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The running vacuum in effective quantum gravity

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Summary. — We briefly review the previous works on the renormalization group in quantum general relativity with the cosmological constant, based on the Vilkovisky and DeWitt version of effective action. On top of that, we discuss the prospects of the applications of this version of renormalization group to the cosmological models with a running Newton constant and vacuum energy density.

1. – Introduction

Nowadays, cosmology seems to be the most promising field of application for semiclassical and quantum gravity models. One of the most difficult problems on the interface of quantum field theory and cosmology is the cosmological constant problem [1]. From the field theory perspective [2], the observed vacuum energy density may not be exactly constant because of the remnant low-energy quantum effects. This observation led to cosmological models with the running cosmological and Newton constants (see [3] and the last paper on the subject [4] for a review and further references).

The main basis of the running cosmology models [2, 5] (see also [3, 6] for the review and discussion of the theoretical side of the problem) is the universal form of the scale-dependent vacuum energy density, owing to the semiclassical gravitational corrections,

$$\rho_\Lambda = \rho_\Lambda^0 + \frac{3\nu}{8\pi G}\big(H^2 - H_0^2\big), \tag{1}$$

where ρ_Λ^0 and H_0^2 are the vacuum energy density and the Hubble parameter at the reference point, *e.g.*, in the present moment of time. ν is a phenomenological parameter which can be associated to the masses of the GUT particles. Regardless the formula (1)

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cannot be proved in the QFT framework, it can be shown that any relevant IR running of the vacuum energy, derived from the covariant semiclassical approach, has the form [6]. There may be other identifications, but the systematic scale-setting procedure suggested in [7] shows that the optimized cosmological identification of the parameter μ is H. In a different physical framework there may be other interpretations of μ. Using the conservation law in the models without the particle creation from vacuum leads to the more "traditional" logarithmic form of the running for the Newton constant [8]

$$G(\mu) = \frac{G_0}{1 + \nu \log(H^2/H_0^2)}. \tag{2}$$

A procedure of scale setting [9] similar to [7] may be useful for the astrophysical applications [10] of the running Newton constant G. There are several derivations of (1) and (2), and all of them are (in some cases, implicitly) based on the quadratic decoupling of massive degrees of freedom in the IR. In other words, the aforementioned running is due to the quantum effects of *massive* particles in the IR, according to the corresponding theorems [11,12]. The massless degrees of freedom do not produce running of the dimensional parameters, such as ρ_Λ and G. On the other hand, it is remarkable that (2) looks exactly as the one-loop running of the massless coupling in the Minimal Subtraction scheme of renormalization, regardless it is derived in a very different framework.

One can ask whether the aforementioned universality of the running of ρ_Λ and G holds in the full quantum gravity (QG), when metric is a quantum field. The complete answer to this question may be given only on the basis of a completely consistent theory of quantum gravity, *i.e.*, something we do not have. However, any kind of a purely metric quantum gravity should have massless degrees of freedom. Thus, one can assume that in the IR the possible massive degrees of freedom (including higher derivative ghosts) decouple and do not play a role in the running. In this way, we arrive at the effective theory of QG, where only the massless modes are active and, consequently, the universal effective theory is the quantum version of general relativity (GR) [13].

It turns out that the running of G and Λ in the effective QG can be achieved only by using the "unique" effective action of Vilkovisky and DeWitt [14,15], in a way this was considered in [16] (previously, there was a detailed one-loop calculation in the quantum GR with a cosmological constant [17]) . In the present contribution we review the recent works [18] and especially [19], trying to give more introductory exposition of the subject.

The manuscript is organized as follows. In the next sect. **2**, we present a very qualitative view on the renormalization group running. In sect. **3**, we consider how the running in effective QG with a nonzero cosmological constant is affected by the power counting and, in particular, explain why the one-loop effects are strongly dominating in the effective renormalization group equations. Section **4** discusses the gauge fixing ambiguity in these equations. The Vilkovisky's construction is described in a very qualitative way in sect. **5**. Section **6** includes a brief survey of the main results for the running [16,19] in cosmology, and sect. **7** aims to show that the running in effective QG is qualitatively different from (1). Finally, in sect. **8** we draw our conclusions and discuss some of the existing perspectives.

2. – Brief review of renormalization group

The renormalization group is a useful and economical way to describe quantum corrections. Thus, it is quite natural trying to use it in quantum gravity (QG).

As an example, consider a fermion loop effect in QED, where the one-loop effective Lagrangian of a gauge vector field has the form

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}[\gamma^\mu(\partial_\mu - A_\mu) - im]\psi. \tag{3}$$

Taking into account the one-loop correction, we get, approximately([1]),

$$\mathcal{L}_{em} = -\frac{1}{4e^2}F_{\mu\nu}\Big[1 - \beta\ln\Big(-\frac{\Box + m^2}{\mu^2}\Big)\Big]F^{\mu\nu}.$$

In the IR, when (Euclidean) momentum satisfies $k^2 \ll m^2$ this becomes an irrelevant redefinition of the charge e. However, in the UV, when $k^2 \gg m^2$ there is an effective logarithmic running $e^2(k) = e_0^2\big(1 - \beta\ln\frac{k^2}{\mu_0^2}\big)$. An illustration of the physical running *vs.* the one in the Minimal Subtraction scheme with μ replacing k, is given in fig. 1.

In more details, the momentum-subtraction β-function leads to the important special cases:

$$\begin{aligned} &\text{UV limit} \qquad p^2 \gg m^2 \Longrightarrow \beta_e^{1UV} = \frac{4e^3}{3(4\pi)^2} + \mathcal{O}\Big(\frac{m^2}{p^2}\Big), \\ &\text{IR limit} \qquad p^2 \ll m^2 \Longrightarrow \beta_e^{1IR} = \frac{e^3}{(4\pi)^2}\cdot\frac{4p^2}{15m^2} + \mathcal{O}\Big(\frac{p^4}{m^4}\Big). \end{aligned} \tag{4}$$

This is nothing else but the standard form of the decoupling theorem by Appelquist and Carazzone [11]. One can demonstrate that the decoupling works in a very similar way in the semiclassical gravity, for both four-derivative [12] and the Einstein-Hilbert terms [21, 22]. This result gives strong, albeit indirect support to the hypothesis of the running (1) and to the respective cosmological applications.

Using renormalization group for the effective couplings one can avoid working with explicit non-localities of the effective action and explore interpolation between the UV and IR. In general QG, there are both massive and massless (graviton) degrees of freedom (see, *e.g.*, [23] and the recent textbook [24] for a pedagogical introduction). In the IR, all the massive modes have to decouple and we end up with the effect of only gravitons. Owing to covariance arguments and assuming locality of the effective QG in the IR, the low-energy effects of QG should be described by the quantum GR.

Moreover, in the pure QG based on GR, the logarithmic form factors are much simpler because of the masslessness of the graviton. Thus, the quantum form factors boil down to the expressions of the type $\ln(-\Box/\mu^2)$, much simpler compared to the massive cases. One of the consequences is that, such an effective QG is valid in a wide interval of energy scales, *i.e.*, between the Planck scale in the UV and the Hubble scale in the IR, where the small cosmological constant becomes an IR regulator for the running owing to the massless degrees of freedom [25]. From the viewpoint of the cosmological applications, this means the running of the parameters of the action produced by quantum gravity within the Minimal Subtraction scheme, can be applied to the whole history of the Universe, say between the inflationary and the present-day epochs.

([1]) The precise one-loop expression can be found in many textbook as integral representation and in [20] in the explicit form.

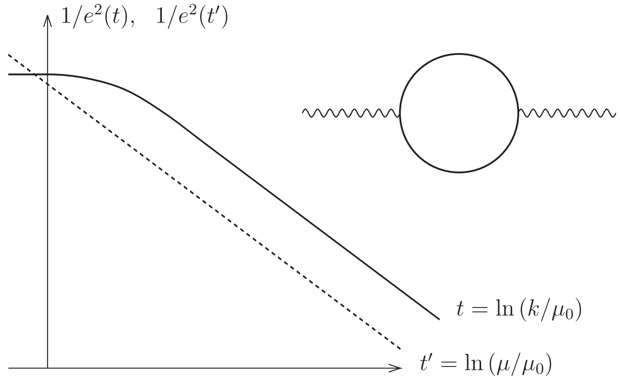

Fig. 1. – The two plots illustrate the difference in the running of the effective charge with $\log(k/\mu)$ in the MS-scheme (dashed line), and in the momentum-subtraction scheme, where we can see with the decoupling in the IR.

Finally, in QED and in the Standard Model, the running is an observable effect. The question is how to apply this idea to QG and what we can learn from doing so.

3. – Gauge invariant renormalizability in QG

Let us start with a brief survey of the gauge invariant renormalizability in quantum gravity. This is, indeed, the fundamental feature of the theory admitting the use of effective approach [26].

Consider a covariant action of gravity

$$S = \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}(g_{\mu\nu}), \tag{5}$$

where $\mathcal{L}(g_{\mu\nu})$ is a scalar Lagrangian that may correspond to Einstein's GR or to another, extended theory of gravity. The gauge transformation $\delta g_{\mu\nu} = R_{\mu\nu,\alpha}\xi^\alpha = -\nabla_\mu\xi_\nu - \nabla_\nu\xi_\mu$ corresponds to the infinitesimal coordinate transformations $x'^\mu = x^\mu + \xi^\mu$.

The gauge invariance of the classical theory means

$$\frac{\delta S}{\delta g_{\mu\nu}} R_{\mu\nu,\alpha} = 0. \tag{6}$$

The gauge invariant renormalizability of QG implies that the same gauge identity is satisfied for the effective action at the quantum level. The first proof of this fact in QG was given by Stelle in 1977 for the fourth-derivative theory [27]. In the recent paper [28] it was generalized (using the Batalin-Vilkovisky technique) to an arbitrary covariant theory of QG (see also [29] for another consideration of the same subject). Also, the textbook-level introduction can be found in [24].

In the effective QG perspective, the gauge invariant renormalizability means that in all loop order, all the divergences in the quantum GR are covariant and local expressions, that can be eliminated by the corresponding covariant local counterterms. Thus, the form of the counterterms is completely defined by the power counting arguments, as described below. On top of that, the dependence on the choice of a gauge fixing condition and the parameterization of the quantum metric can be kept under control by the general theorems [30, 31], as elaborated for QG in [32-35] and, in the general form, in [28, 24].

In details, the gauge fixing and parameterization ambiguities in quantum GR is as follows. Using the Faddeev-Popov approach and the basic parameterization of the metric,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x), \tag{7}$$

we arrive at the total action (using compact DeWitt's notations)

$$S_{tot} = S(h) + \frac{1}{2}\chi^\mu Y_{\mu\nu}\chi^\nu + \bar{C}^\alpha M_\alpha^\beta C_\beta, \qquad \text{where} \qquad M_\alpha^\beta = \frac{\delta\chi^\beta}{\delta h_{\mu\nu}} R_{\mu\nu,\alpha}. \tag{8}$$

The useful choices of gauge fixing conditions and the weight function are the generalized harmonic (Fock-De Donder) gauge $\chi_\mu = \partial^\nu h_{\mu\nu} - \beta\partial_\mu h$ and the more general, background, gauge $\chi_\mu = \nabla^\nu h_{\mu\nu} - \beta\nabla_\mu h$, where $g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$, while the weight operator can be chosen in the form

$$Y_{\mu\nu} = \frac{1}{\alpha} g_{\mu\nu}. \tag{9}$$

In this case, there are two arbitrary gauge-fixing parameters α and β.

The most general (at the one-loop level) background parametrization introduces more arbitrary parameters $r, \gamma_{1,2,\dots,6}$ [35],

$$\begin{aligned} g_{\alpha\beta} \longrightarrow g'_{\alpha\beta} = e^{2\kappa r\sigma}\Big[& g_{\alpha\beta} + \kappa\big(\gamma_1\phi_{\alpha\beta} + \gamma_2\phi g_{\alpha\beta}\big) \\ & + \kappa^2\big(\gamma_3\phi_{\alpha\rho}\phi^\rho_\beta + \gamma_4\phi_{\rho\omega}\phi^{\rho\omega}g_{\alpha\beta} + \gamma_5\phi\phi_{\alpha\beta} + \gamma_6\phi^2 g_{\alpha\beta}\big)\Big], \end{aligned} \tag{10}$$

where $g_{\alpha\beta}$ is the background metric, while $\phi_{\alpha\beta}$ and σ are the quantum fields, and the dimensional coupling is $\kappa^2 = 16\pi G = 16\pi/M_P^2$.

The power counting in QG based on GR is most easily formulated in the case of the basic parametrization (7). For a diagram with an arbitrary number of external lines of $h_{\mu\nu}$ and the number of their derivatives $d(G)$, the superficial degree of divergence $\omega(G)$ is defined by the expression

$$\omega(G) + d(G) = \sum_{l_{int}} (4 - r_I) - 4V + 4 + \sum_V K_V. \tag{11}$$

The first sum is over I internal lines of the diagram, r_I is the inverse power of momentum in the propagator of the given internal line, and V is the total number of vertices. The coefficients K_V represent the numbers of derivatives in the given vertex. In addition, there is the topological relation $L = I - V + 1$, defining the number L of loops.

Our interest in this contribution is the quantum GR with the cosmological constant,

$$S_{EH} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(R + 2\Lambda). \tag{12}$$

In this case, $r_I = 2$ for both quantum metric and ghosts and there are $K_R = 2$ and $K_\Lambda = 0$ vertices. Replacing these numbers into (11), we get

$$\omega(G) + d(G) = 2 + 2L - 2n_\Lambda, \tag{13}$$

where n_Λ is the number of zero-derivatives vertices. Formula (13) is one of the main points of our consideration. This expression shows that

i) Without the cosmological term and the corresponding vertices, *i.e.*, with $n_\Lambda = 0$, there are no logarithmic (with $\omega(G) = 0$) divergences that repeat the form of the classical Lagrangian (12). Instead, all the divergences have higher and higher derivatives, when the number of loops L is growing.

ii) When $\Lambda \neq 0$, the vertices with growing number n_Λ compensate the loop number L. As a result, in all loop orders we meet both cosmological constant and Einstein-Hilbert type counterterms.

iii) The higher loop contributions to the logarithmic divergences are always multiplied by the factors of $(\kappa^2\Lambda)^L$, or, equivalently, of the factors of $(\Lambda/M_P^2)^L$. It is fairly easy to write down renormalization relations for κ^2 and Λ and use them to derive the beta functions for these parameters. Thus, we can conclude that these beta functions are given by the powers series in Λ/M_P^2.

In all known theories, at all energy scales, we have $|\Lambda/M_P^2| \ll 1$. Then, the beta functions for both Newton constant G and the vacuum energy $\rho_\Lambda = \Lambda/(8\pi G)$ have the dominating one-loop parts, while the higher loops are producing very small contributions to the running of the respective parameters. Thus, the main question is whether one can define the one loop beta functions in the non-renormalizable theories in a sufficiently sound way.

4. – Gauge-fixing dependence

The main obstacle to formulate the renormalization group in the effective framework described above is that, for the QG based on GR with the cosmological constant, we meet a serious problem with the gauge- and parametrization ambiguity of the counterterms. The gauge-fixing dependence at the one-loop level was explored directly in several works starting from the pioneer work [36] (see also more detailed calculations in [37-39]). The dependence on the parametrization was explored in [40, 35] and, more recently, in [41] devoted to the effects of QG on the motion of test particles, as suggested in [38]. Let us note that this motion does not depend on both gauge-fixing and parametrization for the reasons explained below.

Indeed, one can explore the mentioned ambiguities without explicit calculations. The point is that, in quantum GR, as in any other gauge theory, the on-shell quantities are well-defined, invariant and universal [28, 30, 31]. At the one-loop level, the on-shell condition has to be derived from the classical equations of motion, in our case for the theory (12). Let us see how it works in practice.

According to the power counting (13), the one-loop divergences have the form

$$\Gamma^{(1)}_{\rm div} = \frac{1}{\epsilon}\int d^4x\sqrt{-g}\{c_1 R^2_{\mu\nu\alpha\beta} + c_2 R^2_{\alpha\beta} + c_3 R^2 + c_4\Box R + c_5 R + c_6\}, \tag{14}$$

where c_k may depend on the gauge-fixing and/or parametrization parameters α_i. Taken the Weinberg's theorem [42], the divergences correspond to the local functional, thus c_k

are c-numbers and $\Gamma^{(1)}_{\rm div}$ is local. Thus, for the two different sets α_i and α_i^0 [35],

$$\Gamma^{(1)}_{\rm div}(\alpha_i) - \Gamma^{(1)}_{\rm div}(\alpha_i^0) = \frac{1}{\epsilon}\int \mathrm{d}^4x\sqrt{-g}(b_1 R_{\mu\nu} + b_2 R g_{\mu\nu} + b_3 g_{\mu\nu}\Lambda)\varepsilon^{\mu\nu}, \tag{15}$$

where $b_k = b_k(\alpha_i)$ depend on the gauge-fixing and parametrization of quantum metric. The classical on-shell is defined by the expression

$$\varepsilon^{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} \sim R^{\mu\nu} - \frac{1}{2}\big(R + 2\Lambda\big)g^{\mu\nu}. \tag{16}$$

Replacing (16) into (15) and comparing the result with (14), the unique two invariant combinations are

$$c_1 \qquad \text{and} \qquad c_{\rm inv} = c_6 - 4\Lambda c_5 + 4\Lambda^2 c_2 + 16\Lambda^2 c_3. \tag{17}$$

This situation gives rise to the well-defined on-shell renormalization group equation for the dimensionless combination $\lambda = 16\pi G\Lambda$ of the parameters G and Λ [32],

$$\mu\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} = -\frac{29}{5}\frac{\lambda^2}{(4\pi)^2}. \tag{18}$$

However, (17) means that there are no well-defined individual running of the physically interesting terms, *e.g.*, ρ_Λ, R, and R^2 in the effective framework based on the conventional perturbative QG. In order to apply the scheme described in the previous section, we need a qualitatively new input. Resolving the problem of gauge-fixing and parametrization ambiguity enables one to use renormalization group in the low-energy QG, that should be an important addition to the existing results in effective approach [13, 43].

5. – Vilkovisky-DeWitt (VdW) effective action in QG

A possible solution of the problem of ambiguity can be based on the Vilkovisky-DeWitt (VdW) scheme for constructing the “unique” effective action. The scheme was suggested by Vilkovisky in [14] and extended by De Witt [15] (see also further discussions in [44-46]). The formalism was applied to quantum gravity in [47] and, in case of the QG theory with a cosmological constant, [16], and more recently in [18, 19]. In particular, [16] and [19] were addressing the renormalization group issue, which we are reviewing here.

The original proposal of [14] makes the one-loop divergences independent on the gauge-fixing and parametrization of the quantum metric. In brief, the construction looks as follows. Consider the parametrization dependence in a non-gauge quantum field model. The one-loop expression for the theory with the classical action $S[\Phi_i]$ has the form

$$\bar\Gamma^{(1)} = \frac{i}{2}\,\mathrm{Ln\,Det}\,S''_{ij}, \qquad \text{where} \qquad S''_{ij} = \frac{\delta^2 S}{\delta\varphi_i\delta\varphi_j}. \tag{19}$$

Changing the variables $\varphi_i = \varphi_i(\varphi'_k)$, we meet

$$\bar\Gamma^{(1)} = \mathrm{Ln\,Det}\Big(\frac{\delta^2 S}{\delta\varphi'_l\delta\varphi'_k}\Big) = \mathrm{Ln\,Det}\Big(S''_{ij}\cdot\frac{\delta\varphi_i}{\delta\varphi'_k}\frac{\delta\varphi_j}{\delta\varphi'_l} + \frac{\delta S}{\delta\varphi_i}\frac{\delta^2\varphi_i}{\delta\varphi'_l\delta\varphi'_k}\Big). \tag{20}$$

One can see that the two one-loop results coincide on the classical equations of motion (on shell), when

$$\varepsilon^i = \frac{\delta S}{\delta \varphi_i} = 0, \tag{21}$$

but are different off-shell. The main idea of the "unique" effective action is to change the definition (19), replacing the usual variational derivatives by the covariant variational derivatives, constructed on the basis of a metric in the space of quantum fields. Then the transformation in (20) becomes of the tensorial type and there is no off-shell difference. It turns out that this scheme really works and the situation is qualitatively the same in the gauge theories, regardless there are serious technical and conceptual complications. The gauge-fixing dependence can be regarded as a particular type of reparametrization of quantum fields. One of the main questions is whether the choice of the metric in the space of quantum fields can affect the result and whether this leads to a new ambiguity (see, *e.g.*, [48]).

6. – Running of G and Λ based on the Vilkovisky effective action

For the effective QG based on Einstein's GR with the cosmological constant, the prescription described above gives the one-loop divergences [16, 18]

$$\bar{\Gamma}^{(1)}_{\rm div} = -\frac{1}{\epsilon}\int \mathrm{d}^4x\sqrt{-g}\left\{\frac{121}{60}C^2 - \frac{151}{180}E + \frac{31}{36}R^2 + 8\Lambda R + 12\Lambda^2\right\}. \tag{22}$$

This, completely invariant, result enables us to construct the renormalization group equations [16, 19]

$$\begin{aligned} \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\left(\frac{1}{16\pi G}\right) &= \frac{8\Lambda}{(4\pi)^2}, \\ \mu\frac{\mathrm{d}\Lambda}{\mathrm{d}\mu} &= -\frac{2}{(4\pi)^2}16\pi G\Lambda^2. \end{aligned} \tag{23}$$

The solutions can be easily found in the form

$$\begin{aligned} G(\mu) &= \frac{G_0}{\left[1 + \dfrac{10}{(4\pi)^2}\gamma_0 \ln\dfrac{\mu}{\mu_0}\right]^{4/5}}, \\ \Lambda(\mu) &= \frac{\Lambda_0}{\left[1 + \dfrac{10}{(4\pi)^2}\gamma_0 \ln\dfrac{\mu}{\mu_0}\right]^{1/5}}, \end{aligned} \tag{24}$$

where $\gamma_0 = 16\pi G_0\Lambda_0^2$ is a dimensionless combination of the initial values of the two running charges. In this way, we arrive at the well-defined running of the Newton and cosmological constants between the Planck and Hubble scales in effective low-energy QG. According to what we explained in the previous sections, the main features of this running are as follows.

i) Universality, that is the independence of the gauge fixing and, in general, of the parametrization of quantum fields.

ii) Let us remember that the higher-loop corrections to eqs. (23) are suppressed by the powers of $\frac{\Lambda}{M_P^2}$. In the present-day Universe this quantity is of the order of 10^{-120}, but even in the inflationary epoch it is at least 10^{-12}. Therefore, these equations describe an effectively exact running, independent on the loop expansion.

Taken that it is certain that below the Planck scale all possible massive gravitational degrees of freedom decouple, the unique assumption that leads to (23) and (24) is that the Vilkovisky's effective action is a "correct" prescription in QG. In our opinion, this is a fundamental assumption that cannot be verified in a purely theoretical framework.

Looking beyond the lowest-order effective framework, the full action of gravity includes higher derivative terms, and can be cast into the form [49, 50]

$$S_{\text{tot}} = \int \mathrm{d}^4x\sqrt{-g}\Big\{ -\frac{1}{\kappa^2}\big(R+2\Lambda\big) - \frac{1}{2\lambda}C^2 + \frac{1}{2\rho}E - \frac{1}{2\xi}R^2 + \sum_{n=1}^{N}[\omega_{n,C}C\Box^n C + \omega_{n,R}R\Box^n R] + \mathcal{O}(R^3_{\dots})\Big\},$$

where λ, ρ, ξ are the dimensionless parameters of the action and $\mathcal{O}(R^3_{\dots})$ stand for the third and higher order in curvatures terms. The renormalization of the higher derivative terms in the effective QG performs similar to the renormalization of the vacuum action of gravity in the semiclassical approach.

Let us consider only one example and refer the reader to [19] for a more complete exposition, including an example of the effectively exact running of the six-derivative coefficient, derived from the two-loop calculations [51].

At the one-loop level we meet an exact beta function for the fourth-derivative term

$$\mu\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} = -\frac{a^2}{(4\pi)^2}\lambda^2, \qquad a^2 = a^2_{\text{QG}} + \frac{1}{5} + \frac{N_f}{10}, \qquad a^2_{\text{QG}} = \frac{121}{30}, \tag{25}$$

where we included the one-loop semiclassical contributions for the sake of completeness and N_f is the number of fermion fields, active (not decoupled) at the corresponding energy scale. The coefficient a^2_{QG} is taken from the expression (22) and is gauge- and parametrization-independent.

7. – On the running vacuum cosmology in the effective QG

We intend to discuss the cosmological applications of the running in effective QG elsewhere and now give just a few general qualitative observations.

Assuming small cosmological constant in the present-day Universe, we obtain

$$\frac{10}{(4\pi)^2}|\gamma_0| \ll 1. \tag{26}$$

On the other hand, we can assume the standard identification of the renormalization group parameter μ with the Hubble parameter, $\mu = H$, in cosmology [2, 6, 7].

In this way, the running solutions (24) lead to the following relations:

$$\frac{1}{G(H)} = \frac{1}{G_0}\left[1 + \frac{8}{(4\pi)^2}\gamma_0 \ln\left(\frac{H}{H_0}\right)\right] \tag{27}$$

and

$$\rho_\Lambda(H) = \rho_\Lambda^0\left[1 + \frac{6}{(4\pi)^2}\gamma_0 \ln\left(\frac{H}{H_0}\right)\right]. \tag{28}$$

At this point we note that the running in the effective QG framework is different from the one in the effective semiclassical approach (2) and (1).

As we have already mentioned earlier, the cosmological and other implications of the running relations (27) and (28) will be discussed elsewhere. Let us just make one important observation. In the semiclassical model of running without the energy exchange between vacuum and matter sectors, (1) can be linked to the running of G by the conservation law [8], that is also an important part of the scale-setting procedure [7] and [9]. It is clear that this is *not* the case for eqs. (27) and (28). The reason is that, along with these two relations, there are also the dependence on μ and, consequently, on H, in the infinite set of higher derivative terms, similar to (25). Thus, the conservation equation should include all these terms too.

On the other hand, the higher derivative terms are strongly Planck-suppressed in the IR (compared to M_P), that is during all the history of the Universe. As a result, we meet a challenging task to construct the cosmological model with a running vacuum which may violate the conservation equation.

8. – Conclusions and discussions

The construction of QG theory which is not restricted to the IR region, is not possible without higher derivative terms. However, at the energies much lower than the Planck scale, the massive modes of gravity decouple and the QG may be described by the quantum GR. We have seen that this fact, together with the Vilkovisky and De Witt version of the "unique effective action" are sufficient to describe the running of cosmological, Newton constant and even of all the parameters of the higher derivative terms in the UV complete action of gravity.

Assuming the Vilkovisky-DeWitt version of unique effective action, we arrive at the well-defined renormalization group equations, which turn out exact, in the sense they are free from the higher-loop corrections. The ambiguity in the definition of the metric in the space of the fields can be fixed by using a reasonable set of assumptions.

Concerning the cosmological and astrophysical applications of the solutions (27) and (28), the perspectives are quite extensive. The main point is that we have, for the first time, the well-defined running of ρ_Λ and G in the effective QG setting, that is based *only* on the three assumptions: 1) accepting the Vilkovisky-DeWitt effective action; 2) assuming that the massive modes in QG decouple in the IR, that is a standard feature to believe it; 3) identification of scale $\mu \sim H$ in cosmology and some working identification (see, *e.g.*, [9, 10]) in astrophysics. In our opinion, those are, by far, the weakest and most "natural" assumptions one can have in the running vacuum models.

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