IL NUOVO CIMENTO **45 C** (2022) 31
DOI 10.1393/ncc/i2022-22031-x

Colloquia: QGRAV 2021

Ghosts in ghost-free analytic infinite derivative gravity

Alexey S. Koshelev

Departamento de Física, Centro de Matemática e Aplicações (CMA-UBI), Universidade da Beira Interior - 6200 Covilhã, Portugal

received 4 February 2022

Summary. — In this note we would like to reiterate concisely the problem of ghosts in what seemed to be ghost-free infinite-derivative theories.

Higher-derivative theories often arise in the literature [1-21] and we start with a simple example: a model example providing such a higher derivative modification can be readily written as follows:

$$S = \int \mathrm{d}^4x \left[\frac{1}{2}\phi(\Box - m^2)f(\Box)^{-1}\phi - V(\phi)\right], \tag{1}$$

where $\Box$ is the d'Alembertian operator. This toy example can be traced to string field theory [22-25] which on top of many attractive features contains analytic infinite derivative (AID) operators or form-factors acting on space-time fields. These operators appear analytic at low energies making IR limit easy to discuss. The characteristic scale, the so-called scale of non-locality $\mathcal{M}$ separates local (IR) and non-local (UV) regimes. Usually we set it to unity but in the full picture we should have $f(\Box/\mathcal{M}^2)$.

In this simple example we may have a propagator of an infinite order in derivatives and there is an interaction term which we assume generates a potential bounded from below. These are the features typical for non-local models that have applications to quantum gravity (renormalizable and ghost-free AID theories) [19] and UV finite non-local scalar theories with an arbitrary potential [20, 26, 27]. For the metric signature $(-+++)$ the above Lagrangian describes a normal non-ghost field if $f = 1$ which is moreover not a tachyon for $m^2 > 0$.

To see why SFT plays the role of the most robust motivation for theories with an infinite number of higher derivatives in the form of analytic form-factors, let us see the kind of Lagrangians which arise in the SFT construction [24, 25],

$$L_{\mathrm{SFT}} = \frac{1}{2}\varphi(\Box - m^2)\varphi - V(e^{-\sigma(\Box)}\varphi), \tag{2}$$

where $\sigma(\Box)$ is some polynomial of the d'Alembertian $\Box$ and V is some interaction, polynomial in its argument and the lowest degree of the field φ in V is 3. In other words, V does not contain quadratic in φ terms. Technically V is not an interaction potential anymore since it has clear momentum dependence. In string theory, the scale of non-locality is the string mass which is theoretically bounded by the Planck mass M_P from above.

From the perspective of the latter Lagrangian we have a theory with AID vertices. One can however easily redefine the field as $\phi = e^{-\sigma(\Box)}\varphi$ and move the AID operators to the term quadratic in fields. This yields

$$L = \frac{1}{2}\phi(\Box - m^2)e^{2\sigma(\Box)}\phi - V(\phi)\,, \tag{3}$$

which is exactly Lagrangian (1) where $f(\Box)^{-1} = e^{2\sigma(\Box)}$.

In a well-defined scenario, one should avoid ghost fields and this is one of the guiding principles limiting our AID theory construction. We know starting from the papers by Ostrogradski [1] that higher derivatives generically introduce ghosts. Since the number of degrees of freedom is counted by the number of poles in the propagator one may try to keep only one pole even with extra derivatives in the Lagrangian. Given the construction above this can be made only if the original operator $(\Box - m^2)$ in the quadratic form is multiplied by a function of the d'Alebertian which has no zeros on the whole complex plane. Mathematically the only possibility for such an extra factor is an exponent of an entire function. This being said as long as the ghost absence question is concerned we can consider $\sigma(\Box)$ to be a generic entire function and not only a polynomial. The presented construction can be trivially elevated to models in a curved background by a simple replacement of the flat space-time d'Alembertian by its covariant counterpart. This will not spoil the ghost-free condition in any way.

We stress however that the non-local function itself depends on the particular vacuum of the potential in order to guarantee the absence of ghosts. Namely, consider a potential that has several vacua which moreover arrange different masses to the scalar field. The presented above construction allows having no ghosts only in a single vacuum in which the mass of the field is given by m^2. In other words, in this vacuum $V''(\phi) = 0$.

As long as $V'' \neq 0$ in a vacuum which is true in a generic situation or around a classical background an attempt to linearize equations results in the following operator in the quadratic form:

$$\mathcal{K}_m(\Box) = \frac{1}{2}(\Box - m^2)e^{2\sigma(\Box)} - \frac{1}{2}m^2\,. \tag{4}$$

Here we use $m^2 = V''(\phi)$ with ϕ being the classical background value of the scalar field or simply a non-trivial vacuum. This particular operator results in infinitely many effective local scalar fields with most of them having complex masses squared.

Suppose we consider a theory in which in all vacua $V''(\phi) = 0$. Then we name these new fields effective because they do not belong to any vacuum of the model in a sense that they cannot be created as states in any present vacuum of the theory. Still, there is nothing wrong with linearizing around a non-vacuum point of evolution especially if the effective mass is a slowly varying quantity as it is for any nearly flat potential (for example, during inflation).

To understand why many fields appear it is useful to represent the operator function of the d'Alembertian utilising the Weierstrass product decomposition for an entire function which tells us that any entire function $\mathcal{G}(z)$ can be presented as

$$\mathcal{G}(z) = \prod_i (z - z_i)^{n_i} e^{g(z)}\,, \tag{5}$$

where z_i are algebraic roots of the equation $\mathcal{G}(z) = 0$, n_i is the root's multiplicity and $g(z)$ is some entire function. In the simplest case all $n_i = 1$. The operator $\mathcal{K}$ outlined above is a manifestly entire function. A further justification to make use of this formula in general comes from the fact that at least the string field theory obtained models certainly contain only entire functions of $\Box$ as operators in field quadratic forms. Therefore, in a generic case for models containing the following term quadratic in field:

$$L = \frac{1}{2}\phi\mathcal{G}(\Box)\phi + \dots\,, \tag{6}$$

with an entire operator function $\mathcal{G}(\Box)$ one can easily achieve two things [28,29]:

- first, the free equation of motion can be easily solved. Indeed, it would look like

  $$\mathcal{G}(\Box)\phi = \prod_i (\Box - m_i^2) e^{g(\Box)}\phi = 0\,. \tag{7}$$

  We assume here for simplicity that the root's multiplicity is always one. Then the solution will be

  $$\phi = \sum_i \phi_i \quad \text{where} \quad (\Box - m_i^2)\phi_i = 0\,. \tag{8}$$

  Moreover, since the original model provides a form-factor with real coefficients in its Taylor expansion around zero, all roots are either real or come in complex conjugate pairs.
  For a pair of complex conjugate roots, numbered say i and j, one should consider correlated initial conditions on functions ϕ_i and ϕ_j such that the sum of these functions is zero. This is a condition for a consistent background solution for the original field ϕ which must be real.
  We note here that more than one real m_i^2 will definitely be a ghost-like excitation and is as such a problematic configuration. However, pairs of complex conjugate masses squared are not necessarily bad and may lead to hassle-free models. Notice that in this case one encounters a somewhat non-canonical complex field model which has the following Lagrangian:

  $$L_i = \frac{1}{2}\left[\mathcal{G}'(m_i^2)\phi(\Box - m_i^2)\phi + \mathcal{G}'(m_i^{2*})\phi^*(\Box - m_i^{2*})\phi^*\right]\,, \tag{9}$$

  which cannot be diagonalized in real fields. Factors of $\mathcal{G}'(m_i^2)$ appear upon computing the Weierstrass decomposition.
  The appearance of complex conjugate poles and their interpretation was already discussed in [3]. In a nut-shell, such poles with masses $m = u \pm i\nu$ with $\nu > 0$

would imply causality violation at distances less than $1/\sqrt{\nu}$. Even generically an alarming symptom, it can be safely ignored given that the imaginary part which would cause troubles is large enough compared to physically important scales of the model. On top of this absence of classical growing modes should be guaranteed to claim safely that newly appeared particles do not interfere with the rest of the model. It is important to verify that this wishful expectation holds.

- Second, one can straightforwardly compute residues at all poles of the propagator. The point is to figure out by use of formula (5) that a residue at the point of m_i^2 is given by $1/\mathcal{G}'(m_i^2)$ which is not zero by construction as long as all roots are assumed to be the simple ones.
  One technically important point here is that in principle fields can be rescaled to a somewhat arbitrary number. This implies that unless we clearly understand the field's normalization, or unless there are no eternal guiding principles for doing that, one can always bring the real part of the residue value to be ± 1.

The novel and intuition breaking thing here is that the very situation of changing the number of degrees of freedom dynamically is extremely unusual and is met here only due to higher derivatives. Moreover, the jump is bizarre from a single excitation to effectively infinitely many of them and most of those extra "would be" excitations have complex masses which are totally alien objects in canonical field theory. What is even more curious, neither vacuum of the presented theory can create anything but one real massless particle.

Hence, in principle, we can say that those effective fields are not excitations in any way and as such just stop discussing them. However, since their appearance, even effective, is not a completely understood process, we are going to follow a safer way and show that these "would be" degrees of freedom are screened by having huge masses compared to real physical scales in the model. This follows from the adjustment of the non-locality scale Λ to be heavier than the Hubble scale during inflation. Also we will formulate a condition allowing no classical growing modes for these new modes.

Masses of effective particles are given by roots of an algebraic equation [28]

$$\mathcal{K}_m(m_k^2) = 0\,. \tag{10}$$

What happens is that the masses of the fields are very large compared to the non-locality scale. This is however not enough to be relaxed about their presence, given that half of them are ghosts.

Additionally we want to see that it is possible to have no growing classical solutions for these new effective modes. This simply boils down to solving the free equations of motion of the form

$$(\Box - m_k^2)\phi = 0 \tag{11}$$

for all modes enumerated by k.

Let us start with the background d'Alembertian operator evaluated on the de Sitter background. Even thought the equation looks familiar it gets a new twist because m_k^2 is complex. Given that the Hubble parameter is denoted as H for our background de

Sitter space-time the solution to the latter equation is given by

$$\phi = e^{-\frac{3}{2}Ht}\left(\alpha J_\rho\left(\frac{ka_0}{H}e^{-Ht}\right)+\beta Y_\rho\left(\frac{ka_0}{H}e^{-Ht}\right)\right) \text{ with } \rho = -\sqrt{\frac{9}{4}-\frac{m_k^2}{H^2}}, \tag{12}$$

where J_ρ, Y_ρ are Bessel functions of the first and second kind and a_0 is the normalization of the scale factor in the metric tensor at $t = 0$ and α, β are integration constants. Absence of growing solutions means that for large times t which correspond to small arguments of the Bessel functions both branches with coefficients α and β at most freeze to constants or have non-growing oscillations. This results in the demand that both functions in the solution grow at most as $e^{3Ht/2}$. Series expansion of Bessel functions with index ρ near the origin tells

$$J_\rho(x) \sim x^\rho,\ Y_\rho \sim x^{-\rho} \tag{13}$$

and it follows from here that we are good to go if $\text{Re}(\rho) \le 3H/2$. Upon some algebra one can figure out that this corresponds to

$$(\text{Im}(m_k^2))^2 < 9H^2\text{Re}(m_k^2). \tag{14}$$

This selects the interior of a parabola-shaped domain on the complex plane. All solutions in (10) must satisfy this condition. This condition prompts for a careful choice of an entire function in the exponent of the infinite-derivative operator and we see that a simple choice of a polynomial does not fulfill the formulated requirement. However, known facts in the complex analysis do not impose any restriction to have such a function and one can try to obtain the desired behavior by combining the Cauchy integral representation for holomorphic functions and the Weierstrass decomposition valid for entire functions [30]. In particular one can deduce the following sufficient condition on the entire function in the exponent of the infinite derivative operator: the absolute value of this function should grow to infinity only along the positive real ray and be bounded in any other direction.

Turning to a usually more simple Minkowski background we see that there is no way out of ghosts. H effectively goes to zero and solutions simply become

$$\phi = \phi_{0+}e^{im_kt} + \phi_{0-}e^{-im_kt} \tag{15}$$

meaning that as long as there is an imaginary part in m_k one readily has exponentially growing solutions. One can boldly say that non-locality forbids pure Minkowski space and advocates any non-zero cosmological constant.

An interpretation of these new fields is still unclear, while there are several including reasonably old studies [3, 31] claiming that they do not spoil unitarity at least at scales below the scale of higher-derivative modification $\mathcal{M}$. Another approach designates such fields as totally virtual degrees of freedom and a term fakeon is used with respect to such excitations [12] especially in describing certain aspects of Lee-Wick type models.

What is important to the currently ongoing study of the infinite-derivative gravity [17], this problem is there as well. Let us recall the logic of the non-local gravity: adding higher curvature terms to the Einstein gravity action is not forbidden but generically creates ghosts; however, due to the success of the findings by Stelle we see that the curvature squared gravity is renormalizable; it is however non-unitary due to a ghost and an attempt to cure the issue adding any arbitrary terms shows that only infinite number of derivatives can be combined in a ghost-free Lagrangian.

Now the general problem gets a new turn. Suppose we fix our gravitational Lagrangian, assuming it is a fundamental theory. Then the only a single chosen gravitational background can be made ghost free. Even though we just have seen that one can contain ghosts in some sense around de Sitter space, other backgrounds were not studied yet.

As the only resolution of this puzzle to the moment we leave it for future study.

* * *

AK is supported by FCT Portugal investigator project IF/01607/2015.

REFERENCES

[1] Ostrogradsky M., *Mémoires sur les équations différentielles relatives au problème des isopérimètres*, *Mem. Ac. St. Petersbourg*, **VI** (1850) 385.
[2] Efimov G. V., *Commun. Math. Phys.*, **5** (1967) 42.
[3] Coleman S., *Resonance poles and resonance multipoles*, in *Theory and Phenomenology in Particle Physics*, edited by Zichichi A. (Academic Press, New York) 1969.
[4] Stelle K. S., *Phys. Rev. D*, **16** (1977) 953.
[5] Fradkin E. S. and Tseytlin A. A., *Phys. Lett. B*, **104** (1981) 377.
[6] Kuzmin Y., *Sov. J. Nucl. Phys.*, **50** (1989) 1011.
[7] Krasnikov N., *Theor. Math. Phys.*, **73** (1987) 1184.
[8] Tomboulis E., *Superrenormalizable gauge and gravitational theories*, arXiv:hep-th/9702146.
[9] Brekke L., Freund P. G., Olson M. and Witten E., *Nucl. Phys. B*, **302** (1988) 365.
[10] Vladimirov V., Volovich I. and Zelenov E., *p-adic Analysis and Mathematical Physics* (World Scientific, Singapore City) 1994, `http://dx.doi.org/10.1142/1581`.
[11] Dragovich B., Khrennikov A. Y., Kozyrev S., Volovich I. and Zelenov E., *Anal. Appl.*, **9** (2017) 87, arXiv:1705.04758 [math-ph].
[12] Anselmi D., *JHEP*, **02** (2018) 141, arXiv:1801.00915 [hep-th].
[13] Nicolis A. , Rattazzi R. and Trincherini E., *Phys. Rev. D*, **79** (2009) 064036, arXiv:0811.2197 [hep-th].
[14] Horndeski G. W., *Int. J. Theor. Phys.*, **10** (1974) 363.
[15] Kobayashi T., *Rep. Prog. Phys.*, **82** (2019) 086901, arXiv:1901.07183 [gr-qc].
[16] Biswas T., Gerwick E., Koivisto T. and Mazumdar A., *Phys. Rev. Lett.*, **108** (2012) 031101, arXiv:1110.5249 [gr-qc].
[17] Biswas T., Koshelev A. S. and Mazumdar A., *Phys. Rev. D*, **95** (2017) 043533, arXiv:1606.01250 [gr-qc].
[18] Keltner L. and Tolley A. J., *UV properties of Galileons: Spectral Densities*, arXiv:1502.05706 [hep-th].
[19] Koshelev A. S., Kumar K. S. and Starobinsky A. A., *Analytic infinite derivative gravity, R^2-like inflation, quantum gravity and CMB*, arXiv:2005.09550 [hep-th].
[20] Koshelev A. S. and Tokareva A., *Phys. Rev. D*, **102** (2020) 123518, arXiv:2006.06641 [hep-th].
[21] Boos J., Frolov V. P. and Zelnikov A., *Phys. Rev. D*, **100** (2019) 104008, arXiv:1909.01494 [hep-th].
[22] Witten E., *Nucl. Phys. B*, **268** (1986) 253.
[23] Witten E., *Nucl. Phys. B*, **276** (1986) 291.
[24] Ohmori K., *A Review on tachyon condensation in open string field theories*, arXiv:hep-th/0102085 [hep-th].

[25] Arefeva I. Y., Belov D. M., Giryavets A. A., Koshelev A. S. and Medvedev P. B., *Noncommutative field theories and (super)string field theories*, https://doi.org/doi:10.1142/9789812777317_0001, arXiv:hep-th/0111208 [hep-th].
[26] Pius R. and Sen A., *JHEP*, **10** (2016) 024, arXiv:1604.01783 [hep-th]; *JHEP*, **09** (2018) 122(E).
[27] Koshelev A. S. and Tokareva A., *Phys. Rev. D*, **104** (2021) 025016, arXiv:2103.01945 [hep-th].
[28] Koshelev A. S., *JHEP*, **04** (2007) 029, arXiv:hep-th/0701103.
[29] Koshelev A. S. and Vernov S. Y., *Class. Quantum Grav.*, **28** (2011) 085019, arXiv:1009.0746 [hep-th].
[30] Lasse Rempe-Gillen, *Proc. London Math. Soc.*, **108** (2011) 1193, arXiv:1106.3439 [math].
[31] Hawking S., Hertog T. and Reall H., *Phys. Rev. D*, **63** (2001) 083504, arXiv:hep-th/0010232.