IL NUOVO CIMENTO **45 C** (2022) 33
DOI 10.1393/ncc/i2022-22033-8

Colloquia: QGRAV 2021

Lorentz non-invariance, locality, unitarity and renormalizability in quantum gravity

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received 4 February 2022

Summary. — We discuss problems of covariant renormalization of Lorentz non-invariant, local and unitary Hořava gravity models in arbitrary spacetime dimensions, including perturbatively renormalizable projectable Hořava gravity in $(2+1)$ and $(3+1)$ dimensions. Renormalization group flow with asymptotically free UV fixed point is presented in $(2+1)$-dimensional theory. A related model of generalized unimodular gravity (GUMG) is shown to have interesting applications in cosmological inflation theory and within the cosmological dark energy problem. The unification of these two models sharing in common Lorentz symmetry violation, fixation of the lapse function and a number of other features is conjectured in the framework of generalized renormalization group approach.

1. – Introduction

The quest for a renormalizable and perturbative quantum gravity theory consistent in the UV domain shows that such a theory can indeed be constructed by introducing higher-order curvature invariants [1-3], but it is doomed to violate unitary in view of ghost modes associated with higher-order derivatives. In spite of various efforts to circumvent this problem or justify the presence of ghosts by special rules of handling them [4, 5] or within the scope of string theory, non-local field models [6, 7], etc., the most widespread point of view is that the absence of ghosts should be a criterion for selecting a healthy stable theory, and if this theory is also local, renormalizable, unitary and consistent in the UV limit, then there is a hope that it can also describe our Nature.

Here we discuss the mechanism that can provide a combination of these properties, based on the breakthrough suggestion of [8] that this can be achieved by dropping the requirement of Lorentz invariance and introducing in the field theory higher-order derivatives only with respect to spatial coordinates. This suggestion turned out to be

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very productive and has led to a new notion of anisotropic scaling invariance and scaling dimensions replacing the conventional physical dimensionality as a criterion of convergence of Feynman diagrams. Application of this criterion within simple power counting arguments led in [8] to the invention of the class of local, unitary quantum gravity models which are considered to be perturbatively renormalizable and, therefore, expected to be consistent in the UV domain. The first part of this paper will be devoted to the discussion of subtle points of this statement, which present certain obstacles to the program of [8]. Namely, this is the problem of irregular propagators which might break the conventional BPHZ scheme of subtracting UV divergences by local counterterms and the problem of local gauge invariance of these counterterms, both of these issues especially inherent in the Hořava gravity. What will be shown is that both problems can be successfully solved for the class of *projectable* Hořava models, but still remain open for non-projectable theories with the dynamical lapse metric variable.

The second part of the paper discusses the model of *generalized unimodular gravity* (GUMG) which can, to a certain extent, be regarded as a twin of the projectable Hořava gravity, because it shares a number of its properties and, moreover, bears fascinating applications in cosmological inflation theory and in the dark energy problem. As we will discuss in conclusions this tale of two models might be accomplished by their unification within the effective field theory framework or the generalized renormalization group.

2. – Renormalizability of the Hořava gravity model

As is well known now [8], the idea of constructing a renormalizable unitary field theory consists in the suggestion to increase the number of spatial derivatives of the fields in the Lagrangian, but to retain only two of their time derivatives

$$\int \mathrm{d}t\,\mathrm{d}^d x\,\big(\,\phi\,\Box\,\phi+...\big)\to\int \mathrm{d}t\,\mathrm{d}^d x\,\big(\,\dot{\phi}^2-\phi(-\Delta)^z\phi+...\big),\quad z>1. \tag{1}$$

On the general power counting ground this provides a better convergence of Feynman integrals and at the same time preserves unitarity, because in each physical mode the propagator continues to have two poles with positive and negative frequencies. Then renormalizability is achieved by choosing sufficiently high integer value of z, and selection of the concrete Lagrangian is done by imposing the relevant global and local symmetry. Obviously the mismatch between the number of spatial and temporal derivatives leads to the loss of Lorentz invariance.

In case of gravity this means that the theory cannot retain full local diffeomorphism invariance, and this local symmetry should be chosen to respect this higher derivative structure in space *vs.* two derivatives in time. This symmetry can obviously be related to the ADM split of the gravitational configuration space into spatial metric γ_{ij}, lapse N and shift N^i functions,

$$\mathrm{d}s^2=-N^2\mathrm{d}t^2+\gamma_{ij}(\mathrm{d}x^i+N^i\mathrm{d}t)(\mathrm{d}x^j+N^j\mathrm{d}t)\ ,\ i,j=1,\ldots,d,$$

where d is the dimensionality of space, which we consider to be rather general in order to learn how the properties of the model depend on spacetime dimensionality $D=d+1$. Under this split the minimal truncation of the diffeomorphism invariance looks like the so-called foliation preserving diffeomorphisms under which spatial coordinates undergo generic time-dependent transformations accompanied by space-independent reparametrization of time,

$$x^i \mapsto x'^i(\mathbf{x},t)\ ,\ t \mapsto t'(t). \tag{2}$$

Lapse and shift functions transform as $N \mapsto N\mathrm{d}t/\mathrm{d}t'$, $N^i \mapsto (N^j\partial x'^i/\partial x^j - \partial x'^i/\partial t)\,\mathrm{d}t/\mathrm{d}t'$, whereas the spatial metric γ_{ij} behaves as a d-dimensional tensor along with the extrinsic curvature tensor of the spatial slice of constant t

$$K_{ij} = \frac{1}{2N}(\dot\gamma_{ij} - \nabla_i N_j - \nabla_j N_i). \tag{3}$$

The lack of Lorentz symmetry can be compensated by the requirement of the new global symmetry borrowed from condensed matter physics —the invariance under anisotropic scaling transformations, which is very important as a selection criterion of viable renormalizable gravitational Lagrangians,

$$x^i \to \lambda^{-1}x^i, \quad t \to \lambda^{-z}t, \quad N^i \to \lambda^{z-1}N^i, \quad \gamma_{ij} \to \gamma_{ij}, \tag{4}$$

$$[x] = -1,\ \ [t] = -z,\ \ [N^i] = z-1,\ \ [\gamma_{ij}] = 0, \qquad [K_{ij}] = z. \tag{5}$$

Here the choice of the parameter z is motivated by the scaling invariance of the Lagrangian (1), and square brackets denote the scaling dimensions of relevant quantities —the notion naturally replacing the notion of their physical dimensionality.

This construction immediately suggests the general structure of the class of gravitational field models proposed in [8], which have by power counting arguments a chance of being perturbatively renormalizable in $(d+1)$-dimensional spacetime, provided $z = d$,

$$S = \frac{1}{2G}\int \mathrm{d}t\,\mathrm{d}^d x\sqrt{\gamma}N\big(K_{ij}K^{ij} - \lambda K^2 - \mathcal{V}\big). \tag{6}$$

Here the kinetic term is quadratic in K_{ij}, that is bilinear in time derivatives of the metric, G is the gravitational coupling constant, λ is another dimensionless constant of the kinetic term, and $\mathcal{V}$ is the potential term which can depend on the metric γ_{ij}, its spatial derivatives and spatial gradients of $\ln N$. For the sake of power counting renormalizability this potential should have scaling dimensionality up to $2d$ implying the presence of $2d$ derivatives. Below we will consider even simpler version of the Hořava gravity —the so-called *projectable* version with frozen time reparametrization invariance corresponding to $N = 1$. Then, say, in (3+1) dimensions relevant and marginally relevant terms of its Lagrangian look as

$$\begin{aligned}\mathcal{V}(\gamma) = 2\Lambda - \eta R + \mu_1 R^2 + \mu_2 R_{ij}R^{ij} + \nu_1 R^3 + \nu_2 R R_{ij}R^{ij}\\ +\nu_3 R^i_j R^j_k R^k_i + \nu_4 \nabla_i R \nabla^i R + \nu_5 \nabla_i R_{jk}\nabla^i R^{jk},\end{aligned} \tag{7}$$

where Λ, η, μ_1, μ_2 are the relevant and $\nu_1,...,\nu_5$ are the marginally relevant coupling constants of the theory.

The prospects of this model turn out, however, to be more complicated than it was originally anticipated in [8]. Even waving aside the difficulty of matching in the low-energy domain the dispersion relations with those of general relativity [9]([1]) (the problem which we do not consider here because we restrict ourselves with only the high-energy domain), the proof of renormalizability cannot really be accomplished by naive power counting arguments. The point is that the degree of divergence of Feynman diagrams (denoted by $\mathcal{D}$) does not *a priori* provide viable renormalizability criteria of the BPHZ mechanism, because in the transition from Lorentz-invariant theories to Hořava models it is now based on counting the anisotropic scaling dimension of their integrand rather than the physical one,

$$\mathcal{D}\int\frac{\mathrm{d}^{d+1}p}{(p^2)^N}=1+d-2N\to\mathcal{D}\int\frac{\mathrm{d}\omega\,\mathrm{d}^d\mathbf{k}}{(\omega^2+\mathbf{k}^{2z})^N}=z+d-2zN. \tag{8}$$

This is because this transition to the integrals over $(d+1)$-dimensional loop momenta $p=(\omega,\mathbf{k})$,

$$\begin{aligned}&\int\prod_{l=1}^{L}\mathrm{d}^{d+1}k^{(l)}\mathcal{F}_n(k)\prod_{m=1}^{M}\frac{1}{\big(P^{(m)}(k)\big)^2}\to\\ &\int\prod_{l=1}^{L}\mathrm{d}\omega^{(l)}\mathrm{d}^d\mathbf{k}^{(l)}\;\mathcal{F}_n(\omega,\mathbf{k})\;\prod_{m=1}^{M}\frac{1}{A_m\big(\Omega^{(m)}(\omega)\big)^2+B_m\big(\mathbf{K}^{(m)}(\mathbf{k})\big)^{2z}},\end{aligned} \tag{9}$$

results in propagators with some constant coefficients A_m and B_m. It turns out that the rules of BPHZ subtraction of UV divergences are guaranteed only when A_m and B_m are both positive [10]. Their values, however, depend on the choice of the model and, moreover, on the choice of gauge conditions used in the Faddeev-Popov (or BRST) gauge fixing procedure. Irregular $1/\omega^2$ and $1/\mathbf{k}^{2z}$ terms in the propagators generically violate the conventional renormalization procedure, so that a subtle step in the proof of renormalizability consists in the search for a special class of gauges in which all propagators are regular —that is with $A_m, B_m>0$.

For the *projectable* Hořava gravity such a two-parameter family of gauges F_i has been found in [10] in the form of the *local* gauge-fixing action with the nonlocal gauge-fixing matrix $\mathcal{O}_{ij}$. Moreover, it was also extended to the class of *background covariant* gauges in the form

$$S_{\rm gf}=\frac{\sigma}{2G}\int\mathrm{d}t\,\mathrm{d}^2x\,\sqrt{\gamma}\,F^i\,\mathcal{O}_{ij}F^i,\qquad F^i=D_tn^i+\frac{1}{2\sigma}\,\mathcal{O}^{-1\,ij}(\nabla_kh^k_j-\lambda\nabla_jh), \tag{10}$$

$$\mathcal{O}_{ij}=\big[-\gamma^{ij}(-\Delta)^{d-1}+\xi\nabla^i(-\Delta)^{d-2}\nabla^j\big]^{-1}, \tag{11}$$

where h_{ij} and n^i are quantum fluctuations on top of background fields γ_{ij} and N^i, $D_tn^i=\partial_tn^i-N^k\,\partial_kn^i+\partial_kN^i\,n^k$, σ and ξ are free gauge-fixing parameters, and all covariant derivatives as well as raising and lowering the indices are done with respect to the background metric γ_{ij}.

([1]) Dispersion relations for transverse-traceless tensor modes, $\omega_{tt}^2=\eta k^2+\mu_2k^4+\nu_5k^6$, and the scalar mode, $\omega_s^2=\frac{1-\lambda}{1-3\lambda}\big(-\eta k^2+(8\mu_1+3\mu_2)k^4+(8\nu_4+3\nu_5)k^6\big)$, show that in the unitarity domain $(1-\lambda)/(1-3\lambda)>0$ for any sign of η the theory has gradient instability for low momenta k either in the tensor sector or the scalar one.

2˙1. *BRST structure of renormalazation.* – The BRST structure of the renormalization in such gauges, which is supposed to lead to covariant counterterms and reduce to the renormalization of the coupling constants in the original action (6)–(7), also requires extension of known results for Lorentz-invariant theories to Hořava gravity. It turns out that such an extension is possible [11] within a much wider class of theories with a generic *closed* algebra of *irreducible* gauge generators which are *linear* in the quantum fields φ. This extension runs via the inclusion into the conventional BRST operator the background field and its BRST partner along with a special choice of the gauge fermion $\Psi_{\rm ext}$ depending on the full set of quantum fields Φ (including together with φ the BRST ghosts and Lagrange multipliers) and their antifields Φ^* playing the role of the sources of the BRST transformations of the full set of quantum fields,

$$Q \to Q_{\rm ext}, \quad \Psi \to \Psi_{\rm ext}[\,\Phi, \Phi^*\,]. \tag{12}$$

The sources J of the original gauge fields φ can also be included into the extended BRST operator $Q_{\rm ext}$,

$$e^{-W/\hbar} = \int D\Phi\, e^{-(S+Q\Psi+J\Phi)/\hbar} \Longrightarrow e^{-W/\hbar} = \int D\Phi\, e^{-(S+Q_{\rm ext}\Psi_{\rm ext})/\hbar}, \tag{13}$$

and the generating functional $W[\,J,\Phi^*]$ can be easily shown to satisfy Slavnov-Taylor and Ward identities, the latter following from the background covariant nature of chosen gauge conditions. Their application to the divergent part of the effective action via the study of the cohomologies of the nilpotent BRST operator Ω yields such BRST structure of the counterterms, that the overall renormalization reduces to a simultaneous local renormalization of the gauge-invariant action of the original gauge fields φ and the gauge fermion $\Psi_{\rm ext}$ which also gets quantum corrections,

$$S[\,\varphi\,] \to S[\,\varphi\,] + \Delta_\infty S[\,\varphi\,], \quad \Psi_{\rm ext}[\,\Phi, \Phi^*\,] \to \boldsymbol{\Psi}_{\rm ext}[\,\Phi, \Phi^*\,] + \Delta_\infty \boldsymbol{\Psi}_{\rm ext}[\,\Phi, \Phi^*\,]. \tag{14}$$

This BRST structure of renormalization, which is the sum of the renormalized classical action and BRST exact term with the renormalized gauge fermion, is achieved via additional renormalization of quantum fields which turns out to be a (generically nonlinear) anti-canonical transformation generated by the gauge fermion $\Psi_{\rm ext}$ itself. It is important that the uncontrollably complicated renormalization of the gauge fermion is immaterial from the viewpoint of physical applications, because anyway the generating functional of physical amplitudes is gauge independent onshell, $\delta_\Psi W\,|_{\,J=\Phi^*=0} = 0$. The remarkable feature of this scheme is that it applies not only to perturbatively renormalizable theories, but also to effective field theories below their cutoff [11]. All this justifies the physically invariant scope of these results and their application, in particular, to Hořava gravity models.

2˙2. *Asymptotic freedom in the* $(2+1)$*-dimensional Hořava gravity.* – In this way perturbative renormalizability and unitarity in UV domain was proven for *projectable* Hořava gravity models in any spacetime dimension [10] and, moreover, their asymptotic freedom is shown in $(2+1)$-dimensional case with the action

$$S = \frac{1}{2G}\int {\rm d}t\, {\rm d}^2x\; N\sqrt{\gamma}\;\left(K_{ij}K^{ij} - \lambda K^2 + \mu R^2\right). \tag{15}$$

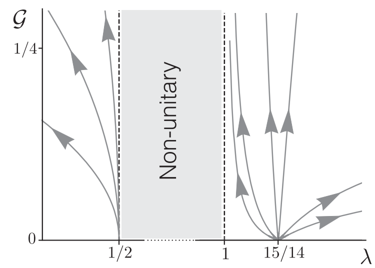

Fig. 1. – RG flow of the couplings in the $(2+1)$-dimensional Hořava gravity. The arrows show the direction of the flow towards the infrared.

Due to the fact that the one-loop effective action under the change of the gauge gets shifted by the equation of motion term $\Gamma_{1-\text{loop}} \to \Gamma_{1-\text{loop}} + \int \mathrm{d}t\, \mathrm{d}^d x\, \Omega_{ij}(\delta S/\delta\gamma_{ij})$, only two coupling constants λ and $\mathcal{G} \equiv G/\sqrt{\mu}$ among G, λ and μ are essential and gauge independent. Their beta functions turn out to be [12]

$$\beta_\lambda = \frac{15-14\lambda}{64\pi}\sqrt{\frac{1-2\lambda}{1-\lambda}}\,\mathcal{G}, \quad \beta_{\mathcal{G}} = -\frac{(16-33\lambda+18\lambda^2)}{64\pi(1-\lambda)^2}\sqrt{\frac{1-\lambda}{1-2\lambda}}\,\mathcal{G}^2, \tag{16}$$

and produce the RG flow shown in fig. 1, which has asymptotically free fixed point at $\lambda = 15/14$ (another fixed point at the boundary $\lambda = 1/2$ of the unitarity domain, $\{\lambda < 1/2\} \cup \{\lambda > 1\}$, is singular and requires two-loop calculation to resolve its singularity).

By the technique of universal functional traces [13] within the background field method all beta functions of the projectable Hořava gravity (6)–(7) were recently computed in the $(3+1)$-dimensional case, and several candidates for asymptotically free UV fixed points have also been found [14].

Unfortunately, all the above conclusions have not yet been extended to non-projectable Hořava models with a dynamical lapse N, because their propagators have in any gauge-irregular terms, and sufficient renormalizability condition is not fulfilled. So even the question of renormalizability in non-projectable models remains open.

3. – Generalized unimodular gravity

Now we turn to the model of generalized unimodular gravity (GUMG) [15,16] descending from maybe the first modification of general relativity (in 1919 due to Einstein [17]) —unimodular gravity (UMG) which is based on the metric restriction to a fixed (unit) determinant $\det g_{\mu\nu} = 1$. This generalization consists in the constraint expressing the lapse function as a rather generic function of the spatial metric determinant

$$(-g^{00})^{-1/2} = N(\gamma), \quad \gamma = \det \gamma_{ij}, \tag{17}$$

UMG obviously being its particular case with $N(\gamma) = 1/\sqrt{\gamma}$. This model has interesting applications in inflation theory and in dark energy problem and, though its quantum properties are not yet studied to the same extent as in the Hořava gravity, the GUMG theory shares with the latter a number of features which makes the tale of these two models rather fascinating. What they share in common is violation of Lorentz and diffeomorphism symmetry, fixing the lapse function (like in the projectable HL model), the same number of degrees of freedom (transverse-traceless tensor plus scalar gravitons), etc.

The GUMG action descending from the Einstein action $S_E[\,g_{\mu\nu}]$,

$$S_{GUMG}[\,g_{ij}, g_{0i}\,] = S_E[\,g_{\mu\nu}]\,\Big|_{(-g^{00})^{-1/2}=N(\gamma)}, \tag{18}$$

generates Einstein equations with effective stress tensor of perfect dark fluid whose energy density ε and pressure p are composed of purely metric degrees of freedom and satisfy the barotropic equation of state with a variable in time parameter w,

$$p = w\varepsilon, \quad w = 2\,\frac{\mathrm{d}\ln N(\gamma)}{\mathrm{d}\ln\gamma}.$$

This model can generate inflation driven by the scalar graviton, playing the role of inflaton [18], and due to varying w might incorporate also a dark energy scenario [16].

3˙1. *Primordial power spectra in GUMG inflation.* – The cosmological perturbation theory for metric fluctuations, decomposed on the inflationary background with the scale factor a into irreducible components, $h_{ij} = a^2(-2\psi\,\sigma_{ij} + 2\nabla_i\nabla_j E + 2\nabla_{(i}F_{j)} + t_{ij})$, generates in an appropriate gauge the Mukhanov-Sasaki equation [19] for the inflaton mode $\vartheta = \theta\,\psi$,

$$\vartheta'' - c_s^2\,\Delta\vartheta - \frac{\theta''}{\theta}\,\vartheta = 0. \tag{19}$$

Here primes denote derivatives with respect to conformal time η, and a nontrivial sound speed c_s and the function θ are determined by the time history of the background scale factor $a(\eta)$ and of the equation of state parameter w,

$$c_s^2 = \frac{w(1+w)}{\mathit{\Omega}}, \quad \theta^2 = 3a^2M_P^2\,\frac{\Omega}{w}, \quad \mathit{\Omega} = 1 + w + 2\frac{\mathrm{d}\ln w}{\mathrm{d}\ln\gamma}.$$

A similar equation in the tensor sector holds with $c_s = 1$ and $\theta = a$. In the unitarity domain, where the theory is free of ghosts and unitary,

$$\frac{w}{\mathit{\Omega}} > 0 \qquad 1 + w > 0,$$

it generates inflationary power spectra of scalar and tensor perturbations typical for the k-inflation models [19]. They read in terms of the energy density ε and the Hubble factor H, $\varepsilon = 3M_P^2H^2$, at the horizon crossing moment for the mode with a wave vector k,

$$\delta_\psi^2(k,\eta) = \frac{1}{36\pi^2}\,\frac{1}{c_s(1+w)}\,\frac{\varepsilon}{M_P^4}\,\bigg|_{c_s k = Ha}, \quad \delta_t^2(k,\eta) = \frac{2}{\pi^2}\,\frac{H^2}{M_P^2}\,\bigg|_{k=Ha}. \tag{20}$$

These primordial spectra will be slightly red tilted in accordance with the present day phenomenology of inflation theory if the function $N(\gamma)$ in the definition of the GUMG action (18) is parameterized in terms of the spectral tilt $1 - n_s \simeq 0.04 \ll 1$ as

$$N(\gamma) = \frac{1}{\sqrt{\gamma}}\Big[1 + \sqrt{\frac{\gamma}{\gamma_*}} + B\left(\frac{\gamma}{\gamma_*}\right)^{3/2-4(1-n_s)/3}\Big], \quad w \simeq -1 + \sqrt{\frac{\gamma}{\gamma_*}}, \tag{21}$$

with some constant $B = O(1)$ and the spatial volume of the Universe $\sqrt{\gamma_*} = a_*^3$ at the end of inflation. Quite remarkably this result satisfies the naturalness condition for the anticipated (but not yet measured) small tensor to scalar ratio $r = \delta_t^2/\delta_\psi^2$, because in this model this ratio looks, in terms of the canonically assumed value of the inflation e-folding number $\mathcal{N} = \frac{1}{6}\ln(\gamma_*/\gamma\,|_{\,c_s k=Ha}) \simeq 60$, as $r \sim 54(e^{-\mathcal{N}})^{4(1-n_s)} \sim 10^{-3}$ —phenomenological exponentially big red shift number $z = e^{\mathcal{N}}$ does not contradict moderately small value of r.

3˙2. *GUMG theory as k-essence and self-gravitating media.* – It turns out that the k-inflation type of the spectra (20) is not accidental because the GUMG model can be shown to be equivalent to a special type of k-essence models [19] or more generally models of self-gravitating media [20]. This equivalence can be derived by first embedding the GUMG model into diffeomorphism invariant theory of four scalar Stueckelberg fields, three of those being able to decouple from the system, so that the remaining Stueckelberg field ϕ with a nonlinear and translation non-invariant in the ϕ-space kinetic term becomes a matter source in the covariant Einstein theory

$$S_K[\,g_{\mu\nu},\phi\,] = S_E[\,g_{\mu\nu}\,] + \int \mathrm{d}^4x\sqrt{-g}\,K(\phi)P(X), \quad X \equiv g^{\mu\nu}\nabla_\mu\phi\,\nabla_\nu\phi. \tag{22}$$

Under the appropriate choice of $P(X)$ and $K(\phi)$ this action is equivalent to (18) in the sense that it generates the same equations of motion in the coordinate gauge $\phi = x^0$ for which $X = g^{00} = -1/N^2$. The function $P(X)$ is directly related to the function $N(\gamma)$ in the GUMG action (18)

$$P(X) = \left(\frac{-X}{\Gamma(1/\sqrt{-X})}\right)^{1/2}, \quad \Gamma(N(\gamma)) \equiv \gamma,$$

in terms of $\Gamma(N)$ —the function inverse to $N(\gamma)$, whereas the construction of $K(\phi)$ is more complicated and described in [20]. The perfect fluid of this k-essence has pressure, density and the speed of sound [19]

$$p = w\varepsilon = K(\phi)\,P(X), \quad \varepsilon = \frac{p}{w}, \quad c_s^2 = \frac{\partial p/\partial X}{\partial\varepsilon/\partial X}, \tag{23}$$

the resulting $c_s^2 = w(1+w)/\Omega$ exactly matching with (20) in view of the relation $P(X(\gamma)) = 1/\sqrt{\gamma}N(\gamma)$.

Reconstruction of $K(\phi)$ in the GUMG model of inflation with function (21) can be done within perturbation theory in two small parameters

$$\delta = \sqrt{\frac{-\nabla^\mu\phi\nabla_\mu\phi}{\gamma_*}} \ll 1, \quad \epsilon = \frac{3H_0\phi}{\sqrt{\gamma_*}} \equiv \frac{\phi}{\phi_0} \sim \frac{a^3}{a_*^3} \ll 1,$$

the second one expressing smallness of the scale factor a during inflation compared to its value a_* at the end of inflation stage, $\sqrt{\gamma_*} = a_*^3$. H_0 here is the Hubble factor at the onset of inflation. Within this perturbation theory the Lagrangian of the k-essence scalar field takes the form

(24)
$$\mathcal{L}_K(\phi, \nabla_\mu\phi\nabla^\mu\phi) = \frac{3M_P^2H_0^2}{\sqrt{\gamma_*}}\,\Big(\sqrt{-\nabla_\mu\phi\nabla^\mu\phi} - \sqrt{\gamma_*}\Big) + O\Big(\epsilon, \epsilon^{3+8\,\frac{n_s-1}{3}}, \delta^2, \delta^{3+8\,\frac{n_s-1}{3}}\Big),$$

where we disregard higher-order integer and fractional powers of these smallness parameters. The leading term here is the Lagrangian of the so-called cuscuton model [21] with the nonlinear square-root kinetic term and the constant potential. With a special choice of the potential cuscuton models generate dark energy mechanism similar to that of the Dvali-Gabadadze-Porrati model [22]. They are also known for being not dynamical, because of ultralocal in time equation of motion, which is interpreted as infinite speed of sound of the "propagating" excitations of ϕ.

On the contrary, the above ϵ and δ corrections are critically important in the GUMG model to make its scalar mode dynamical and generate small red tilt $n_s - 1 \simeq -0.04$ in the primordial CMB spectrum. Smallness of this tilt actually allows one to expand these corrections, which yields logarithms $\ln(\phi/\phi_0)$ and $\ln(-X)$ resembling Coleman-Weinberg potential and gradient expansion of quantum corrections. This suggests the hypothesis that the GUMG theory might be the effective field theory of some fundamental quantum model.

4. – Conclusions and discussion

Except for the final remark above we have not yet associated the projectable Hořava gravity with the GUMG theory at the quantum level. Such an association is still possible in the form of the hypothesis that reduction of the foliation preserving diffeomorphisms in Hořava gravity to even narrower class of coordinate transformations —spatially transverse diffeomorphisms— might still lead to the renormalizable theory with the action (6)–(7) in which all couplings, except G, become functions of γ and the projectability condition $N = 1$ is replaced by (17),

$$g_A = (\lambda, \Lambda, \eta, \mu_1, \mu_2, \nu_1, ...) \to g_A(\gamma) = \big(\lambda(\gamma), \Lambda(\gamma), \eta(\gamma), \mu_1(\gamma), \mu_2(\gamma), \nu_1(\gamma), ...\big),$$
$$N \to N(\gamma).$$

These generalized couplings $N(\gamma)$ and $g_A(\gamma)$ would perform infinite resummation of marginally relevant invariants of the same anisotropic scaling dimensionality, and their running in both the Wilsonian energy scale k_* and the dimensionless variable γ will be determined by RG equations with generalized beta functions $\beta_A(g, \partial_\gamma g, \partial_\gamma^2 g, ...)$ algebraically depending not only on the set of g, but also on their first-order and higher derivatives in γ,

(25)
$$k_*\frac{\partial g_A}{\partial k_*} = \beta_A(g, \partial_\gamma g, \partial^2_\gamma g, ...).$$

Such differential RG equations in partial derivatives were suggested for nonrenormalizable theories with infinite set of charges in [23], and perhaps in this Hořava gravity context they might find viable applications. If this hypothesis materializes, then the choice of the sequence of coefficient functions $N(\gamma), \lambda(\gamma), ...$, will be based not on purely phenomenological ground, as the one suggested above from the viewpoint of the needs of inflation theory, but from fundamental field theoretical arguments. In particular, they will follow from fixed points of these generalized RG flows, which will be given by the solutions of a nontrivial set of ordinary differential equations $\beta_A(g, \partial_\gamma g, \partial_\gamma^2 g, ...) = 0$. This construction might eventually provide an alternative to the non-projectable Hořava gravity, whose renormalizability still remains an open issue.

At the moment, however, too many conjectures are underlying the above picture attempting to unify these two Hořava and GUMG models, in order we could further guess on the viability of such a hypothesis. So, instead, we would just summarize the reliable facts surveyed above. As was shown, perturbative renormalizability of the projectable Hořava gravity in any spacetime dimension provides salvation of local, unitary quantum gravity theory consistent in the UV domain. The regularity of its propagators guarantees BPHZ renormalization and the choice of background covariant gauges provides local gauge invariance of counterterms via the use of BRST extension of the theory and the cohomological properties of its nilpotent BRST operator. Asymptotic freedom of the $(2+1)$-dimensional projectable Hořava gravity in the UV limit was proven, and a similar property has recently been found in the $(3+1)$-dimensional case [14]. As a possible twin of the projectable Hořava gravity, the generalized unimodular gravity is shown to have interesting applications in the phenomenology of cosmological inflation and k-essence theories. These results show that violation of the Lorentz invariance which, however, does not rule out its subsequent recovery via Stueckelberg-type covariantization, indeed, it serves as a palladium of locality, unitarity and renormalizability in quantum gravity, and further exciting revelations are expected on this road.

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The author is deeply grateful for long-term productive collaboration on the above issues to Diego Blas, Mario Herrero-Valea, Alexander Kamenshchik, Nikita Kolganov, Alexander Kurov, Dmitrii Nesterov, Sergey Sibiryakov, Christian Steinwachs and Alexander Vikman. The work was partially supported by the Russian Foundation for Basic Research grant 20-02-00297 and by the Foundation for Theoretical Physics Development "Basis".

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