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Nonlocal vertices, UV "opaqueness" and causality

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Summary. — Nonlocal field theories have seen a revival of interest in recent years. Such theories possess nonlocal vertices characterized by some nonlocality length scale ℓ. This amounts to loss of resolution for scales finer than ℓ. There are many physical arguments for expecting such resolution loss for ℓ of the order of Planck length. The wavelet decomposition of fields, which analyses field configurations in terms of successive resolution scales, is a natural framework for addressing this as outlined here. Some recent results on unitarity and causality in the presence of nonlocal vertices are also briefly mentioned.

1. – Introduction

After an initial period of exploration in the forties and fifties, there has been renewed interest in recent years in investigating nonlocal field theories. In these theories the point vertices of local field theory are replaced by nonlocal vertices. There are several physical reasons for this interest. Such nonlocal vertices introduce a universal nonlocality length scale ℓ and, provided they satisfy certain properties, can solve the UV problem. The prime example where this is realized are the Feynman rules of string field theory (SFT), where the vertices are nonlocal. Due to the great complexity of the SFT Feynman rules, such vertices are more conveniently studied in model field theories of, say, scalar fields with nonlocal vertices of similar type [1-3].

Nonlocal vertices smear interactions over the scale ℓ. This amounts to loss of resolution inside regions of size $\sim \ell^d$, since the interior of such regions cannot be adequately resolved by the nonlocal vertices. This is a prime feature of nonlocal vertices characterized by some fundamental scale ℓ and is present regardless of the manner in which the vertices were obtained in any particular model. It implies that the occurrence of nonlocal vertices can be viewed in a rather wider context. There is indeed an extensive body of

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work [4] spanning decades that shows that, in the presence of gravity, there is a limit to the resolution any localization experiment can achieve. Indeed, since both the probing and probed particles cary energy and, hence, gravitate, when it comes to energies for localization within a scale of the order of the Planck length local gravitational collapse and horizon formation must occur preventing finer resolution. This "UV opaqueness" is necessarily present due to the universal nature of the gravitational coupling. Here we outline a formalism suitable for describing such excision of regions of certain size in space. The framework of wavelet decompositions would appear to be the natural framework for this task. A very brief listing of some recent general results on unitarity and causality in the presence nonlocal vertices is presented at the conclusion.

2. – Field wavelet decompositions

We fix some UV length scale ℓ, which here may be naturally taken to be of the order of Planck length or some unification scale. With $\hat{\ell}$ denoting this scale in dimensionless units([1]) we set

$$\hat{\ell} = 2^{-\hat{l}} \tag{1}$$

with integer $\hat{l}$.

A wavelet decomposition is implemented through the construction of a Multi-Resolution Analysis (MRA), which decomposes a function space $\mathcal{H}$ into a sequence of orthogonal spaces of increasingly finer resolution,

$$\begin{aligned} \mathcal{H} &= V_{\hat{l}} \oplus W_m \oplus W_{m+1} \oplus W_{m+2} \oplus \cdots \\ &= V_{\hat{l}} \bigoplus_{m \geq \hat{l}} W_m \ . \end{aligned} \tag{2}$$

The starting space is the "scaling space", denoted $V_{\hat{l}}$, which we take here to refer to the scale ℓ, whereas the spaces W_m refer to scales $m \geq \hat{l}$ of progressively finer resolutions. A basis set for this decomposition is constructed by translations and dilations from a scaling mother function $\sigma(x)$ and a set of $2^d - 1$ mother wavelet functions $\upsilon^q(x)$ on $\mathbb{R}^d$, *i.e.*, given a mother scaling function $\sigma(x)$ and corresponding $2^d - 1$ mother wavelet functions $\upsilon^q(x)$ on $\mathbb{R}^d$, the basis set is given by

$$\sigma_{\hat{l}n}(x) = 2^{d\hat{l}/2}\sigma(2^{\hat{l}}x - bn), \tag{3}$$

$$\upsilon^q_{mn}(x) = 2^{dm/2}\upsilon^q(2^m x - bn) \ . \tag{4}$$

with

$$x \in \mathbb{R}^d, \qquad n \in \mathbb{Z}^d, \qquad \hat{l} \leq m \in \mathbb{Z}, \qquad 1 \leq q \leq 2^d - 1, \qquad 0 < b \in \mathbb{R}^+. \tag{5}$$

Here b is the translation parameter, which, as customary, in the following we set equal to unity. The scaling set $\sigma_{\hat{l}n}(x)$ constitutes a basis in $V_{\hat{l}}$, whereas, for each m, the set

([1]) Expressing ℓ in powers of 2 is the standard practice in wavelet theory. We could, of course, by choice of units, set $\ell = 1$, $\hat{l} = 0$, but it is preferable to keep explicit reference to this UV scale.

$\upsilon^q_{mn}(x)$ constitutes a basis for the m-th resolution space W_m. The spaces are mutually orthogonal and one has the orthogonality relations

$$\int \mathrm{d}^d x \overline{\sigma}_{\hat{l}n}(x) \sigma_{\hat{l}k}(x) = \delta_{nk}, \tag{6}$$

$$\int \mathrm{d}^d x \overline{\sigma}_{\hat{l}n}(x) \upsilon^q_{mk}(x) = 0, \quad m \geq \hat{l}, \tag{7}$$

$$\int \overline{\upsilon}^q_{mn}(x) \upsilon^{q'}_{m'k}(x) = \delta_{mm'} \delta_{nk} \delta_{qq'}. \tag{8}$$

The mother functions $\sigma(x)$ and $\upsilon(x)$ are generally well localized around the origin within a length of order ℓ; they may, in particular, be of compact support. Physical requirements dictate that the FT $\widehat{\sigma}(k)$ be of sufficiently rapid decay along the Euclidean momentum axis and an entire function on $\mathbb{C}^d$.

A field configuration $\phi(x)$ on $\mathbb{R}^d$ has then the wavelet expansion

$$\phi(x) = \sum_n \phi_n \sigma_{\hat{l}n}(x) + \sum_{q,m,n} \phi^q_{mn} \upsilon^q_{mn}(x) \tag{9}$$

with coefficients $\phi_n = \langle \sigma_{\hat{l}n}, \phi \rangle$, $\phi^q_{mn} = \langle \upsilon^q_{mn}, \phi \rangle$ and summations over n, m, q as defined in (5).

Equation (9) is a decomposition in successively finer resolutions. $\sum_n \phi_n \sigma_{\hat{l}n}(x)$, the scaling part of the field, represents the "coarse" part of the field, which represents all features of ϕ down to scale ℓ. The wavelet parts probe inside such regions with successively finer resolution: the υ^q_{mn} wavelet terms probe scales of order ℓ^m, $m \geq 1$. It is a remarkable, and very nontrivial, fact that this separation of scales is accomplished in an *orthogonal* exact decomposition. The demonstration that this possible [5] in the eighties led to the explosive development of wavelet theory and applications [5, 6].

Being orthonormal and complete, the basis (3), (4) furnishes a complete resolution of the identity and the coefficients set $\{\phi_n, \phi^q_{mn}\}$ provide an equivalent (discrete) complete representation of the $\phi(x)$: in the functional integral integration over the field ϕ can be replaced by integration over the infinite set $\{\phi_n, \phi^q_{mn}\}$ as the dynamical degrees of freedom. For our purposes here, however, it will be often more convenient to revert to use of $\phi(x)$ as the dynamical variable. The connection is simply provided by the projection operators onto the subspaces $V_{\hat{l}}$ and W_m of the decomposition (9) given by

$$P_{\hat{l}}(x,y) = \sum_n \sigma_{\hat{l}n}(x) \overline{\sigma}_{\hat{l}n}(y), \qquad Q_m(x,y) = \sum_{q,n} \upsilon^q_{mn}(x) \overline{\upsilon}^q_{mn}(y). \tag{10}$$

In terms of these (9) assumes the form

$$\phi(x) = \int \mathrm{d}^d y P_{\hat{l}}(x,y)\phi(y) + \sum_m \int \mathrm{d}^d y Q_m(x,y)\phi(y) \equiv (P_{\hat{l}}\,\phi)(x) + \sum_m (Q_m \phi)(x). \tag{11}$$

Introducing the projection $Q_{\hat{l}}\,(x,y)$ to the direct sum of all W_m subspaces, *i.e.*, $Q_{\hat{l}}\,(x,y) \equiv \sum_{m \geq \hat{l}} Q_m(x,y)$, one has

$$P_{\hat{l}} + Q_{\hat{l}} = 1. \tag{12}$$

By (6)–(8), one indeed has $(P^2_{\hat{l}})(x,y) = P_{\hat{l}}(x,y)$, $(Q^2_{\hat{l}})(x,y) = Q_{\hat{l}}(x,y)$, and $(P_{\hat{l}}Q_{\hat{l}})(x,y) = 0$. The projections $P_{\hat{l}}$ and $Q_{\hat{l}} = 1 - P_{\hat{l}}$ thus decompose the field configuration space $\mathcal{S}$ into two orthogonal subspaces: the subspace $V_{\hat{l}}$ of fields representing features down to scale ℓ; and the subspace $\mathcal{W}_{\hat{l}} = \bigoplus_{m\geq\hat{l}} W_m$ of fields that can represent features from ℓ down to arbitrarily small length scales. Thus, from (11)

$$\phi(x) = (P_{\hat{l}}\,\phi)(x) + (Q_{\hat{l}}\,\phi)(x). \tag{13}$$

3. – Field theoretic models with limited UV resolution

3˙1. *Basic model.* – As seen above wavelet decompositions allow one to selectively pick out parts of a field configuration pertaining to a limited range of resolution. Define a field $\varphi(x)$ that contains only scaling parts: $\varphi(x) = \sum_n \varphi_n \sigma_{\hat{l}n}(x)$ with Euclidean action

$$S = \int \mathrm{d}^d x \left(\frac{1}{2}\varphi K \varphi + \mathcal{L}_I(\varphi)\right), \tag{14}$$

where $\mathcal{L}_I$ is some local (polynosmial) interaction Lagrangian and

$$K(\partial) = (-\Delta + m^2) \tag{15}$$

with $\Delta = \delta^{\mu\nu}\partial_\mu\partial_\nu$. This implements the idea that regions of length scale ℓ become "opaque" and cannot be probed by interactions so as to achieve finer resolution. Quantization is performed via the Euclidean path integral

$$Z[J] = \int [D\varphi] \exp\left\{ -\int \left(\frac{1}{2}\varphi K\varphi + \mathcal{L}_I(\varphi) + J\varphi\right)\right\}. \tag{16}$$

In (16) the measure is defined by: $[D\varphi] \equiv \prod_n \mathrm{d}\varphi_n$, since $\{\varphi_n\}$ are the independent degrees of freedom of the field φ, with the action expressed in terms of $\{\varphi_n\}$; *e.g.*, for the φ kinetic term in (14)

$$\frac{1}{2}\varphi K\varphi = \frac{1}{2}\sum_{n,n'} \varphi_n K_{nn'} \varphi_{n'}, \qquad K_{nn'} = \langle \sigma_{\hat{l}n}, K\sigma_{\hat{l}n'}\rangle, \tag{17}$$

and similarly for the other terms in the action. As already noted, it is generally more convenient, however, to work with a field $\phi(x)$ containing all scales. This is done via the projection operators $P_{\hat{l}}$, $Q_{\hat{l}}$. Define a field $\chi(x) = \sum_{q,m,n} \chi^q_{mn} \upsilon^q_{mn}(x)$ containing only wavelet parts. φ and χ are independent, indeed orthogonal fields. We next include in (16) also integration over the fields $\chi \in \mathcal{W}_{\hat{l}}$ defined by $[D\chi] \equiv \prod_{q,m,n} \mathrm{d}\chi^q_{mn}$. This has no effect since the fields χ do not appear in the action and are thus decoupled. We may now define a field ϕ by adding φ and χ,

$$\phi = \varphi + \chi, \qquad P_{\hat{l}}\phi = \varphi, \quad Q_{\hat{l}}\phi = \chi, \tag{18}$$

which contains components at all length scales. Note that, with the inclusion of integrating over the decoupled χ fields in (16), the integration measure can be expressed in terms of the usual formal ϕ-measure,

$$[D\varphi][D\chi] = [D\phi] = \prod_x \mathrm{d}\phi(x);$$

whereas the action (14), expressed in terms of ϕ, becomes

$$S = \int \mathrm{d}^d x \left(\frac{1}{2} \phi P_{\hat{l}}^\dagger K P_{\hat{l}}\, \phi + \mathcal{L}_I(P_{\hat{l}}\, \phi) \right), \tag{19}$$

$$\int \mathrm{d}^d x \left(\frac{1}{2} \phi K \phi - \phi K Q_{\hat{l}}\, \phi + \frac{1}{2} \phi\, Q_{\hat{l}}^\dagger K Q_{\hat{l}}\, \phi + \mathcal{L}_I(P_{\hat{l}}\, \phi) + J P_{\hat{l}}\, \phi \right), \tag{20}$$

where in the second equality we used the fact that $\varphi = P_{\hat{l}}\, \phi = \phi - \chi = \phi - Q_{\hat{l}}\, \phi$. If we now treat the first term in (20) as defining the bare propagator K^{-1} and all other terms as interactions, it is not hard to see that all contributions from the new 2-point interactions thus generated, *i.e.*, $KQ_{\hat{l}}$ and $Q_{\hat{l}}^\dagger K Q_{\hat{l}}$, cancel, cf. [7]. The resulting effective rules for the action (20) are thus the same as the Feynman rules as for the action,

$$S = \int \mathrm{d}^d x \left(\frac{1}{2} \phi K \phi + \mathcal{L}_I(P_{\hat{l}}\, \phi) \right) + J P_{\hat{l}}\, \phi. \tag{21}$$

This has the ordinary propagator $\Delta(k) = K^{-1}(k) = \frac{1}{k^2+m^2}$ for a scalar field $\phi(x)$ but nonlocal vertices arising from the presence of the projections $P_{\hat{l}}$ in them. These vertices are given by entire functions [7]. Thus (21) has the standard form of the actions studied as models of nonlocal Feyman rules of nonlocal field and String field theories [1, 2].

Now, for explicit calculations, one needs the explicit form of the mother scaling function $\sigma(x)$ such that the set $\{\sigma_{\hat{l}n}(x)\}$ constructed by translations and dilations form a complete orthonormal basis for the scaling space $V_{\hat{l}}$, cf. (3), in spacetime dimension d. To form a complete MRA one also needs the $2^d - 1$ mother wavelet functions $\upsilon^q(x)$ on $\mathbb{R}^d$ such that the complete wavelet set $\{\upsilon^q_{mn}(x)\}$ resolves scales $\leq \hat{l}$, *i.e.*, the space $W_{\hat{l}}$, while maintaining the orthogonality relations (6)–(8). This is the hard part in obtaining an explicit MRA directly in dimension d —in fact such constructions are known only for $d \leq 3$. In almost all wavelet work the mother scaling/wavelet functions are constructed as the tensor product of one dimensional functions. This, however, is not suitable here since the direct product picks a particular frame in d-dimensional Euclidean space and thus manifest $O(d)$ invariance is lost.

3˙2. *Generalized model.* – In our case, however, this technical issue is easily evaded: by construction the fields $\chi \in \mathcal{W}_{\hat{l}}$, though used in intermedia steps for convenience, do not appear in the final result, cf. (21). This is of course as it should be, since, by the original physical reasoning, such regions are unresolvable. In other words, in mathematical terms, we do not need a complete MRA. The *orthogonal* decomposition of the field space $\mathcal{S} = V_{\hat{l}} \bigoplus \mathcal{W}_{\hat{l}}$, however, is guaranteed to exist [5, 6] even if we do not have an explicit construction of the wavelet bases $\{\upsilon^q_{mn}(x)\}$ decomposing $\mathcal{W}_{\hat{l}}$. Let $\sigma(x) = \sigma(|x|)$ be a radially symmetric normalized function on $\mathbb{R}^d$ and such that its FT is entire. One may then again construct the set $\sigma_{\hat{l}n}(x) \equiv 2^{d\,\hat{l}/2}\sigma(2^{\hat{l}}x - n)$ by translations and dilations. This set does not constitute an orthonormal set but such a set $\tilde{\sigma}_{\hat{l}n}(x)$ may be obtained by orthogonalization, specifically, symmetric (Löwdin) orthogonalization, which treats all elements of $\sigma_{\hat{l}n}(x)$ on an equal footing. Using the orthonormal $\tilde{\sigma}_{\hat{l}n}(x)$ one can construct projection operators $V_{\hat{l}}$ as before, see (10), and proceed in an analogous manner. The details are given in [7].

4. – Concluding remarks

Theories of the type presented here, cf. (21), are a subclass of the general class of actions with ordinary physical propagators and nonlocal vertices. Assuming the vertices are sufficiently convergent along the Euclidean axis and satisfy a number of other properties, such as possessing FT that are entire functions on $\mathbb{C}^d$, such theories have good properties: their analytically continued amplitudes onto the Minkowski axis are UV finite and unitary [1, 2]. For vertices convergent also along the Minkowski directions one may formally define the theory directly in Minkowski space. One may then ask whether the resulting theory is equivalent to the one defined in Euclidean space. As demonstrated in recent work [8], it is not. In particular, the resulting amplitudes are not unitary beyond tree level. This is because the infinite arcs in a Wick rotation make non-vanishing contributions due to the entire function nature of the vertices. The correct prescription for defining the theory is indeed to start in Euclidean space $\mathbb{R}^d$ and analytically continue the external momenta to Minkowski space.

In the presence of nonlocal vertices causality is always a concern. Acausal effects are indeed generally present. The physically relevant question is: how big can they be? In discussing such effects in scattering experiments the appropriate use of wave packets is crucial since particles can be experimentally detected only with limited resolution. Realistically, the typical extent of the wave packets L can be assumed to satisfy $L/\ell \gg 1$ for nonlocality scale ℓ of Planck scale order. A detailed analysis for arbitrary tree diagrams in terms of deviations from the Bogoliubov Causality Condition is carried out in [8]. The result is that for the most part acausal effects are exponentially suppressed, though in cases where time-like vertex separations are involved the suppression can be polynomial in (ℓ/L).

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