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Presentation and development of a Physics experiment devised for didactic learning: The Lambert-Beer-Bouguer law

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Summary. — The Lambert-Beer-Bouguer law refers to the decay of the intensity of a radiation travelling through a dissipative medium. Due to the simple phenomenology and to the easiness of the realization of an experimental apparatus for its verification and measure, such law represents, from a didactic point of view, a powerful tool to illustrate the basic features of the exponential decay behaviour to undergraduate students. This paper describes how such purpose can be reached by employing simple equipment and very accessible formalism.

1. – Introduction

The exponential decay describes the behaviour of many natural phenomena [1, 2], finding application also in very wide and general fields such as transients in electronics, radioactive decays, chemical reaction kinetics, diffusion phenomena (including pandemic dynamics). Actually, from the mathematical point of view, the exponential decay is the solution of a differential equation describing a very common occurrence: the equation describing a quantity I decreasing as a function of a variable x, whose decrease rate ($\mathrm{d}I/\mathrm{d}x$) is directly proportional to the current value of the quantity I itself,

$$\frac{\mathrm{d}I}{\mathrm{d}x} = -\alpha I. \tag{1}$$

In eq. (1), α is a positive constant, quantifying the decay rate (if the independent variable x is the time, the inverse of α is often referred to as the characteristic decay

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time). The general solution of eq. (1) is the function

$$I(x) = I(0)e^{-\alpha x}, \tag{2}$$

where $I(0)$ is the value of I at $x = 0$. Equation (2) represents the exponential decay law. Among the many phenomena described by eq. (2), we selected the absorption of light during its propagation inside a dissipative medium as a prototypal example to illustrate to undergraduate students some aspects of such important law, because of the easiness of the realization of the experiment and because of its effectiveness in illustrating the transition from the discrete to continuous approach.

2. – Results and discussion

In the case of absorption of light, eq. (2) is the Lambert-Beer-Bouguer law [3]; in it, I is the intensity of light as a function of the coordinate x along the propagation direction, $x = 0$ representing the separation between the air (where the light comes from) and the medium, and α represents the absorption coefficient (characteristic of the medium, and, in general, depending of the light wavelength, but constant on x if the medium is homogeneous). We proposed, in our classrooms, an experiment to verify the Lambert-Beer-Bouguer law by using simple instrumentation, whose basic components were: a commercial halogen lamp as a light source, a standard digital lux-meter for intensity measurements, Plexiglas prisms as (discrete) components of the optical medium. For the following remarks, it is important to underline that the exit face of each prism was covered with a tape to "artificially" increase the absorption coefficient to an easily detectable level. Further details (for example, on the specific features of the equipment, or on the details to produce incident plane waves) can be found in ref. [4]. The measurements as a function of the coordinate x can be realized by simply changing the number n_p of consecutive plexiglass prisms and measuring any time the intensity at the end of the series: the corresponding x value is the sum of the thicknesses of the employed prisms ($n_p{\cdot}L$ if all the prisms have the same thickness L). An example of experimental results measured by using prisms with equal thicknesses, is reported in fig. 1(a), where I is normalized to $I(0)$ (measured without prisms) and x is normalized to the single prism thickness, L. The experimental points are well fitted by an exponential decay law, confirming the Lambert-Beer-Bouguer law, and the absorption coefficient can be easily estimated as the opposite of the slope of the plot of $\ln(I/I(0))$ *vs.* x (shown in the inset of fig. 1(a)), which linearizes the exponential trend. Under this point of view, this is a useful experimental exercise and a check of a physical law for a laboratory course.

A further step is to consider a remark that we often received from the students in different form. Let us consider eq. (1) in terms of relative variation (loss) of intensity, $\mathrm{d}I/I$, as a function of the traversed path inside the material, $\mathrm{d}x$. The j-th "small" prism can be considered as our elementary piece of material (a didactic locution often used in the physics textbooks); in this respect, being I_j the intensity of the light emerging from the j-th prism, it is natural to consider $\mathrm{d}I = \Delta I_j = I_j - I_{j-1}$, and $\mathrm{d}x = L$. Consequently, eq. (1) reads

$$\frac{I_j - I_{j-1}}{I_{j-1}} = -\alpha L. \tag{3}$$

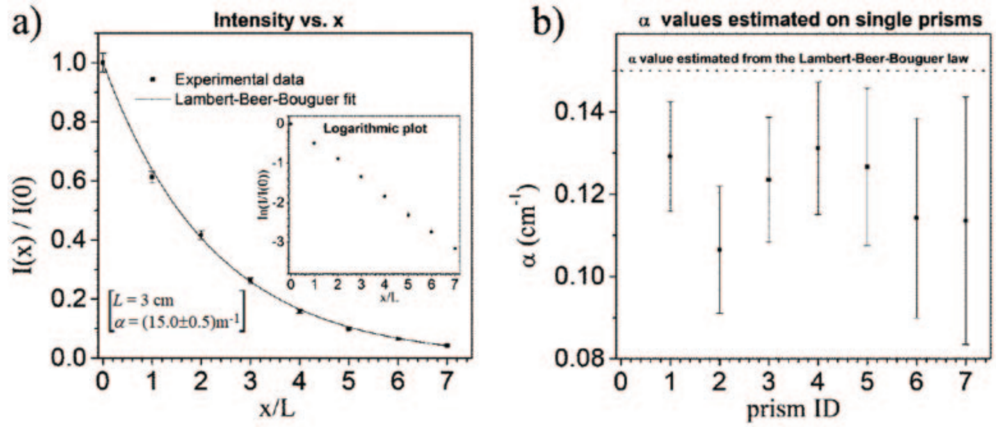

Fig. 1. – (a) Experimental data of light intensity I (normalized to $I(0)$) *vs.* x (normalized to the prism thickness L) inside the series of Plexiglas prisms; in the inset, the same data plot in logarithmic scale for the intensity. (b) Estimations of α values through the discrete approach described in the main text; the horizontal dotted line indicates the estimation obtained with the exponential decay fit.

Equation (3) would give a simple method to infer α from the measurement on a single prism, since the other quantities in it (the entering and emerging light intensities and the thickness L) are easily measurable. The data acquired for the exponential plot of fig. 1(a) already contain the measurements of the intensities entering and emerging in each prism, and the (same) thickness is known. We are therefore already able to compute α through eq. (3) on each cube: Figure 1(b) is a plot of such estimations, compared with the estimation of α from the exponential law as previously described (represented by a horizontal line in fig. 1(b)). As can be seen, this method largely fails, giving a systematic underestimation of the absorption coefficient.

To shed light on what is occurring, through experimental checks rather than employing mathematical demonstrations, we proposed to the students to repeat the measurements, and the comparison between the results coming from the exponential fit and from the single cube evaluation, by changing the following details: 1) using the same kind of prisms, but with half thickness; 2) replacing the tape that we had used to increase α, as described above, with a more transparent one, therefore reducing the effective α value. Note that both expedients have the effect to reduce the product $\alpha \cdot L$, which, according to eq. (3), implies a reduction of the relative intensity loss in each cube. The consequence is a stronger reduction of the relative underestimation of the absorption coefficient with the discrete approach.

To be more quantitative: in the first version of the experiment we had $\alpha \cdot L \approx 0.45$ and we got an α underestimation of about 20%; with prism of half thickness we had $\alpha \cdot L \approx 0.21$ with an α underestimation of about 10%; with a more transparent tape we had $\alpha \cdot L \approx 0.15$ with an α underestimation of about 7%.

Despite the simplicity of the employed instrumentation, the accuracy of the performed experiments (together with the systematicity of the trend of the results) is good enough to dissipate the idea that the discrepancy between the estimations produced by the exponential fit and the discrete evaluation could be due to mere experimental errors or fluctuations. However, such doubt (that can be justifiable in undergraduate students) can be definitely cancelled by numerical simulations of $I(x)$ through the Lambert-Beer-Bouguer law for different values of the product $\alpha \cdot L$, *i.e.*, repeating the two kind of estimation on dataset without errors. Example of such simulations are reported in ref. [4], and were illustrated to students.

From these observations, we can assert that the observed discrepancy between the two estimations of α entirely lies in the discrete nature of the second approach, whose results coincide with the one inferred from the exponential decay fit only when $\alpha \cdot L \ll 1$: which, exactly, corresponds to the continuous limit. Equation (3) suggests that the relative intensity loss per unit length should be independent on the length of the path and equal to the absorption coefficient. The developed checks and considerations, however, demonstrated that it is true that such value is independent on the path, but it approaches the right α value only when $\alpha \cdot L \ll 1$.

The effectiveness of our approach in dissipating the above mentioned doubt was quantitatively tested by a questionnaire on 65 undergraduate students in Physics (the physics of the problem had been already illustrated to them). Among several questions, we asked also about the validity of an α estimation from only two experimental points: 62 students answered "yes, it is formally valid; simply, the experimental accuracy is less than the one obtained by a regression on several points", and only 2 gave the correct answer "no, there is a systematic error because of the non-linearity of the relation" (only one student chose a different wrong answer among those proposed). After having illustrated the experiment and the remarks that we developed in this study, by reproposing the question we got an essentially reversed result (59 students over 65 marked the correct answer).

It can be actually useful to observe once more that plotting the light intensity on a logarithmic scale produces a linear plot, in which the angular coefficient estimations obtained by the globally interpolating line or by considering pairs of points are obviously the same (inside, of course, the experimental errors). This means that the discretization process "works" if applied to the logarithmic version of eq. (3), *i.e.*, $\Delta \ln(I_j) = -\alpha \cdot L$. So, the underestimation produced by eq. (3) arises because $\Delta \ln(I_j)$ and $\Delta I_j / I_{j-1}$ are the same only in the limit of small variation of I, which is exactly what the mathematical analysis teaches.

3. – Conclusions

By performing the described experiment and developing the illustrated considerations, a student can understand how a discrete experimental estimation on single constituting units can produce affordable results only if the mathematical continuity limit is approached. The exponential decay is very effective in illustrating the non-obvious differences between the discrete and the continuous approach; and the Lambert-Beer-Bouguer law, in turns, allows an easy test of the exponential decay behaviour, offering very simply measurable quantities, an understandable way to transit from the discrete to continuous (in the experiment, such transition is practically realized when $\alpha \cdot L \ll 1$), and an easy comparison between the results for the calculation of discrepancies.

REFERENCES

[1] Istratov A. A. and Vyvenko O. F., *Rev. Sci. Instrum.*, **70** (1999) 1233.
[2] Hobbie R. K. and Roth B. J., *Exponential Growth and Decay*, in *Intermediate Physics for Medicine and Biology* (Springer, New York, NY) 2007, pp. 31–47.
[3] Kocsis L., Herman P. and Eke A., *Phys. Med. Biol.*, **51** (2006) N91.
[4] Di Capua R., Offi F. and Fontana F., *Eur. J. Phys.*, **35** (2014) 045025.