

Cosmological perturbations in Palatini formalism

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received 4 February 2022

Summary. — In this article, we discuss cosmological perturbations of scalar-tensor theories in Palatini formalism. First we introduced an action where the Ricci scalar is conformally coupled to a function of a scalar field and its kinetic term and there is also a k-essence term consisting of the scalar and its kinetic term. This action has three frames that are equivalent to one another: the original Jordan frame, the Einstein frame where the metric is redefined, and the Riemann frame where the connection is redefined. We calculated the quadratic action and the sound speed of scalar and tensor perturbations in three different frames and show explicitly that they coincide. Furthermore, we showed that for such action the sound speed of gravitational waves is unity. Thus, this model serves as dark energy as well as an inflaton despite the presence of the dependence of the kinetic term of a scalar field in the non-minimal coupling, different from the case in metric formalism. We then proceeded to construct the L3 action called Galileon terms in Palatini formalism and compute its perturbations. We found that there are essentially 10 different (inequivalent) definitions in Palatini formalism for a given Galileon term in metric formalism. We also saw that, in general, the L3 terms have a ghost due to Ostrogradsky instability and the sound speed of gravitational waves could potentially deviate from unity, in sharp contrast with the case of metric formalism. Interestingly, once we eliminate such a ghost, the sound speed of gravitational waves also becomes unity. Thus, the ghost-free L3 terms in Palatini formalism can still serve as dark energy as well as an inflaton, like the case in metric formalism.

1. – Introduction

There have been long attempts to extend general relativity since the Brans-Dicke theory. Such scalar-tensor theories have recently received renewed attention because a scalar dynamical degree of freedom may possibly be the cause of dark energy and well be that of inflation, both of which enjoy strong observational support. In a scalar-tensor theory, a scalar field does not necessarily couple to gravity (a tensor field) minimally but non-minimally. One major example is the Higgs inflation, which was originally proposed by introducing such a non-minimal coupling of a scalar field to the Ricci scalar.

When one introduces such a non-minimal coupling between the scalar and tensor fields, two different approaches are commonly considered in the literature: one is called the metric formalism and the other the Palatini formalism. In the former formalism, an affine connection is not an independent variable in the action but given solely by the metric, that is, one *a priori* decides to use the Levi-Civita connection. On the other hand, in the latter formalism, a connection is regarded as an independent variable in a Jordan-frame action, and is fixed *or solved* through Euler-Lagrange equations, which are given by taking the variations of the action with respect to not only the metric (and matter) but also the connection.

At present, since Einstein gravity fits observations very well, one cannot judge which formalism is adopted by nature. Such judgment can be done only through comparing theoretical predictions in both formalisms with the observations that may arise beyond General Relativity.

For this purpose, people have recently constructed viable scalar-tensor theories and made theoretical predictions for inflation in Palatini formalism and investigated the difference between those in metric and Palatini formalisms. One promising example is Higgs inflation in Palatini formalism. In the case of Higgs inflation, where the Higgs field directly couples to the Ricci scalar, one can always go into the Einstein frame through a conformal transformation. Of course, the correspondence between the Einstein and Jordan frames is different in metric and Palatini formalisms. That is, when we start from an original action in the Jordan frame in general, the corresponding action in the Einstein frame is different between these formalisms. By the use of the standard formulae for inflationary predictions in the Einstein frame, one can make different predictions for metric and Palatini formalisms.

However, in a wider class of scalar-tensor theories, the Einstein frame does not necessarily exist. Therefore, it is quite useful and interesting to discuss cosmological perturbations directly in the Jordan frame in the context of Palatini formalism without resorting to the conformal transformation into the Einstein frame. Furthermore, as far as we are aware of, nobody has yet derived quadratic actions for cosmological scalar and tensor perturbations in the Jordan frame in the context of Palatini formalism of scalar-tensor theory. Such quadratic actions are indispensable for determining normalizations of the perturbations and manifestly give their sound speeds. Thus, the main purpose of this article is to derive quadratic actions for cosmological scalar and tensor perturbations and to discuss their properties based on the quadratic actions directly in the Jordan frame as well as in the Einstein frame where the metric is redefined, and in the Riemann frame where the connection is redefined, in the context of Palatini formalism.

As an example of the absence of the Einstein frame, we consider so-called the Galileon (Kinetic Gravity Braiding) term as well as the non-minimal coupling of the Ricci scalar and the k-essence term in our action. A term corresponding to $\square\phi$ in metric formalism is not uniquely defined in Palatini formalism due to the covariant derivative not being compatible with the metric, *i.e.*, the presence of non-metricity. We have found that there are essentially 10 different (inequivalent) definitions in Palatini formalism for such a term, and have included all of them in our action. In the case of metric formalism, it is known that the sound speed of tensor perturbations (gravitational waves) is still unity even if one includes the Galileon term in an action. Therefore it can serve as dark energy even after the observation of GW170817 and GRB170817A, which strongly constrains the sound speed of gravitational waves. We are going to address whether the Galileon term would modify the sound speed of gravitational waves in Palatini formalism or not, which is crucial for serving as dark energy.

This article is based on the paper [1]⁽¹⁾

2. – K-essence with its non-minimal coupling to Ricci scalar

In this section, we will first consider an action that has a conformally coupled Ricci scalar as follows:

$$(1) \quad S_4^{\text{Jordan}} := \int d^4x \sqrt{-g} \left[G_4(\phi, X) \overset{\Gamma}{R} + K(\phi, X) \right],$$

with $X := -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$.

In the metric formalism, in order to keep the equation of motion second-order, it is known that one must introduce a “counter term”, namely

$$(2) \quad \mathcal{L}_4^{\text{metric}} = G_4(\phi, X) \overset{g}{R} + G_{4X} [(\phi_{\mu\nu})^2 - (\phi_\mu^\mu)^2],$$

where $\phi_\mu := \partial_\mu\phi$ and $\phi_{\mu\nu} := \nabla_\mu \overset{g}{\nabla}_\nu \phi$. However, for the Palatini Lagrangian (1) this counter term is unnecessary to keep the equation of motion to be second-order. Furthermore, as we will mention later, the covariantization of the counter term in Palatini formalism may make the connection dynamical and introduce new degrees of freedom.

This action (1) can be investigated through three methods. In the first method, as most literature considers, one can conduct a conformal transformation of the metric to the Einstein frame and calculate everything there. This method is useful when there exists an Einstein frame. The second way is to directly calculate within the Jordan frame (1). Although tedious, this is the most straightforward method. Finally, another less-known method is to solve the connection, which is non-dynamical, and substitute the solution to the action. The resultant (on-shell) action is written fully in terms of Riemann geometry. This results in an action that is neither Einstein nor Jordan. We shall call this frame, where the connection rather than the metric is redefined, the Riemann frame.

Since all three frames are nothing but (invertible) redefinitions of physical variables, one expects to see that the results of calculations in three different frames coincide. We shall see this in the following sections.

2.1. Analysis in Einstein frame. – Similarly to the usual case of metric formalism, consider the conformal transformation of the action (1) under $\tilde{g}_{\mu\nu} = G_4 g_{\mu\nu}$. The action then becomes

$$(3) \quad S_4^{\text{Einstein}} = \int d^4\tilde{x} \sqrt{-\tilde{g}} \left[\tilde{g}^{\mu\nu} \overset{\Gamma}{R}_{\mu\nu} + \frac{K(\phi, G_4 \tilde{X})}{G_4^2(\phi, G_4 \tilde{X})} \right],$$

where $\tilde{X} := -\frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\partial}_\mu\phi\tilde{\partial}_\nu\phi = G_4^{-1}X$ in which \tilde{x}^μ is the coordinate in the Einstein frame; see below.

This term is none other than the Einstein-Hilbert term plus a k-essence term. Thus, since we are considering no torsion, the connection could be uniquely solved as

$$(4) \quad \Gamma_{\mu\nu}^\lambda = \{\lambda_{\mu\nu}\}_{\tilde{g}} := \frac{1}{2}\tilde{g}^{\lambda\sigma}(\partial_\mu\tilde{g}_{\nu\sigma} + \partial_\nu\tilde{g}_{\mu\sigma} - \partial_\sigma\tilde{g}_{\mu\nu}).$$

⁽¹⁾ See also the references and acknowledgments in ref. [1].

Substituting the solution, we obtain the Einstein-frame action written purely with the conformal metric $\tilde{g}_{\mu\nu}$ as

$$(5) \quad S_4^{\text{Einstein}}|_{\Gamma=\{\}} = \int d^4\tilde{x} \sqrt{-\tilde{g}} \left[\tilde{g}^{\mu\nu} \tilde{R}_{\mu\nu} + \tilde{K}(\phi, \tilde{X}) \right],$$

where $\tilde{R}_{\mu\nu}$ is the Ricci tensor purely defined with the Levi-Civita connection of the new metric $\tilde{g}_{\mu\nu}$. We have also defined $\tilde{K}(\phi, \tilde{X}) = \frac{K(\phi, X)}{G_4^2(\phi, X)}$.

Then, from the standard formulae in the metric formalism, the quadratic action for tensor perturbations \tilde{h}_{ij} is given as

$$(6) \quad \delta^{(2)} S_4^{\text{Einstein, tensor}} = \frac{1}{4} \int d\tilde{t} d^3\tilde{x} \tilde{a}^3 \left[\tilde{h}'_{ij}{}^2 - \frac{1}{\tilde{a}^2} (\tilde{\partial}_k \tilde{h}_{ij})^2 \right],$$

where the prime represents the derivative with respect to \tilde{t} , which is the cosmic time in the Einstein frame, and $\delta^{(2)}$ implies quadratic in perturbation. The sound speed of tensor perturbations, *i.e.*, the velocity of the gravitational waves, is unity. This is in sharp contrast with the case of metric formalism where, in general, the sound speed differs from unity due to the X dependence in G_4 and the associated counter term in eq. (2).

The quadratic action for the curvature (scalar) perturbations $\tilde{\zeta}$ is given as

$$(7) \quad \delta^{(2)} S_4^{\text{Einstein, scalar}} = \int d\tilde{t} d^3\tilde{x} \tilde{a}^3 \left[\tilde{\mathcal{G}}_S \tilde{\zeta}'^2 - \frac{\tilde{\mathcal{F}}_S}{\tilde{a}^2} (\tilde{\partial}_i \tilde{\zeta})^2 \right],$$

with

$$(8) \quad \tilde{\mathcal{F}}_S = \frac{6\tilde{X}K_{\tilde{X}}}{-\tilde{K} + 2\tilde{X}\tilde{K}_{\tilde{X}}} = 2\tilde{\epsilon} = \frac{6X(2KG_{4X} - K_X G_4)}{(K - 2XK_X)G_4 + 3KG_{4X}X},$$

$$(9) \quad \tilde{\mathcal{G}}_S = \frac{6(\tilde{X}\tilde{K}_{\tilde{X}} + 2\tilde{X}^2\tilde{K}_{\tilde{X}\tilde{X}})}{-\tilde{K} + 2\tilde{X}\tilde{K}_{\tilde{X}}} \\ = \frac{6X}{(G_4 - G_{4X}X)^2 \{-K(G_4 + 3G_{4X}) + 2XK_X G_4\}} \\ \times [-6X^2 K G_{4X}^3 + X(8K + 5XK_X)G_4 G_{4X}^2 + (K_X + 2K_{XX}X)G_4^3 \\ - 2\{K(G_{4X} + 2G_{4XX}) + XK_X(3G_{4X} - XG_{4XX}) + X^2 K_{XX} G_{4X}\} G_4^2],$$

where $\tilde{\epsilon} := -\frac{\tilde{H}'}{\tilde{H}^2}$ and all of the quantities on the right-hand sides should be understood as background ones. The sound speed of the curvature perturbations will thus be

$$(10) \quad \tilde{c}_S^2 = \frac{2\tilde{\epsilon}}{\tilde{\mathcal{G}}_S} = \frac{\tilde{K}_{\tilde{X}}}{\tilde{K}_{\tilde{X}} + 2\tilde{X}\tilde{K}_{\tilde{X}\tilde{X}}} \\ = (G_4 - G_{4X}X)^2 (-2KG_{4X} + K_X G_4) \\ \times [-6X^2 K G_{4X}^3 + X(8K + 5XK_X)G_4 G_{4X}^2 + (K_X + 2K_{XX}X)G_4^3 \\ - 2\{K(G_{4X} + 2XG_{4XX}) + XK_X(3G_{4X} - XG_{4XX}) + X^2 K_{XX} G_{4X}\} G_4^2]^{-1},$$

where all of the quantities should be understood as background ones.

2.2. Analysis in Jordan frame. – Let us return to the original Lagrangian,

$$(11) \quad S_4^{\text{Jordan}} = \int d^4x \sqrt{-g} \left[G_4(\phi, X) R + K(\phi, X) \right].$$

By directly solving the equation of motion for the connection, the solution becomes

$$(12) \quad \Gamma_{\mu\nu}^\lambda = \{\lambda_{\mu\nu}\}_g + \frac{1}{2} g^{\lambda\sigma} (2g_{\sigma(\mu} \partial_{\nu)} \ln G_4 - g_{\mu\nu} \partial_\sigma \ln G_4),$$

which indeed coincides with the solution that was obtained in the Einstein frame (4). Notice that when $\partial_\mu \phi = 0$ the connection reduces to that of Levi-Civita, and G_4 becomes effectively Planck mass. As a result, when the dynamics of the scalar ends, such as after inflation, the theory becomes that of Einstein practically.

Substituting the solutions, we obtain the quadratic action for the tensor-perturbations,

$$(13) \quad \begin{aligned} \delta^{(2)} S_4^{\text{Jordan, tensor}} &= \delta^{(2)} \left\{ \int d^4x \sqrt{-g} \left(G_4 R + K \right) \right\}, \\ &= \frac{1}{4} \int dt d^3x G_4 a^3 \left[\dot{h}_{ij}^2 - \frac{1}{a^2} (\partial_k h_{ij})^2 \right]. \end{aligned}$$

Therefore, the sound speed of tensor perturbation is indeed unity, which coincides with the analysis of the earlier section in the Einstein frame. In the similar way, we obtain the quadratic action for the scalar-perturbations as

$$(14) \quad \delta^{(2)} S_4^{\text{Jordan, scalar}} = \delta^{(2)} \left(\int d^4x \sqrt{-g} G_4 R \right) = \int dt d^3x a^3 \left[\mathcal{G}_S \dot{\zeta}^2 - \frac{\mathcal{F}_S}{a^2} (\partial_i \zeta)^2 \right].$$

Here $\mathcal{F}_S = G_4(t) \tilde{\mathcal{F}}_S$ and $\mathcal{G}_S = G_4(t) \tilde{\mathcal{G}}_S$ while $\tilde{\mathcal{F}}_S$ and $\tilde{\mathcal{G}}_S$ being precisely that of (8) and (9) and thus the sound speed c_S coinciding with (10). Both of the quadratic actions are the same with those of Einstein frame up to the conformal redefinition of

$$d\tilde{t} = \sqrt{G_4(t)} dt, \quad d\tilde{x} = dx, \quad \tilde{a}(\tilde{t}) = \sqrt{G_4(t)} a(t), \quad \tilde{h}_{ij} = h_{ij}, \quad \tilde{\zeta} = \zeta.$$

Since the quadratic action of tensor and scalar perturbations of the Jordan frame are precisely that of the Einstein frame, the observables are the same.

2.3. Analysis in Riemann frame. – Instead of considering the Einstein or Jordan frame, one may transform the connection and analyze in a frame where the dynamics of the metric and scalar are equivalent to that in the Jordan frame but written instead in Riemann geometry, *i.e.*, the connection is fixed as Levi-Civita. We shall call this frame the Riemann frame.

First noticing that the connection is not dynamical, one may substitute the solution of the connection (12) to the action (1) This results in

$$(15) \quad \begin{aligned} \mathcal{L}_4^{\text{Riemann}} &= \sqrt{-g} \left[G_4 \overset{g}{R} + \frac{3}{2} \frac{(\overset{g}{\nabla} G_4)^2}{G_4} + K \right] \\ &= \sqrt{-g} \left[G_4 \overset{g}{R} - \frac{3}{2G_4} (2G_4^2 \phi X + 2G_4 \phi G_{4X} \phi^\alpha \phi_{\alpha\beta} \phi^\beta - G_{4X}^2 \phi^\alpha \phi_{\alpha\beta} \phi^{\beta\gamma} \phi_\gamma) + K \right]. \end{aligned}$$

This action is dynamically equivalent to (1). In other words, the metric and the scalar follows the same equation of motion for both Lagrangians (1) and (15). Interestingly, this action is a qDHOST of class ${}^2N\text{-I/Ia}$.

By defining the following functions:

$$\begin{aligned} f &= G_4, \\ \bar{K} &= K - \frac{3G_{4\phi}^2 X}{G_4} + 6X \frac{\partial}{\partial \phi} \int \frac{G_{4\phi} G_{4X}}{G_4} dX, \\ Q &= -3 \int \frac{G_{4\phi} G_{4X}}{G_4} dX, \\ A_4 &= \frac{3G_{4X}^2}{2G_4}, \end{aligned}$$

the action (15) reduces to the DHOST action considered in as

$$(16) \quad \mathcal{L} = fR^g + \bar{K} + Q\Box^g\phi + A_4\phi^\mu\phi_{\mu\nu}\phi^{\nu\rho}\phi_\rho.$$

We find that the quadratic action for tensor perturbations in Riemann frame is

$$(17) \quad \delta^{(2)}\mathcal{L}_4^{\text{Riemann,tensor}} = \frac{1}{8} \int dt d^3x M^2 a^3 \left[\dot{h}_{ij}^2 - \frac{1}{a^2} (\partial_k h_{ij})^2 \right] = \delta^{(2)}\mathcal{L}_4^{\text{Jordan,tensor}},$$

which indeed coincides with the analysis of the Jordan frame. Similarly, the quadratic action for scalar perturbations becomes

$$(18) \quad \begin{aligned} \delta^{(2)}\mathcal{L}_4^{\text{Riemann,scalar}} &= a^3 \frac{M^2}{2} \{ -6(\dot{\zeta} - \beta_1 \dot{\alpha})^2 + 12H[(1 + \alpha_B)\dot{\zeta} - \beta_1 \dot{\alpha}]\alpha \\ &\quad + H^2(\alpha_K - 6 - 12\alpha_B)\alpha^2 + 4[\dot{\zeta} - \beta_1 \dot{\alpha} - H(1 + \alpha_B)\alpha]\beta \\ &\quad + \frac{1}{a^2} [2(\partial\zeta)^2 + 4(1 + \alpha_H \partial_i \zeta \partial_i \alpha + \beta_3 (\partial_i \alpha)^2)] \}. \end{aligned}$$

Variation of β gives $\alpha = \frac{\dot{\zeta}}{H(1 + \alpha_B - \dot{\beta}_1)}$ with $\tilde{\zeta}$ being the redefined variable of $\tilde{\zeta} = \zeta - \beta_1 \alpha$. This results in the quadratic action of

$$(19) \quad \delta^{(2)}S_4^{\text{Riemann,scalar}} = \int d^2x dt a^3 \frac{M^2}{2} \left[A_{\tilde{\zeta}} \dot{\tilde{\zeta}}^2 + B_{\tilde{\zeta}} \frac{(\partial_i \tilde{\zeta})^2}{a^2} \right],$$

with

$$(20) \quad A_{\tilde{\zeta}} = \frac{1}{(1 + \alpha_B - \frac{\dot{\beta}_1}{H})^2} \left[\alpha_K + 6\alpha_B^2 - \frac{6}{a^3 H^2 M^2} \frac{d}{dt} (a^3 H M^2 \alpha_B \beta_1) \right],$$

$$(21) \quad B_{\tilde{\zeta}} = 2 - \frac{2}{aM^2} \frac{d}{dt} \left[\frac{aM^2(1 + \alpha_H + \beta_1)}{H(1 + \alpha_B) - \dot{\beta}_1} \right].$$

After some lengthy computation, we see that this indeed coincides with (8) and (9), *i.e.*, $\frac{M^2}{2} A_{\tilde{\zeta}} = \mathcal{G}_S$ and $\frac{M^2}{2} B_{\tilde{\zeta}} = \mathcal{F}_S$. Furthermore, we see that indeed the computation

done in the Riemann frame is precisely that in the Jordan frame since the quadratic action above is precisely that of the Jordan frame (14).

To conclude, we have seen that the calculations in all frames, namely Einstein, Jordan, and Riemann frames, give the same quadratic actions, power spectra, and sound velocities for both the scalar and tensor perturbations. In the literature, perturbations in the Einstein frame are heavily investigated, due to it being simple and straightforward. However, one may wonder what could be said for theories that do not have an Einstein frame. This we will investigate in the next section.

3. – Possible Galileon terms in Palatini formalism

In this section we will re-think the covariantization of the flat space-time action

$$(22) \quad \mathcal{L}_3^{\text{flat}} = G_3 \eta^{\mu\nu} \partial_\mu \partial_\nu \phi,$$

with $G_3 = G_3(\phi, -\frac{1}{2}\eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi)$.

In the usual metric formalism, the covariant action is straightforwardly obtained and unique, which is,

$$(23) \quad \mathcal{L}_3^{\text{metric}} = G_3 g^{\mu\nu} \nabla_\mu^g \nabla_\nu^g \phi.$$

However, in Palatini formalism, due to the metric incompatibility of the connection, one need to consider additional terms such as

$$(24) \quad \mathcal{L}_3^{\text{Palatini}} = \begin{cases} G_3 g^{\mu\nu} \nabla_\mu^\Gamma \nabla_\nu^\Gamma \phi, \\ G_3 \nabla_\mu^\Gamma (g^{\mu\nu} \nabla_\nu^\Gamma \phi), \\ G_3 \nabla_\mu^\Gamma \{ \nabla_\nu^\Gamma (g^{\mu\nu} \phi) \}, \\ G_3 g^{\mu\nu} g^{\alpha\beta} \nabla_\mu^\Gamma (g_{\alpha\beta} \nabla_\nu^\Gamma \phi), \\ \vdots \end{cases}$$

One may wonder how many possible terms there could emerge, or even if it is finite at all. To write down the possible terms, one must note the following three points.

Firstly, one notices that the covariant derivative ∇^Γ acting on any rank tensor relates to the one of Levi-Civita ∇^g as

$$(25) \quad \nabla^\Gamma = \nabla^g + \text{terms containing } Q_\lambda^{\mu\nu},$$

since any (torsionless) affine connection can always be rewritten as

$$(26) \quad \Gamma_{\mu\nu}^\lambda = \{ \lambda_{\mu\nu} \}_g + \frac{1}{2} (Q_{\mu\nu}^\lambda + Q_{\nu\mu}^\lambda - Q_{\mu\nu}^\lambda)$$

with $Q_\sigma^{\mu\nu} := \nabla_\sigma^\Gamma g^{\mu\nu}$. Secondly, the covariantized terms can be schematically written as

$$(27) \quad G_3 \cdot \nabla^\Gamma \cdot \nabla^\Gamma \cdot \phi,$$

with \cdot representing an arbitrary number of metrics. The first derivative acts either on a metric, a covariant derivative, or ϕ , whereas the second derivative acts on a metric or ϕ . All of the space-time indices must be contracted to make the resultant terms scalars.

With the above in hand, the covariantization in Palatini formalism of the flat action (22) is constructed through all the possible contractions in the form of $\overset{g}{\nabla}\overset{g}{\nabla}\phi$, $Q \times \partial\phi$, $Q \times Q$, $\overset{g}{\nabla}Q$, which are the following 1+9 terms:

- 1) $\overset{g}{\square}\phi$,
- 2) $Q^\mu \partial_\mu \phi$, $\bar{Q}^\mu \partial_\mu \phi$,
- 3) $\phi Q_{\alpha\beta\gamma} Q^{\alpha\beta\gamma}$, $\phi Q_{\alpha\beta\gamma} Q^{\beta\gamma\alpha}$, $\phi Q^\mu Q_\mu$, $\phi Q_\mu \bar{Q}^\mu$, $\phi \bar{Q}_\mu \bar{Q}^\mu$,
- 4) $\phi \overset{g}{\nabla}_\mu Q^\mu$, $\phi \overset{g}{\nabla}_\mu \bar{Q}^\mu$,

where we have defined the Weyl vector and the other trace vector of non-metricity as

$$(28) \quad Q_\mu := \frac{1}{4} Q_{\mu\nu}{}^\nu, \quad \bar{Q}^\mu := Q_\nu{}^{\nu\mu}.$$

Thus the most general Palatini L_3 action consists of 10 different terms and is given as

$$(29) \quad \mathcal{L}_3^{\text{Jordan}} := G_{3,0} \overset{g}{\square}\phi + G_{3,1} Q^\mu \partial_\mu \phi + G_{3,2} \bar{Q}^\mu \partial_\mu \phi \\ + G_{3,3} \phi Q_{\alpha\beta\gamma} Q^{\alpha\beta\gamma} + G_{3,4} \phi Q_{\alpha\beta\gamma} Q^{\beta\gamma\alpha} + G_{3,5} \phi Q^\mu Q_\mu + G_{3,6} \phi Q_\mu \bar{Q}^\mu \\ + G_{3,7} \phi \bar{Q}_\mu \bar{Q}^\mu + G_{3,8} \phi \overset{g}{\nabla}_\mu Q^\mu + G_{3,9} \phi \overset{g}{\nabla}_\mu \bar{Q}^\mu$$

$$(30) \quad =: \sum_{i=0}^9 G_{3,i} \overset{\Gamma}{\square}_{(i)} \phi,$$

where the arguments of all the functions are ϕ and X , *i.e.*, $G_{3,i} = G_{3,i}(\phi, X)$, etc. Under the flat space-time limit of $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$ and $\Gamma_{\mu\nu}^\lambda \rightarrow 0$, the Palatini L_3 action (30) indeed reduces to the flat space-time action (22).

Furthermore, recall that the Riemann tensor is the form of $\overset{\Gamma}{R} \sim \partial\Gamma + \Gamma\Gamma$. Similarly, the terms $Q \times Q$ and $\overset{g}{\nabla}Q$ also inhere such structure of $\Gamma\Gamma$ and $\partial\Gamma$ respectively. One then might guess that these terms might affect the results of the cosmological perturbations significantly, such as the speed of gravitational waves. Especially, one may wonder if the speed of gravitational waves could deviate from unity, different from the case of metric formalism. We shall investigate these issues in the next section.

4. – Tensor and scalar perturbations with the Galileon terms in Palatini formalism

Here, noting the previous section, we consider the following Lagrangian:

$$(31) \quad \mathcal{L}_{3+4}^{\text{Jordan}} := \mathcal{L}_3^{\text{Jordan}} + \mathcal{L}_4^{\text{Jordan}} = G_4 \overset{\Gamma}{R} + K + \sum_{i=0}^9 G_{3,i} \overset{\Gamma}{\square}_{(i)} \phi,$$

where G_4 , K , $G_{3,i}$ are understood to be functions of ϕ and $X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$.

Unlike what was considered in sect. 2, the conformal transformation of this Lagrangian, due to the existence of the L3 term and its connection dependence, does not lead to the Einstein frame. Similarly, analysis in the Jordan frame will be tedious. We therefore shall resort to analysis in the Riemann frame.

The connection for this Lagrangian (31) can be solved as

$$(32) \quad \Gamma_{\mu\nu}^\lambda = \{\mu\nu\}_g + \frac{1}{D}[\{A^X\partial^\lambda X + A^\phi\partial^\lambda\phi\}g_{\mu\nu} + 2\{B^X\partial_{(\mu}X + B^\phi\partial_{(\mu}\phi\}\delta_{\nu)}^\lambda],$$

where A , A^X , B^ϕ , B^X , D are some functions of ϕ and X , and their explicit forms are given in ref. [1]. When $G_{3,i} = 0$, this indeed reduces to (12). Again notice that under $\partial_\mu\phi = 0$ the connection reduced to that of Levi-Civita.

Substituting the solutions of the connection to (31) the Riemann frame of this action after some calculation becomes

$$(33) \quad \mathcal{L}_{3+4}^{\text{Riemann}} = G_4\overset{g}{R} + K + G_{3,0}\overset{g}{\square}\phi + E_{\phi\phi} + E_{\phi X}\phi^\alpha\phi_{\alpha\beta}\phi^\beta + E_{XX}\phi^\alpha\phi_{\alpha\beta}\phi^{\gamma\delta}\phi_\delta,$$

where $E_{\phi\phi}$, $E_{\phi X}$, E_{XX} are some functions of ϕ and X , and their explicit forms are given in ref. [1]. Again, indeed under $G_{3,i} = 0$, this reduces to the action (15).

Unlike the action (1) we previously considered, this action, in general, will have ghost degrees of freedom, namely the Ostrogradsky instability. To eliminate this Ostrogradsky ghost, one must impose the condition

$$(34) \quad E_{XX} = \frac{3G_{4X}^2}{2G_4}.$$

As a result, the theory (31) will have at most 2 tensor and 1 scalar degrees of freedom. This again falls into the qDHOST class of ²N-1/Ia. Thus, similar to sect. 2 the tensor perturbation of this theory, under the condition (34), has the sound velocity of unity, which coincides with that in metric formalism. Thus, contrary to the naive expectation, once one removes the ghost degree of freedom, the L3 terms in Palatini formalism can still serve as dark energy as well as an inflaton.

Furthermore, again, under the redefinition of functions

$$(35) \quad f = G_4, \quad Q = G_{3,0} + \int E_{\phi X}dX, \quad A_4 = \frac{3G_{4X}^2}{2G_4} = E_{XX},$$

the action (33) reduces to the DHOST action (16) which can then be used to estimate the scalar perturbations. Due to tediousness, we shall omit the explicit form of the sound speed of scalar perturbations, however it can be computed from (20) and (21) following the lines of sect. 2.3.

5. – Conclusion and discussions

In this article, we considered the cosmological perturbations of scalar-tensor theories in the Palatini formalism. First of all, we discuss the action (1), where the Ricci scalar is conformally coupled to a function of a scalar and its kinetic term, and there is k-essence

action consisting of a scalar and its kinetic term. We have found that for such a non-minimally coupled theory of (1), there are three (classically) equivalent frames, Jordan, Einstein, and Riemann; have computed their quadratic formulae for tensor and scalar perturbations; and have shown their equivalence. Notably, the tensor modes propagate with the sound velocity of unity, which is different from the metric formalism counterpart. Thus, this model can serve as dark energy as well as an inflaton despite the presence of X dependence in the G_4 term.

Next we considered the extension of the \mathcal{L}_3 terms called Galileon terms to the Palatini formalism as in (31), which does not have an Einstein frame. A term corresponding to $\square\phi$ in metric formalism is not uniquely defined in Palatini formalism due to the covariant derivative not being compatible with the metric, that is, non-metricity. We found that there are essentially 10 different (inequivalent) definitions in Palatini formalism for such a term. By including all of them in our action, we have also computed its perturbations. One might expect that the L3 terms can generate a ghost due to Ostrograsky instability and the sound speed of gravitational waves could potentially deviate from unity, in sharp contrast to the case of metric formalism. However, imposing the ghost-free conditions leads to the speed of the tensor modes to be unity, whereas the scalar-perturbations differ in general. This fact is quite interesting because the ghost-free L3 terms in Palatini formalism can still serve as dark energy as well as an inflaton.

Similar to sect. 3, one may want to consider higher terms associated with the scalar such as $(\nabla^\Gamma \nabla^\Gamma \phi)^2$ to implement \mathcal{L}_4 terms or \mathcal{L}_5 terms. This, however, introduces the kinetic term for the connection in general. Thus the theory will exhibit more than 3 degrees of freedom. This implies that one cannot analyze neither in the Einstein frame nor in the Riemann one. Furthermore, one can say that such theory is similar to quadratic gravity in Palatini/metric-affine formalism, which also has gained increasing interest in recent years. However, these are left for future work.

Finally, we would also like to comment that scalar-tensor theories in Palatini formalism are yet to be fully analyzed. Up to our knowledge, neither generalized scalar-tensor theories with a dynamical nor a non-dynamical connection that admit second-order equations of motion are known. It will be interesting to follow Lovelock's and Horndeski's footsteps to find such a theory. However, this is also left for future work.

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MY greatly acknowledges the organizers Dr. Luca Buoninfante and Dr. K. Sravan Kumar for organizing this wonderful workshop, which is one of the best online workshops he has ever attended.

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