the non-trivial roots of  $\zeta(s)$  are related to the prime numbers*, so that the trivial roots must not be present in the expansion (39). This expansion is analogous to that of a polynomial of degree  $n$ , which can be expanded as a finite product of its roots. Consequently, Equation (39) states that the function  $\xi(s)$  is like a polynomial of infinite degree. Generally speaking, we can assert that the convergence of the factorization of  $\xi(s)$  depends by the density of its zeros, which cannot be too much dense.

By summarizing, *it can be demonstrated that  $\xi(s)$  has an infinity of zeros  $\rho = \sigma + it$ . These zeros are symmetrical with respect to the real axis and to the line  $\sigma = 1/2$ .*

### 3.3.2 The roots $\rho$ of the function $\xi(s)$

In order to investigate the convergence of the product (39), it is necessary to establish the positions of the roots  $\rho$  of  $\xi(\rho) = 0$ . To this end, Riemann considered the Euler product formula (14), that is,

$$\zeta(s) = \prod_{p \text{ prime}} \left(\frac{1}{1 - p^{-s}}\right) \quad \Re(s) > 1. \tag{40}$$

Equation (40) shows immediately that  $\zeta(s)$  has no zeros in the halfplane  $\Re(s) > 1$ , because a convergent infinite product can be zero only if one of its factors is zero. From Equation (33), it follows that none of the roots  $\rho$  of  $\xi(\rho) = 0$  lie in the halfplane  $\Re(s) > 1$ . In fact, the factors other than  $\zeta(s)$  have only a simple zero at  $s = 1$ . From the functional equation  $\xi(s) = \xi(1 - s)$ , it follows that  $1 - \rho$  is a root if and only if  $\rho$  is. Consequently, none of the roots  $\rho$  lie in the halfplane  $\Re(s) < 0$ . The conclusion is that *all the roots  $\rho$  of  $\xi(s) = 0$  lie in the critical strip  $0 \leq \Re(s) \leq 1$* . Moreover, Hadamard showed that no zero  $\rho$  is present on the line  $\Re(s) = 1$ , and, consequently, on the line  $\Re(s) = 0$ . Therefore, the roots of  $\xi(s) = 0$  (and the non-trivial zeros of  $\zeta(s)$ ) strictly lie in the critical strip  $0 < \Re(s) < 1$ . It has to be noticed that the demonstration that no zeros are present on the line  $\Re(s) = 1$  is essential in order to achieve the P.N.T. [4].

At this point, Riemann gave an estimation of the quantity of roots  $\rho$  in the critical strip whose imaginary parts lie between 0 and  $T$ , that is,

$$N_{\rho}(T) = \frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} \tag{41}$$

with an order of magnitude of the relative error equal to  $1/T$ . To this end, Riemann stated that the number of roots in this region is equal (with the previous relative error) to the integral

$$N_{\rho}(T) = \int_{\gamma} \frac{\xi'(s) ds}{2\pi i \xi(s)} \tag{42}$$

where  $\gamma$  is the boundary of the rectangle  $\{0 \leq \Re(s) \leq 1, 0 \leq \Im(s) \leq T\}$ , but the proof of this statement was not provided by Riemann, which probably took it for granted, but only given by von Mangoldt in 1905.

### 3.4 The Riemann hypothesis (RH)

Once estimated the number of roots  $\rho$  of  $\xi(s)$ , that is,  $\xi(\rho) = 0$ , whose imaginary part lies between 0 and  $T$ , Riemann stated that "it is very probably that all the roots  $\rho$  have a real part equal to  $\Re(\rho) = 1/2$ ". This is equivalent to assume that the relative error in the approximation of the number of zeros of  $\xi(\frac{1}{2} + it)$  for  $0 \leq t \leq T$  by (41) approaches zero as  $T \rightarrow \infty$  [10]. This is the so-called Riemann Hypothesis (RH), but no indication of a possible proof was give by Riemann himself, whose initial attempts were failed. In any case, such a proof was not necessary for his main scope, which consisted in the achievement of a formula for computing the number of primes less than a given magnitude.

Since Riemann, this hypothesis has not yet been proved or disproved, even if some partial results have been obtained. In brief, Hardy [14] proved in 1914 that  $\xi(\frac{1}{2} + it)$  has infinitely many roots for real values of  $t$ . Selberg [15] showed in 1942 that these roots are at least  $KT \log T$  for some positive  $K$  and all large  $T$ . Previously, Bohr and Landau [16] proved in 1914 that the number of complex roots  $t$  of the relation  $\xi(\frac{1}{2} + it) = 0$  in the range  $\{0 \leq \Re(t) \leq T, -\epsilon \leq \Im(t) \leq +\epsilon\}$  is equal, for any  $\epsilon < 0$ , to the quantity (41) with a relative error which approaches zero as  $T \rightarrow \infty$ . Levinson in 1974 showed that at least  $1/3$  of the non-trivial zeros falls on the critical line [17]. Successively, this result has been improved by arriving to more than 40% of the zeros [18].

#### 3.4.1 Computational evaluations of the RH

A computational approach can be made by calculating the first non trivial zeros with some powerful formulas, in order to verify their position on the critical line  $\Re(s) = 1/2$ . Obviously, such a research can only confirmed the RH up a certain limit, but the RH cannot be finally proved. However, remarkable advancements have been done in this field, thanks to a more and more powerful computational capacity due to the availability of modern computers, parallelization of efficient programs, and so on. These improvements as reported in Table 3 [19] <sup>1</sup>

**Table 3:** History of the numerical verification of the RH through the computation of the first  $n_z$  non-trivial zeros  $\rho$  of  $\zeta(1/2 + i\rho)$

Year	$n_z$	Authors
1859	$\geq 2$	B.Riemann
1903	15	J.P.Gram [20]
1916	79	R.J.Backlund [21]
1925	138	J.I.Hutchinson [22]
1935	1 041	E.C.Titchmarsh [23]
1953	1 104	A.M.Turing [24]
1956	15 000	D.H.Lehmer [25]
1956	25 000	D.H.Lehmer [26]
1958	35 337	N.A.Meller [27]
1966	250 000	R.S.Leghman [28]
1968	3 500 000	J.B.Rosser, J.M. Yoke, L.Schoenfeld [29]
1977	40 000 000	R.P.Brent [30]
1979	80 000 001	R.P.Brent [31]
1982	200 000 001	R.P.Brent, J.van de Lune, H.J.J.te Riele, D.T.Winter [32]
1983	300 000 001	J.van de Lune, H.J.J.te Riele [33]
1986	1 500 000 001	J.van de Lune, H.J.J.te Riele, D.T.Winter [34]
2001	10 000 000 000	J.van de Lune (unpublished)
2003	900 000 000 000	S.Wedeniowski [35]
2004	10 000 000 000 000	X.Gourdon [19]

Such computations show that the RH is certainly satisfied until greater and greater values [36]. The first important result was obtained by Gram in a 1903 paper [20], where he approximately computed the first 15 non-trivial zeros, in such a way this list included all the roots up to height 50. To this end, he used a method featuring the Euler-Maclaurin summation [10], which is able to give a satisfactory evaluation the functions  $\zeta(s)$  and  $\Gamma(s)$  until a number of significant digits. The Euler-Maclaurin formula of  $\zeta(s)$ , by starting from the Equation (12), was evaluated at  $s = 1/2 + it$ , so obtaining the exact, but not absolutely converging, series expansion

$$\zeta(s) = \sum_{j=1}^{n-1} j^{-s} + \frac{n^{-s}}{2} + \frac{n^{1-s}}{s-1} + \sum_{k=1}^{\infty} R_{k,n}(s) \tag{43}$$

where the term  $R_{k,n}$  of the remainder is function of the Bernoulli numbers  $B_k$ . Gram showed that the calculation of the zeros for  $0 \leq \Im(s) \leq 50$  requires the evaluation of  $\zeta(s)$  with at least  $n = 20$  terms. In fact, it can be viewed that the number  $n$  of terms required to estimate  $\zeta(s)$  up a reasonable precision linearly grows with  $\Im(s)$ , so that the method based on the Euler-Maclaurin formula, even if simple, can be used only for small imaginary parts. In any case, Equation (43) allows to verify the RH up to a certain height  $T$ , by excluding the

<sup>1</sup>see also <http://numbers.computation.free.fr/Constants/Miscellaneous/zetazeroscompute.html>

existence of other zeros of  $\zeta(s)$  in the range  $0 \leq |\Im(s)| \leq T$ . In particular, the recognition of the zeros in this interval can be done by considering a real function  $Z(t)$  that is defined in such a way  $Z(t) = 0$  if and only if  $\zeta(\frac{1}{2} + it) = 0$ , that is,

$$Z(t) = e^{i\theta(t)} \zeta\left(\frac{1}{2} + it\right) \tag{44}$$

with

$$\theta(t) = \Im \left[ \log \left( \Gamma \left( \frac{1 + 2it}{4} \right) \right) \right] - \frac{t}{2} \log(\pi) \tag{45}$$

Consequently, we can find the critical zeros by investigating when  $Z(t)$  changes its sign [37]. The points  $t$  for which  $Z(t) = 0$  are called *Gram points*  $g_n$ , and checking if  $\zeta(\frac{1}{2} + ig_n) = 0$  at the  $n$ -th Gram point  $g_n$  is called *Gram test*. Hutchinson showed in 1925, by starting from the Gram’s computations, that RH is valid up to  $T = 300$  [22]. Table 4 shows the values of the first  $n = 16$  zeros  $\rho_k, k = 1, \dots, n$  on the critical line.

**Table 4:** The first  $n = 16$  non-trivial zeros  $\rho_k, k = 1, \dots, n$  of  $\zeta(1/2 + i\rho_k)$

Zeros number	Zeros value
$\rho_1$	$1/2 + 14.1347i$
$\rho_2$	$1/2 + 21.0220i$
$\rho_3$	$1/2 + 25.0109i$
$\rho_4$	$1/2 + 30.4249i$
$\rho_5$	$1/2 + 32.9351i$
$\rho_6$	$1/2 + 37.5862i$
$\rho_7$	$1/2 + 40.9187i$
$\rho_8$	$1/2 + 43.3271i$
$\rho_9$	$1/2 + 48.0052i$
$\rho_{10}$	$1/2 + 49.7738i$
$\rho_{11}$	$1/2 + 52.9703i$
$\rho_{12}$	$1/2 + 56.4462i$
$\rho_{13}$	$1/2 + 59.3470i$
$\rho_{14}$	$1/2 + 60.8318i$
$\rho_{15}$	$1/2 + 65.1125i$
$\rho_{16}$	$1/2 + 67.0798i$

### 3.4.2 The Riemann-Siegel formula

As previously stated, the Equation (44) is difficult to be directly utilized for high values of  $t$ . However, by starting from 1932, the computations have been made faster due to an approximation of  $Z(t)$  computed by means of the *Riemann-Siegel formula*. In fact, Siegel in 1932 published a paper [38], where the zeta function was analyzed based on Riemann’s private papers in the archives of the University Library at Göttingen [39]. In fact, the integral in the Equation (25) can be evaluated by utilizing a different integral contour than in [9] and then applying the Cauchy residue theorem [10]. By setting  $\Re(s) = 1/2$ , we obtain a more useful expression for  $Z(t)$ , that is,

$$Z(t) = 2 \sum_{n^2 < \frac{t}{2\pi}} n^{-1/2} \cos(\theta(t) - t \log n) + R(t) \tag{46}$$

where  $Z(t)$  and  $\theta(t)$  are defined in (44) and (45), respectively. By using Equation (46), the efforts of the researchers were directed to bound the remainder term  $R(t)$ , so improving the determination of the zeros, especially with Turing’s method [24]. After Turing, more and more large scale computations were successively performed, especially at the beginning of the 21st century. In the project ZetaGrid, lead by Wedeniwski, more than 10 000 computers in over 70 countries performed a distributed computation based on the software developed by van de Lune, te Riele and Winter [34]. In this way, more than  $9 \cdot 10^{11}$  zeros were found on the critical line and the RH was verified until  $|\Im(s)| < 57\,292\,877\,670.307$  [35].

**Table 5:** Complexity of the methods for evaluating the RH on the critical strip: A) The number of operations needed to verify the RH for the first  $N$  zeros; B) The number of operations needed to compute  $\pi(x)$ ; C) The number of bits of storage needed to compute  $\pi(x)$ .

Type of operation/Algorithm	Euler-Maclaurin	Riemann-Siegel	Odlyzko-Schönhage
A	$O(N^{2+\epsilon})$	$O(N^{3/2+\epsilon})$	$O(N^{1+\epsilon})$
B	$O(x^{2/3+\epsilon})$	$O(x^{3/5+\epsilon})$	$O(x^{1/2+\epsilon})$
C	$O(x^\epsilon)$	$O(x^\epsilon)$	$O(x^{1/4+\epsilon})$

In 2004 Demichel and Gourdon [19] utilized the *Odlyzko-Schönhage method* to verify that the first  $10^{13}$  zeros lie on the critical line. In fact, in 1988 Odlyzko and Schönhage [40] developed a different approach for the evaluation of the zeta function based on the fast Fourier transform. Such an approach does not outperform the Riemann-Siegel method when a single value is evaluated, but it significantly improves the verification time of the RH in case of the first  $N$  zeros are considered [36]. Thanks to this method, Odlyzko has been able to compute some billions of zeros around the very large values of  $10^{22}$  and  $10^{23}$ , in order to achieve the distribution of the zeros at great heights [41]. This experiments was designed by supposing that a deviation from the RH could be quite probable for these large values of  $|\Im(s)|$ , but no deviation was really found [42]. Table 5 shows the number of elementary operations required by the three algorithms for verifying the RH on the critical line.

### 3.5 The connection between $\zeta(s)$ and the primes

#### 3.5.1 The step function $J(x)$

The essence of the relationship between the zeta function  $\zeta(s)$  and the primes is showed by the Euler product formula reported in Equation (14), that is,

$$\zeta(s) = \prod_{p \text{ prime}} \left( \frac{1}{1 - p^{-s}} \right) \quad (\Re(s) > 1). \quad (47)$$

We have seen in Section 3.3 that taking the logarithm of both sides of Equation (47) does not change its convergence. Consequently, we can utilize the series  $\log(1 - x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \dots$ , so obtaining

$$\log \zeta(s) = -\log \sum_p (1 - p^{-s}) = \sum_p p^{-s} + \frac{1}{2} \sum_p p^{-2s} + \frac{1}{3} \sum_p p^{-3s} + \dots = \sum_p \sum_n \left( \frac{1}{n} \right) p^{-ns} \quad (\Re(s) > 1) \quad (48)$$

since the double series is absolutely convergent for  $\Re(s) > 1$ .

The greatly ingenious Riemann's intuition was to put

$$p^{-s} = s \int_p^{+\infty} x^{-s-1} ds, \quad p^{-2s} = s \int_{p^2}^{+\infty} x^{-s-1} ds, \dots \quad (49)$$

as we have, for example,

$$s \int_p^{+\infty} x^{-s-1} ds = s \left( -\frac{1}{s} \right) \left[ \frac{1}{x^s} \right]_p^{+\infty} = -(0 - p^{-s}) = p^{-s}. \quad (50)$$

At this point, Riemann was able to introduce a new step function  $J(x)$ , which is correlated to the step function of the primes  $\pi(x)$  in this way

$$J(x) = \pi(x) + \frac{1}{2} \pi(x^{1/2}) + \frac{1}{3} \pi(x^{1/3}) + \dots + \frac{1}{n} \pi(x^{1/n}) + \dots, \quad (51)$$

that is,  $J(x)$  is a step function starting at 0 for  $x = 1$ , and increasing with a jump of 1 at each prime  $p$ , with a jump of  $\frac{1}{2}$  at each prime square  $p^2$ , with a jump of  $\frac{1}{3}$  at each prime cube  $p^3$ , and so on. In order to better manage this step function, the value of  $J(x)$  at each jump  $x_0$  is defined to be halfway between its old and new values, that is,  $J(x_0) = \frac{1}{2} [J(x_0 - \epsilon) + J(x_0 + \epsilon)]$ . This means that the function  $J(x)$  can be written as

$$J(x) = \frac{1}{2} \left[ \sum_{p^n < x} \frac{1}{n} + \sum_{p^n \leq x} \frac{1}{n} \right] = \begin{cases} 0 & \text{if } 1 \leq x < 2 \\ \frac{1}{2} & \text{if } x = 2 \\ 1 & \text{if } 2 < x < 3 \\ 1 + \frac{1}{2} = \frac{3}{2} & \text{if } x = 3 \\ 2 & \text{if } 3 < x < 4 \\ 2 + \frac{1}{4} = \frac{9}{4} & \text{if } x = 4 \\ 2 + \frac{1}{2} = \frac{5}{2} & \text{if } 4 < x < 5 \\ 3 & \text{if } x = 5 \\ 3 + \frac{1}{2} = \frac{7}{2} & \text{if } 5 < x < 7 \\ \dots & \text{and so on.} \end{cases} \quad (52)$$

By starting from Equations (48) and (50), Riemann found the following marvellous relation

$$\frac{1}{s} \log \zeta(s) = \int_0^{+\infty} J(x) x^{-s-1} dx \tag{53}$$

The great importance of the Equation (53) is that it gives a relationship between the zeta function  $\zeta(s)$  and the step function  $J(x)$ , which is related to the prime number function  $\pi(x)$ . The successive step of the Riemann’s analysis was then its inversion, in order to explicate the dependence of  $J(x)$  with respect to  $\zeta(s)$ .

### 3.5.2 Fourier inversion of $J(x)$ and the Perron formula

In the Fourier analysis, a function  $\phi(x)$  can be expressed as a superposition of complex exponentials (under suitable conditions)

$$\phi(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Phi(\lambda) e^{ix\lambda} d\lambda \tag{54}$$

where the coefficients  $\Phi(\lambda)$  of the expansion (54) can be written as

$$\Phi(\lambda) = \int_{-\infty}^{+\infty} \phi(x) e^{-i\lambda x} dx. \tag{55}$$

In the 1859 analysis, Riemann was able to apply the Fourier inversion to the Equation (53), by putting  $s = a + ib$ , where  $a > 1$ . Then, he put  $\lambda = \log x$  and, consequently,  $x = e^\lambda$  and  $d \log x = dx/x$ , in such a way Equation (53) becomes

$$\frac{\log \zeta(s)}{s} = \int_{-\infty}^{+\infty} J(x) x^{-s} d \log x \tag{56}$$

and then

$$\frac{\log \zeta(a + ib)}{a + ib} = \int_{-\infty}^{+\infty} J(e^\lambda) e^{-(a+ib)\lambda} d\lambda = \frac{1}{2\pi} \int_{-\infty}^{+\infty} 2\pi J(e^\lambda) e^{-(a+ib)\lambda} d\lambda \tag{57}$$

At this point, we can apply the inversion formula (55) to the term  $\phi(\lambda) = 2\pi J(e^\lambda)e^{-a\lambda}$ , so obtaining

$$2\pi J(e^\lambda)e^{-a\lambda} = \int_{-\infty}^{+\infty} \frac{\log \zeta(a + ib)}{a + ib} e^{b\lambda} db \tag{58}$$

from which, by changing the variable again, that is,  $x = e^\lambda$ , we obtain

$$J(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\log \zeta(a + ib)}{a + ib} x^{a+ib} db = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \log \zeta(s) x^s \frac{ds}{s} \quad (a > 1). \tag{59}$$

where the integral in (59) means the limit as  $T \rightarrow \infty$  of the integral itself over the vertical line segment from  $a - iT$  to  $a + iT$ . Riemann did not care about the applicability of the Fourier analysis to the function  $J(e^x)e^{-ax}$ , and stated simply that Equation (59) holds in full generality. However, such a statement can be rigorously proved. Successively, by starting from Equation (59), Riemann utilized an alternative formula for  $\log \zeta(s)$  and consequently he obtained a simpler expression for  $J(x)$ , which is the main result of his investigation.

The alternative formula utilized by Riemann for  $\log \zeta(s)$  is based on the function  $\xi(s)$  defined in Section 3.3, and in particular on the Equations (33) and (39), which are reported here

$$\begin{cases} \xi(s) = \Gamma\left(\frac{s}{2}\right) (s-1) \pi^{-s/2} \zeta(s) \\ \xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \end{cases} \tag{60}$$

By taking the logarithm in Equation (60), we obtain

$$\begin{cases} \log \zeta(s) = \log \xi(s) - \log \Gamma\left(\frac{s}{2}\right) + \frac{s}{2} \log \pi - \log (s-1) \\ \log \xi(s) = \log \xi(0) + \sum_{\rho} \log \left(1 - \frac{s}{\rho}\right) \end{cases} \tag{61}$$

If we put the second relation of (61) into the first one, we have

$$\log \zeta(s) = \log \xi(0) + \sum_{\rho} \log \left(1 - \frac{s}{\rho}\right) - \log \Gamma\left(\frac{s}{2}\right) + \frac{s}{2} \log \pi - \log (s-1). \tag{62}$$

At this point, we can substitute Equation (62) into the integral of (59) and integrating termwise. However, the direct substitution leads to divergent integrals. In order to overcome this issue, Riemann first integrated by parts in the Equation (59) (it can be demonstrated that this is possible), so obtaining

$$J(x) = \frac{1}{2\pi i} \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[ \frac{\log \zeta(s)}{s} \right] x^s ds \quad (a > 1). \tag{63}$$

and successively operating the substitution for  $\log \zeta(s)$ . This operation allows to express  $J(x)$  as the sum of five terms, because the integral of a finite sum is always the sum of the integrals provided that the latter converges. Consequently, the derivation of the formula for  $J(x)$  depends by the evaluation of these five integrals. This can be done by means of the Perron formula [43] that was the key to obtain the Riemann’s final relation, that is,

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{x^s ds}{s} = \begin{cases} 0, & x = (0, 1) \\ \frac{1}{2}, & x = 1 \\ 1, & x > 1 \end{cases} \quad x > 0, a > 0 \tag{64}$$

**3.5.3 The main term of  $J(x)$**

It can be shown that the principal term in the expansion of  $J(x)$  is the term corresponding to  $-\log(s - 1)$  in the expansion (62) of  $\log \zeta(s)$ , that is,

$$\frac{1}{2\pi i} \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[ \frac{\log(s - 1)}{s} \right] x^s ds \quad (a > 1). \tag{65}$$

The outstanding Riemann’s result was that for  $x > 1$  the value of the definite integral of (65) is the logarithm integral  $\text{li}(x)$ , which is equal to the function  $\text{Li}(x)$  in the Equation (5) when the integral starts from 0, that is,

$$\text{li}(x) = \lim_{\epsilon \rightarrow 0} \left[ \int_0^{1-\epsilon} \frac{dt}{\log t} + \int_{1+\epsilon}^x \frac{dt}{\log t} \right]. \tag{66}$$

The demonstration of this results requires many computations of complex analysis. In brief, for  $x < 1$ , Riemann noticed that all the terms of the Equation (62) (except the term  $\log \xi(0)$ ) could be expressed with the function  $F(\beta)$  defined as

$$F(\beta) = \frac{1}{2\pi i} \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[ \frac{\log(\frac{s}{\beta} - 1)}{s} \right] x^s ds \quad (a > 1). \tag{67}$$

In the case of the main term (65),  $F(1)$  is then the desired result. In fact, after a lot of mathematical passages, he found that

$$F(1) = \int_0^{1-\epsilon} \frac{dt}{\log t} + \int_{1-\epsilon}^{1+\epsilon} \frac{t-1}{\log t} \cdot \frac{dt}{t-1} + \int_{1+\epsilon}^x \frac{dt}{\log t} + i\pi \tag{68}$$

where the second integral is on the semicircle in the upper halfplane. Because when  $\epsilon \rightarrow 0$  the ratio  $\frac{t-1}{\log t}$  approaches 1 along this semicircle, it follows that the integral approaches  $\int_{1-\epsilon}^{1+\epsilon} \frac{dt}{t-1} = -i\pi$ . Consequently, the limit for  $\epsilon \rightarrow 0$  of Equation (68) is

$$F(1) = \text{li}(x). \tag{69}$$

**3.5.4 The term of  $J(x)$  involving the roots  $\rho$**

In the previous section, we shown that the main term for the Riemann’s computation of the function  $J(x)$  (which is related to the prime number function  $\pi(x)$ ) is just the logarithmic integral function  $\text{li}(x)$  conjectured by Gauss. For the P.N.T., such a function is the limit for  $x \rightarrow \infty$  of  $\pi(x)$ . Consequently, the contribution to  $J(x)$  (and then to  $\pi(x)$ ) of the other four terms of Equation (62) becomes negligible for  $x \rightarrow \infty$ . In particular, the demonstration of the P.N.T. implies that

$$\pi(x) = \text{li}(x) + O \left\{ x \exp \left[ -c (\log x)^{3/5} (\log \log x)^{-1/5} \right] \right\}, \quad c > 0 \tag{70}$$

However, if the RH holds, the approximation of  $\pi(x)$  by means of  $\text{li}(x)$  will become much more precise, that is, of the order of  $\sqrt{x}$ ,

$$\pi(x) = \text{li}(x) + O \left( x^{1/2} \log x \right) \tag{71}$$

The above inequality holds for the values  $x \leq 10^{27}$  that are reported in Table 1, which is a confirmation that the RH is valid for this low-valued interval of  $\pi(x)$ . In fact, the number of digits changing from  $\pi(x)$  to  $\text{li}(x)$  are more or less an half than the digits of the real number  $x$ . From the previous analysis, it follows that the difference between  $\text{li}(x)$  and  $\pi(x)$  is directly related to the positions of the non-trivial zeros of  $\zeta(s)$  in the critical strip, that is, the approximation given by  $\text{li}(x)$  would be optimal if all the zeros lie on the critical line as conjectured by the RH. The contribution of the non-trivial zeros  $\rho$  to  $J(x)$  (and then to  $\pi(x)$ ) is due to the term  $\sum_{\rho} \log \left(1 - \frac{x}{\rho}\right)$  in the expansion (62) of  $\log \zeta(s)$ , that is, we obtain

$$-\frac{1}{2\pi i} \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[ \frac{\sum_{\rho} \log \left(1 - \frac{x}{\rho}\right)}{s} \right] x^s ds \quad (a > 1). \tag{72}$$

By interchanging the operations of summations over the roots  $\rho$  and integration, the operations of the Section 3.5.3 can be substantially repeated, because the Equation (72) matches the expression (67). In particular, the summation can be decomposed by expliciting the roots  $\rho$  and  $1 - \rho$ , that is,

$$\sum_{\rho} \log \left(1 - \frac{x}{\rho}\right) = \sum_{\Im(\rho) > 0} \left[ \log \left(1 - \frac{x}{\rho}\right) + \log \left(1 - \frac{x}{1-\rho}\right) \right]. \tag{73}$$

Consequently, we have two integrals to be evaluated on complex paths by applying the Residue Theorem, and finally the following result is reached

$$-\frac{1}{2\pi i} \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[ \frac{\sum_{\rho} \log \left(1 - \frac{x}{\rho}\right)}{s} \right] x^s ds = - \sum_{\Im(\rho) > 0} [\text{li}(x^{\rho}) + \text{li}(x^{1-\rho})] \tag{74}$$

in which the logarithmic integral function  $\text{li}(x)$  is evaluated on complex values depending by the non-trivial roots  $\rho$ . As Riemann noticed, the summation (74) is only conditionally convergent, so that it has to be termwise evaluated in the order of increasing  $\Im(\rho)$ , otherwise it can take on any arbitrary real value. The computation of (74) requires that  $\Re(\rho) > 0$ , but Riemann did not show that this is true for all the roots  $\rho$ . This fact was demonstrated by Hadamard [4] together with the condition  $\Re(\rho) < 1$ , which is essential to demonstrate the P.N.T., so that the non-trivial roots  $\rho$  strictly lie in the critical strip  $0 < \Re(\rho) < 1$ .

### 3.5.5 The remaining terms of $J(x)$

Concerning the other three contributions to  $J(x)$  in the Equation (62), the term arising from  $(s/2) \log \pi$  drops out after the division by  $s$  and the differentiation with respect to  $s$ . Instead, the term arising from the constant  $\log \xi(0)$  gives, by computing the complex integral,

$$-\frac{1}{2\pi i} \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[ \frac{\log \xi(0)}{s} \right] x^s ds = -\frac{1}{2\pi i} \frac{\log \xi(0)}{\log x} x^s \Big|_{a-i\infty}^{a+i\infty} + \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\log \xi(0)}{s} x^s ds = \log \xi(0) \tag{75}$$

by applying the Perron formula (64) and considering that  $\frac{\log \xi(0)}{s} x^s \Big|_{a-i\infty}^{a+i\infty} = 0$ .

Moreover,  $\xi(0) = \Gamma(0) \pi^{-0} (0 - 1) \zeta(0) = -\zeta(0) = 1/2 \implies \log \xi(0) = -\log 2$ .

Finally, the last term will be

$$\frac{1}{2\pi i} \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[ \frac{\log \Gamma \left(\frac{s}{2}\right)}{s} \right] x^s ds \quad (a > 1). \tag{76}$$

This term can be evaluated by taking into account that

$$\Gamma(s) = \prod_{n=1}^{\infty} \frac{n^{1-s} (n+1)^s}{s+n} = \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right)^{-1} \left(1 + \frac{1}{n}\right)^s \tag{77}$$

and consequently

$$\log \Gamma \left(\frac{s}{2}\right) = \sum_{n=1}^{\infty} \left[ -\log \left(1 + \frac{s}{2n}\right) + \frac{s}{2} \log \left(1 + \frac{1}{n}\right) \right] \tag{78}$$

By substituting the expression (78) into (76), we obtain

$$\begin{aligned} \frac{d}{ds} \left( \frac{\log \Gamma(s/2)}{s} \right) &= \sum_{n=1}^{\infty} \frac{d}{ds} \left( \frac{-\log(1 + s/2n) + \frac{s}{2} \log(1 + 1/n)}{s} \right) = \\ &= \sum_{n=1}^{\infty} \frac{d}{ds} \left( \frac{-\log(1 + s/2n)}{s} + \frac{1}{2} \log\left(1 + \frac{1}{n}\right) \right) = \sum_{n=1}^{\infty} \frac{d}{ds} \left( \frac{-\log(1 + s/2n)}{s} \right) \end{aligned} \tag{79}$$

because the second term is composed by constant values having null derivative. Consequently, we have to evaluate integrals that are similar to (65) and (72). However, in this case, the term  $\frac{dt}{\log t}$  of the integral in the expression (66) will be multiplied by the decreasing term  $\frac{1}{t(t^2-1)}$ , in such a way the term (76) becomes negligible. In summary, the expression for  $J(x)$  found by Riemann is given by

$$J(x) = \text{li}(x) - \sum_{\Im(\rho) > 0} [\text{li}(x^\rho) + \text{li}(x^{1-\rho})] + \int_x^\infty \frac{dt}{t(t^2-1)\log t} - \log 2 \tag{80}$$

where the first term is the main term given by the logarithmic integral function  $\text{li}(x)$ , the second term is the correction term that is function of the non-trivial zeros  $\rho$  of the zeta function, the third term is negligible, and the fourth term is simply a constant term. Noticeably, the correction term has a minus sign, but since it can be both positive and negative, the main term  $\text{li}(x)$  could be incremented or decremented.

### 3.5.6 The final expression for $\pi(x)$ and its connection with the non trivial zeros of $\zeta(s)$

The last part of the Riemann’s paper [9] was devoted to finally obtain a formula for the prime number function  $\pi(x)$ , by starting from the expression (80) of the intermediate function  $J(x)$ . From Equation (52), we know that the function  $J(x)$  is a step function that increases of 1 at the same points of  $\pi(x)$ , that is, in correspondence with each prime  $p$ , and in addition it increases of  $1/n$  at each prime power  $p^n$ . Consequently,  $J(x)$  increases more rapidly than  $\pi(x)$ , in order to compensate the decreasing trend of the primes. Since the number of prime squares less than  $x$  is obviously equal to the number of primes less than  $x^{1/2}$ , that is,  $\pi(x^{1/2})$ , and in the same way the number of prime  $n$ -th powers  $p^n$  less than  $x$  is  $\pi(x^{1/n})$ , it follows that the relationship between  $J(x)$  and  $\pi(x)$  is given by

$$J(x) = \pi(x) + \frac{1}{2} \pi(x^{1/2}) + \frac{1}{3} \pi(x^{1/3}) + \dots + \frac{1}{n} \pi(x^{1/n}) + \dots \tag{81}$$

Noticeably, the summation in Equation (81) is finite for any given  $x \in \mathbb{R}$ , because  $x^{1/n} < 2$  for  $n$  sufficiently large. In order to obtain the prime number function  $\pi(x)$  as a function of  $J(x)$ , the relation (81) has to be inverted. The inversion of (51) was made by Riemann by considering the Möbius inversion formula. In order to have an idea of this procedure, we progressively eliminate all the terms from the right side of Equation (81), apart from  $\pi(x)$ . The first step is done by computing the term  $\frac{1}{2} J(x^{1/2})$  and subtracting it from Equation (81), that is,

$$\begin{cases} \frac{1}{2} J(x^{1/2}) = \frac{1}{2} \pi(x^{1/2}) + \frac{1}{4} \pi(x^{1/4}) + \frac{1}{6} \pi(x^{1/6}) + \frac{1}{8} \pi(x^{1/8}) + \dots \\ \text{and therefore} \\ J(x) - \frac{1}{2} J(x^{1/2}) = \pi(x) + \frac{1}{3} \pi(x^{1/3}) + \frac{1}{5} \pi(x^{1/5}) + \frac{1}{7} \pi(x^{1/7}) + \frac{1}{9} \pi(x^{1/9}) + \dots \end{cases}$$

The result of the first step is the deletion of the terms where the  $n$  denominator is a multiple of 2. Then, in the second step, we put  $J_2(x) = J(x) - \frac{1}{2} J(x^{1/2})$ , that is, the term at the left side of the second relation, compute  $\frac{1}{3} J_2(x^{1/3})$ , and subtract it from the previous result

$$\begin{cases} \frac{1}{3} J_2(x^{1/3}) = \frac{1}{3} J(x^{1/3}) - \frac{1}{6} J(x^{1/6}) = \\ = \frac{1}{3} \pi(x^{1/3}) + \frac{1}{6} \pi(x^{1/6}) + \frac{1}{9} \pi(x^{1/9}) + \frac{1}{12} \pi(x^{1/12}) + \frac{1}{15} \pi(x^{1/15}) + \frac{1}{18} \pi(x^{1/18}) + \dots + \\ - \frac{1}{6} \pi(x^{1/6}) - \frac{1}{12} \pi(x^{1/12}) - \frac{1}{18} \pi(x^{1/18}) + \dots = \frac{1}{3} \pi(x^{1/3}) + \frac{1}{9} \pi(x^{1/9}) + \frac{1}{15} \pi(x^{1/15}) + \dots \\ \text{and therefore} \\ J(x) - \frac{1}{2} J(x^{1/2}) - \frac{1}{3} J_2(x^{1/3}) + \frac{1}{6} J(x^{1/6}) = \pi(x) + \frac{1}{5} \pi(x^{1/5}) + \frac{1}{7} \pi(x^{1/7}) + \frac{1}{11} \pi(x^{1/11}) + \dots \end{cases}$$

The result of the second step is the deletion of the terms whose denominator is multiple of 3, apart from those already deleted in the first step. At this stage, all the terms whose denominator is both multiple of 2 and 3 have



been cancelled. In the same way as in the previous steps, we now put  $J_3(x) = J(x) - \frac{1}{2} J(x^{1/2}) - \frac{1}{3} J(x^{1/3}) + \frac{1}{6} J(x^{1/6})$ , then consider the next prime  $p = 5$ , and compute  $\frac{1}{5} J_3(x^{1/5})$ , that is,

$$\left\{ \begin{aligned} \frac{1}{5} J_3(x^{1/5}) &= \frac{1}{5} J(x^{1/5}) - \frac{1}{10} J(x^{1/10}) - \frac{1}{15} J(x^{1/15}) + \frac{1}{30} J(x^{1/30}) = \\ &= \frac{1}{5} \pi(x^{1/5}) + \frac{1}{10} \pi(x^{1/10}) + \frac{1}{15} \pi(x^{1/15}) + \frac{1}{20} \pi(x^{1/20}) + \frac{1}{25} \pi(x^{1/25}) + \frac{1}{30} \pi(x^{1/30}) + \frac{1}{35} \pi(x^{1/35}) + \dots + \\ &\quad - \frac{1}{10} \pi(x^{1/10}) - \frac{1}{20} \pi(x^{1/20}) - \frac{1}{30} \pi(x^{1/30}) - \frac{1}{40} \pi(x^{1/40}) - \dots - \frac{1}{15} \pi(x^{1/15}) - \frac{1}{30} \pi(x^{1/30}) - \frac{1}{45} \pi(x^{1/45}) - \dots + \\ &\quad + \frac{1}{30} \pi(x^{1/30}) + \frac{1}{60} \pi(x^{1/360}) + \dots = \frac{1}{5} \pi(x^{1/5}) + \frac{1}{25} \pi(x^{1/25}) + \frac{1}{35} \pi(x^{1/35}) + \dots \end{aligned} \right.$$

and therefore

$$\left\{ \begin{aligned} J(x) - \frac{1}{2} J(x^{1/2}) - \frac{1}{3} J(x^{1/3}) + \frac{1}{6} J(x^{1/6}) - \frac{1}{5} J(x^{1/5}) + \frac{1}{10} J(x^{1/10}) + \frac{1}{15} J(x^{1/15}) - \frac{1}{30} J(x^{1/30}) = \\ = \pi(x) + \frac{1}{7} \pi(x^{1/7}) + \frac{1}{11} \pi(x^{1/11}) + \frac{1}{13} \pi(x^{1/13}) + \dots \end{aligned} \right.$$

After this step, we have deleted all the terms whose denominator includes a factor given by the primes 2, 3 or 5. Then, the procedure goes on by replacing in each step the function  $J(x)$  with  $\frac{1}{p} J(x^{1/p})$ , so that we successively move on the left side all the factors of  $n$  given by the primes already covered, while the other factors of  $n$  lie still on the right side. When the procedure ends, all the terms in the right side of Equation (51) have been deleted, except the term  $\pi(x)$ . Consequently, the final result is

$$\pi(x) = J(x) - \frac{1}{2} J(x^{1/2}) - \frac{1}{3} J(x^{1/3}) - \frac{1}{5} J(x^{1/5}) + \frac{1}{6} J(x^{1/6}) - \frac{1}{7} J(x^{1/7}) + \frac{1}{10} J(x^{1/10}) + \dots \tag{82}$$

where, in the right side, the denominator of each term has only factors that are distinct primes. Moreover, the sign is negative if the denominator is the product of an odd number of distinct primes, where the sign is positive if the product is given by an even number of distinct primes [10].

Equation (82) can be shortened as follows

$$\pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} J(x^{1/n}) \tag{83}$$

where the terms  $J(x^{1/n})$  become null if  $x^{1/n} < 2$ , and  $\mu(n)$  is the Möbius function, which is defined as

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes} \\ 0 & \text{if } p^2 | n \text{ for some prime } p. \end{cases} \tag{84}$$

We can notice that the summation (83) is finite for each fixed value of  $x$ . In fact, for each  $x$ , a positive integer  $n_0$  exists such that we have  $x^{\frac{1}{n_0}} > 2$ , but  $x^{\frac{1}{n}} < 2$  if  $n > n_0$ . As reported in Equation (52), the function  $J(x) = 0$  if  $x < 2$ .

The Möbius function appears many times in this study. For example, this generalization of Equation (83) is valid in the halfplane  $\Re(s) > 1$ , due to the Möbius inversion property

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \left( 1 - \frac{1}{p^s} \right). \tag{85}$$

In order to obtain an exact formulation for the prime number function  $\pi(x)$ , we put in Equation (83) the expression for  $J(x)$  given by Equation (80). If we only consider the main term of (80), we already obtain an estimation of  $\pi(x)$  that is better than (66), that is,

$$\pi(x) \sim \text{li}(x) + \sum_{n=2}^{\infty} \frac{\mu(n)}{n} \text{li}(x^{1/n}) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \text{li}(x^{1/n}) \tag{86}$$

as reported in Table 6.

Evidently, the correction term  $\sum_{n=2}^{\infty} \frac{\mu(n)}{n} \text{li}(x^{1/n})$  decreases the estimation  $\text{li}(x)$ . This is due to the Möbius function  $\mu(n)$ , whose main terms give a negative contribution. Consequently, in the case of the  $\text{li}(x)$  function overestimates the prime number function  $\pi(x)$ , as usually happens, this correction removes such an overestimation. This is evident by considering the scores of Table 6, where the Riemann's estimation fluctuates around the real values of  $\pi(x)$ . As a result, the plot of the approximation (86) does not lie above the step function representing  $\pi(x)$ , but it appears to be a smoothing function around the staircase of the primes. In order to obtain the exact

**Table 6:** Errors of the approximations (86) and (66) of the prime number function  $\pi(x)$ .

$x$	Riemann's error	Gauss' error
1 000 000	30	130
2 000 000	-9	122
3 000 000	0	155
4 000 000	33	206
5 000 000	-64	125
6 000 000	24	228
7 000 000	-38	179
8 000 000	-6	223
9 000 000	-53	187
10 000 000	88	339

reconstruction of  $\pi(x)$ , we must also take into account the term driven by the non-trivial zeros  $\rho$  of the  $\zeta(s)$  function, that is, if  $N$  is large enough that  $x^{1/(N+1)} < 2$ ,

$$\pi(x) = \text{li}(x) + \sum_{n=2}^N \frac{\mu(n)}{n} \text{li}\left(x^{1/n}\right) + \sum_{n=1}^N \sum_{\rho} \text{li}\left(x^{\rho/n}\right) + \text{lesser terms} \tag{87}$$

being the contribution of the lesser terms negligible.

In fact,  $\log \xi(0) = \log\left(\frac{1}{2}\right) = -0.69315$  and  $\int_x^\infty \frac{dt}{t(t^2-1)\log t} < 5 \cdot 10^{-5}$  for  $x > 100$ .

### 3.5.7 The Chebyshev's weighted prime counting function

However, the contribution of the non-trivial root term  $\sum_{n=1}^N \sum_{\rho} \text{li}\left(x^{\rho/n}\right)$  is not easily predictable. In fact, Lehmer demonstrated that the series  $\sum [\text{li}(x^\rho) + \text{li}(x^{1-\rho})]$  is only absolutely convergent, but in general it diverges [25]. In particular, it is conditionally convergent, that is, the value of its sum for any  $x$  depends on the cancellation of signs among the terms of the series. As a consequence, it can be demonstrated that many individual terms  $\text{li}(x^\rho)$  grow in magnitude at least as fast as  $x^{1/2} \log x$ , and would therefore be expected to be as significant for large  $x$  as the term  $-\frac{1}{2} \text{li}(x^{1/2})$  and more significant than any of the following terms of Equation (86). This can be an explanation of why the prime number counting function  $\pi(x)$  exceeds the logarithmic integral function  $\text{li}(x)$  for very large  $x$ , notwithstanding the negative contribution of the main terms of the sum  $\sum_{n=2}^\infty \frac{\mu(n)}{n} \text{li}\left(x^{1/n}\right)$ .

In any case, the term  $\sum_{n=1}^N \sum_{\rho} \text{li}\left(x^{\rho/n}\right)$ , which includes the non-trivial zeros of  $\zeta(s)$ , can be viewed as the *spectrum of the primes*, in an analogous way as the Fourier development, because this term is able to rebuild all the steps of the prime number function  $\pi(x)$ , by starting from their smoothing approximation (86). In order to better make explicit this fact, we consider the *Chebyshev's weighted prime counting function*, that is,

$$\Psi(x) = \sum_{p^m \leq x} \log p = \sum_{n \leq x} \Lambda(n) \tag{88}$$

where  $\Lambda(n)$  is the *von Mangoldt function*, which is defined on the naturals [18]

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m, p \text{ prime} \\ 0 & \text{otherwise.} \end{cases} \tag{89}$$

The function  $\Psi(x)$ , which is similar to  $J(x)$ , as it is better shown in the following, is a step function that is incremented of  $\log p$  at each prime power  $p^m$ . In these points, as  $J(x)$ , the value of  $\Psi(x)$  is defined as a halfway between the old and the new values, that is,  $\Psi(x) = \frac{1}{2} [\Psi(x - \epsilon) + \Psi(x + \epsilon)]$ . In practice,  $\Psi(x)$  is the logarithm of the LCM (Least Common Multiple) of all the integers between 1 and  $x$ . The advantage of the function  $\Psi(x)$  is that the weight  $\log p$ , by which we count the prime powers in Equation (88), exactly counterbalances the progressive decreasing of the primes. In fact, if the RH holds, we have

$$\Psi(x) = x + O\left(x^{1/2} (\log x)^2\right) \tag{90}$$

This means that the function  $\Psi(x)$  is essentially square root close to the function  $f(x) = x$ , that is, the 45 degree straight line, if the RH holds.

### 3.5.8 The von Mangoldt formula

In order to show that the functions introduced by Chebyshev and von Mangoldt allow a simpler relation between the succession of primes and the non-trivial zeros of  $\zeta(s)$ , we reconsider Equation (80) that gives the expression of  $J(x)$  as a function of  $\text{li}(x)$  and the non-trivial zeros  $\rho$ , that is,

$$J(x) = \text{li}(x) - \sum_{\rho} \text{li}(x^{\rho}) + \int_x^{\infty} \frac{dt}{t(t^2 - 1) \log t} - \log 2 \tag{91}$$

We now differentiate Equation (91), so obtaining

$$\frac{dJ(x)}{dx} = \frac{1}{\log x} - \sum_{\rho} \frac{x^{\rho-1}}{\log x} - \frac{1}{x(x^2 - 1) \log x} \tag{92}$$

and consequently

$$\log x \frac{dJ(x)}{dx} = 1 - \sum_{\rho} x^{\rho-1} - \frac{1}{x(x^2 - 1)}. \tag{93}$$

Let consider now Equation (53), which related the function  $J(x)$  with the zeta function  $\zeta(s)$ , that is,

$$\frac{\log \zeta(s)}{s} = \int_0^{+\infty} J(x) x^{-s-1} dx \implies \log \zeta(s) = \int_0^{+\infty} \left( \frac{dJ(x)}{dx} \right) x^{-s} dx = \int_0^{+\infty} x^{-s} dJ(x) \tag{94}$$

and successively make an inversion of Equation (94). In this case, the function  $\log \zeta(s)$  has logarithmic singularities in all the roots of the zeta function  $\zeta(s)$ . Moreover, as a function of the complex variable  $s$ , this function is very difficult to manage outside the halfplane  $\Re(s) > 1$ . On the other hand, its derivative  $\frac{\zeta'(s)}{\zeta(s)}$  is analytic in the entire plane, except for poles at the non-trivial roots  $\rho_i$ , at the pole  $s = 1$ , and at the zeros on the negative real axes, which are located at the points  $s = -2n$ .

Consequently, it is more convenient to handle the derivative of Equation (94). Being  $x^{-s} = e^{-s \log x}$ , we obtain the following relation

$$\frac{\zeta'(s)}{\zeta(s)} = - \int_0^{+\infty} x^{-s} (\log x) \left( \frac{dJ(x)}{dx} \right) dx = - \int_0^{+\infty} x^{-s} (\log x) dJ(x) \tag{95}$$

The measure  $(\log x) dJ(x)$  assigns the weight  $\log(p^n) \cdot (1/n)$  to prime powers  $p^n$  and the weight zero to all other points. Consequently, this measure can be written as  $d\Psi(x)$ , where  $\Psi(x)$  is the Chebyshev's function (88), because  $\Psi(x)$  starts at zero and has a jump of  $\log(p^n) \cdot (1/n) = \log p$  at each prime power  $p^n$ . Such an equivalence allows to write Equation (95) as

$$\frac{\zeta'(s)}{\zeta(s)} = - \int_0^{+\infty} x^{-s} d\Psi(x). \tag{96}$$

Equation (96) is similar to Equation (94), but if  $J(x)$  is replaced by  $\Psi(x)$ , then the uncomfortable  $\log \zeta(s)$  is replaced by the more tractable function  $\frac{\zeta'(s)}{\zeta(s)}$ . Now, if we start from Equation (92), that is,

$$dJ(x) = \frac{1}{\log x} - \sum_{\rho} \frac{x^{\rho-1}}{\log x} - \frac{1}{x(x^2 - 1) \log x} dx, \quad x > 1 \tag{97}$$

we obtain, because  $\frac{1}{x(x^2-1)} = \frac{1}{x} \cdot \left( \frac{x^{-2}}{1-x^{-2}} \right) = x^{-3} (1 + x^{-2} + x^{-4} + \dots) = \sum_{n=1}^{\infty} x^{-2n-1}$ , the following relation

$$d\Psi(x) = (\log x) dJ(x) = \left( 1 - \sum_{\rho} x^{\rho-1} - \sum_n x^{-2n-1} \right) dx, \quad x > 1 \tag{98}$$

and consequently we obtain the von Mangoldt formula, that is,

$$\Psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} + \sum_n \frac{x^{-2n}}{2n} + \frac{\zeta'(0)}{\zeta(0)}, \quad x > 1 \tag{99}$$

where the constant value is  $\frac{\zeta'(0)}{\zeta(0)} = \log 2\pi$ . We can immediately notice that Equation (99) includes the same information of the Equation (87) found by Riemann, but the utilization of the step function  $\Psi(x)$ , which

is equivalent to the prime number function  $\pi(x)$ , allows to avoid the uncomfortable logarithmic integral function  $\text{li}(x)$ . Another observation is that the third term is very similar to the second one, apart from the non-trivial roots  $\rho$  of  $\zeta(s)$ , which are replaced by the trivial zeros in the positions  $s = -2n$ . However, the contribution of such a term rapidly becomes null.

If the RH holds, each non-trivial zero on the critical line  $\rho = 1/2 + it$  corresponds to another conjugate zero that is symmetrically placed on the same line, that is,  $\bar{\rho} = 1 - \rho = 1/2 - it$ . Consequently, the second term of Equation (99) is given by a series of conjugate pairs, whose sum is a real number, that is,

$$\frac{x^\rho}{\rho} + \frac{x^{1-\rho}}{1-\rho} = \left( \frac{\sqrt{x}}{\frac{1}{4} + t^2} \right) [\cos(t \log x) + 2t \sin(t \log x)]. \tag{100}$$

Each of these real values gives the correction that is required for recovering the staircase trend of prime number function  $\pi(x)$ , by starting from its smooth approximation. In fact, as shown by Equation (100), this correction is able to reconstruct the staircase trend of  $\pi(x)$  because it is given by a sum of trigonometric functions, which was named by Riemann as *periodic terms*, in such a way it will become possible to obtain a spectrum of primes similar to that of the Fourier analysis [44].

### 4 A possible strategy for approaching the RH

A possible strategy for approaching the RH can be suggested by the observation reported in [1] that the sequence of primes is the result by a set of approximating sequences of integer numbers. In particular, the Prime Characteristic

Function  $\xi_p(n) = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$  is the limit of a succession of sequences  $\xi(k, n)$ ,  $k = 1, \dots, \infty$  defined as

$$\xi(k, n) = \begin{cases} \psi(0, n) & \text{if } p(0)^2 \leq n < p(1)^2 \\ \psi(1, n) & \text{if } p(1)^2 \leq n < p(2)^2 \\ \dots & \\ \psi(k-1, n) & \text{if } p(k-1)^2 \leq n < p(k)^2 \\ \psi(k, n) & \text{if } n \geq p(k)^2 \end{cases} \tag{101}$$

where

$$\psi(k, n) = \begin{cases} 0 & \text{if } p(i)|n \text{ for some } i = 1, \dots, k \\ 1 & \text{otherwise.} \end{cases} \tag{102}$$

Consequently, the sequences  $\xi(k, n)$  are the partial characteristic functions at the end of the  $k - th$  step of the Sieve of Eratosthenes procedure. Therefore, each step is characterized by a partial sequence that includes real primes and composites numbers that have not been already deleted by the Sieve procedure. If we remember the Equation (14) that puts in connection the zeta function with the distribution of the primes, that is,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left( \frac{1}{1 - p^{-s}} \right), \tag{103}$$

then we can take into account the possibility to define other zeta functions, the so-called Generalized Zeta Functions (GZFs) [8], by considering different number successions than the primes, but such that it can be easier the proof that its zeros lie on the critical line. For example, we can take the succession of the naturals, that is

$$\zeta_0(s) = \sum_{n=1}^{\infty} 1 = \sum_{n=1}^{\infty} \frac{1}{n^{(s-0)}} = \prod_n \left( \frac{1}{1 - n^{-s}} \right), \tag{104}$$

where the subscript 0 indicates that this sequence appears at the beginning, that is, at the step 0, of the Sieve procedure. Consequently, once that the RH has been proved for this initial sequence, it could be possible to associate a different GZF to each partial numerical sequence produced by the other progressive steps of the Sieve procedure, and then applied the Induction Principle, by assuming that the RH is true for the GZF  $\zeta_n(s)$  associated to the sequence of the  $n - th$  step of the Sieve Procedure, and successively proving that the RH is true for the GZF  $\zeta_{n+1}(s)$  associated to the sequence of the  $(n + 1) - th$  step.

## 5 Conclusions and future developments

This Report has proposed a possible novel approach to the RH based on the partial numerical successions generated by the successive steps of the Sieve of Eratosthenes procedure. Given the flourishing theory assembling the most disparate Generalized Zeta Functions (GZFs) starting from the Riemann's zeta function, we can try to define the GZFs in correspondence to each of these partial successions and apply the Induction Principle to the succession of the GZFs. Future developments will concern the evaluation of the possibility to define such GZFs in view of the application of the Induction Principle.

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