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Perspectives and applications of chiral quantum walks

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Summary. — Continuous-time quantum walks (CTQWs) are quantum systems undergoing a unitary evolution on discrete structures, in analogy to classical random walks on graphs. They are widely studied in quantum information science to model quantum transport, to design quantum algorithms and as a universal paradigm for quantum computation. Although their definition assumes a real, symmetric generator, their generalization to complex, hermitian generators is possible, and in fact it can be uniquely derived from minimal requests on the correspondence with their classical analogues. This leads to *chiral* CTQWs, whose peculiarities are related to the concept of Aharonov-Bohm phases. In this article, we review the ideas behind the generalization of CTQWs to chiral quantum walks and the new features possessed by the latter, providing some examples in which the effect of such phases enhances the performance of these simple, yet powerful quantum models.

1. – Introduction

Quantum walks [1-5] can be thought of as a quantum analogue of classical Markov chains with a finite or countable state space or, perhaps more intuitively, as the quantum version of a classical random walk on a graph. Their evolution can be defined at discrete time steps [6-8] or continuously in time [9]. In the former case, a coin degree of freedom is often employed to provide a non-trivial dynamics for the quantum amplitude over the graph, and one speaks of discrete-time quantum walks (DTQW). Continuous-time quantum walks (CTQWs), instead, are characterized by a continuous, unitary time evolution which, at least in their simplest form, is generated by a Hamiltonian dictated solely by the connections in the underlying graph. The two models are quite distinct in their formalism and properties and, despite the fact that they show similar behavior on certain graphs (*e.g.*, ballistic spread of probability on a line graph), no universal relation

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between them has been proved on non-regular graphs. In this article, we shall focus on continuous-time quantum walks, whose implementation is perhaps more straightforward to imagine [10], since it requires designing a (usually finite-dimensional) quantum system with a given Hamiltonian, as explained in detail in sect. **2**.

Since their first appearance, quantum walks have emerged as an outstanding simple paradigm in quantum information science to study coherent transport of probability on discrete structures, both for quantum technologies and as purported models for excitonic transfer by photosynthetic complexes [11,12], to generalize quantum algorithms (*e.g.*, for the search problem, see [13-15]) and as models for universal quantum computation [16]. In condensed matter physics, CTQWs arise as the generalization of the tight-binding model to non regular structures and their theory allows one to explore the time evolution of particular initial states, often assumed to be localized states on one site, in addition to the more familiar aspects of the band structure.

In sect. **3**, we will consider how they can be generalized by removing the real constraint on the Hamiltonian generator, while keeping a straightforward relation with the topology of the underlying graph. In doing so, we arrive at *chiral* CTQWs, which have additional degrees of freedom that can be interpreted as Aharonov-Bohm phases along the loops of the graph, as detailed in sect. **4**. Having characterized the additional parameters that can be tweaked at will, at least theoretically, in a chiral quantum walk on a given graph, it is natural to ask if these degrees of freedom can help improve the performance of known quantum walk solutions to standard tasks: we will explore this possibility in sect. **5** by providing two instances of chiral evolution that permit directional quantum transport and quantum search to the speed limit, respectively.

2. – Classical and quantum walk on graphs

An undirected, simple graph G is defined by a couple of finite or countable sets (V, E), such that the elements of V are called *vertices* (or *sites*) and the elements of E are unordered pairs of distinct vertices, called *edges*. If every vertex is drawn as a dot, an edge of the graph will be an arc of a curve joining the corresponding distinct dots. Two vertices are called *adjacent* if they are connected by an edge.

We can picture a classical random walk on a graph G with N vertices by considering a particle, or *walker*, which initially sits on a specific vertex of the graph at $t = 0$. Then, during an infinitesimal time lapse $\mathrm{d}t$, the particle can hop to an adjacent vertex with probability $\gamma\mathrm{d}t$, where γ is a transition rate which, for the case of *unweighted* graphs that we shall consider, is assumed to be the same for all edges of G. At each instant in time, the distribution of the particle over the vertices of G can be described by an N-dimensional real vector $\underline{p}(t)$ such that the j-th entry $p_j(t)$ is precisely the probability to find the walker in vertex j at time t. According to the dynamics we just described, it is a simple task to show that the time evolution of this vector will be given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\underline{p}(t) = -\gamma\mathbf{L}\cdot\underline{p}(t), \tag{1}$$

where $\mathbf{L} = \mathbf{D} - \mathbf{A}$ is an $N \times N$ matrix called the Laplacian of G. Here $\mathbf{D}$ is a diagonal matrix such that $[\mathbf{D}]_{jj} = d_j$ is the *connectivity* of vertex j, *i.e.*, the total number of edges connected to it, while $\mathbf{A}$ is the *adjacency matrix* of G, whose entries are either 0 or 1 and $[\mathbf{A}]_{jk} = 1$ if and only if the vertices j and k are connected by an edge in G. By construction, $\mathbf{A}$ is symmetric and traceless, since a simple graph cannot have self-loops.

As a consequence, $\mathbf{L}$ is a *bistochastic* matrix, *i.e.*, the sum of the entries in each row and the sum of the entries in each column is 0. It is worth mentioning that the form of eq. (1) with the Laplacian is strongly constrained by the requirement that the total probability must be conserved and that, apart for the rate γ, the random walk should be specified just by the structure of the graph.

In the spirit of this classical construction, one can define a continuous-time quantum walk on the same graph, following [9], by considering an N-dimensional *complex* vector $\underline{\psi}(t) \in \mathbb{C}^N$ which encodes the quantum *amplitudes* for the walker to be found in each vertex of G, and postulate a Schrödinger evolution which closely mimics eq. (1),

$$i\hbar \frac{\mathrm{d}}{\mathrm{dt}} \underline{\psi}(t) = -\gamma \mathbf{L} \cdot \underline{\psi}(t). \tag{2}$$

Hence, we are considering an N-dimensional quantum system whose evolution is governed by the time-independent Hamiltonian $\mathbf{H} = -\gamma\mathbf{L}$ and for which we have a preferred basis, called the localized basis, whose elements are wavevectors of the form $|j\rangle$, $j = 1, \dots, N$, localized at each vertex j of G. A direct consequence of the fact that such a $\mathbf{H}$ is real is that, given two generic quantum states $|a\rangle$ and $|b\rangle$ for the system (not necessarily localized), the transition probability between them $P_{a\to b}(t) = |\langle b| \exp(-i\mathbf{H}t/\hbar)|a\rangle|^2$ is always time-symmetric, *i.e.*, $P_{a\to b}(t) = P_{a\to b}(-t)$. This, combined with the generally valid equality $P_{a\to b}(t) = P_{b\to a}(-t)$ yields

$$\mathbf{H} = \mathbf{H}^* = \mathbf{H}^T \;\Rightarrow\; P_{a\to b}(t) = P_{b\to a}(t). \tag{3}$$

In particular, transition probabilities between any two sites of a graph will always be symmetric under the exchange of the initial and the final sites. On graphs that possess a reflection symmetry, the evolution will thus be non-chiral whenever the starting condition is on the reflection axis, meaning that it will not distinguish between left or right. This point is most clearly illustrated by an example involving a cycle graph, as the one shown in fig. 1. The entries of the Laplacian for such a graph are given by $\mathbf{L}_{jj} = 2$ and $\mathbf{L}_{j,j+1} = \mathbf{L}_{j,j-1} = -1$, with $j = 1, \dots, N$ and identifying site $N+1$ with site 1 and site -1 with site N, while all the other entries are null. Equation (3) then implies that, starting from the localized state $|1\rangle$ at vertex 1, the probability for the quantum walker to be found at vertex k is equal at all times to the probability to be found at vertex $N-k+2$, for any $1 < k < N/2$. In particular, when N is odd, the probability at any site apart from the initial one can never grow larger than $\frac{1}{2}$, since there is always a symmetric site with identical probability.

Another important point is that whenever the graph is *regular*, *i.e.*, all the vertices are connected to the same number of edges, $\mathbf{D}$ is proportional to the identity matrix, therefore $\mathbf{L}$ and $\mathbf{A}$ will generate exactly the same quantum evolution, apart for an unobservable overall phase. For non regular graphs, instead, there is no *a priori* reason to prefer $\mathbf{L}$ over $\mathbf{A}$ as the generator of the *quantum* evolution, whereas the classical one, as noted before, must be generated by the Laplacian in order to conserve the total probability. In the special instance of cubic lattices, it can be shown that in the continuous limit and for $N \to \infty$ eq. (2) reduces to the Schrödinger equation for a free particle whose mass is inversely proportional to γ, but such a relation is not generically meaningful for non-regular graphs.

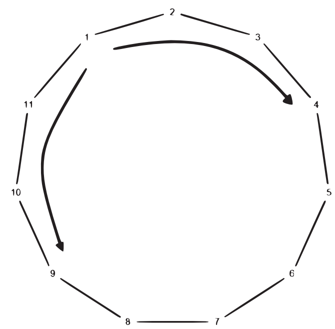

Fig. 1. – Cycle graph with 11 vertices. For a quantum walk starting in a localized state at vertex 1, the probabilities to be observed at vertices 4 and 9 are the same at all times, as a consequence of the time-reversal symmetry of the evolution and the symmetries of the graph.

3. – Generalization to chiral quantum walks

Considering these ambiguities and coming back to the analogy between eq. (1) and a discrete Schrödinger equation, which motivated the association([1]) $\mathbf{H} = -\mathbf{L}$, it is natural to wonder whether this correspondence is forced upon us by general principles or it entails a particular choice among more general relations between the Laplacian of the graph and the generator of the unitary quantum evolution. Indeed we saw that the constraint on the Hamiltonian being real, it bears deep consequences on the quantum evolution, whereas to define a valid unitary evolution, only hermiticity is required for the generator. A reasonable starting point to generalize eq. (2) would be to list all the fundamental requirements for a generic hermitian matrix $\mathbf{H}$ to be the generator of a quantum walk on a graph and then to derive the most general relation between $\mathbf{H}$ and the Laplacian of the graph fulfilling all the constraints. The main ones are the following:

1) In order for $\mathbf{H}$ to reflect the topology of the graph, we should have $[\mathbf{L}]_{jk} \neq 0$ if and only if $[\mathbf{H}]_{jk} \neq 0$ $(j \neq k)$.

2) The equation relating $\mathbf{H}$ and $\mathbf{L}$ could have many hermitian solutions for the matrix $\mathbf{H}$, but one of them must be $\mathbf{H} = -\mathbf{L}$ in order to encompass the usual definition of a continuous-time quantum walk according to eq. (2).

3) Since the off-diagonal matrix elements of the Hamiltonian in quantum mechanics are interpreted as transition amplitude rates for the QW, while those of $-\mathbf{L}$ play the role of transition probability rates for the classical random walk, they should be related by Born's rule.

The solution to these constraints was derived in [17] and reads as follows:

$$[\mathbf{L}]_{kj} = [\mathbf{H}^2]_{jj}\delta_{jk} - \mathbf{H}_{jk}\mathbf{H}_{kj}. \tag{4}$$

Equation (4) possesses several valuable features. Given a generic hermitian matrix $\mathbf{H}$, it yields a unique bistochastic matrix $\mathbf{L}$ which can serve as the generator of a classical

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([1]) From now on, we shall assume $\gamma = \hbar = 1$ for convenience, for example by choosing appropriate units for time.

stochastic evolution according to eq. (1). Moreover, when $\mathbf{H} = -\mathbf{L}$ is a Laplacian of an unweighted graph, eq. (4) is self-consistent, thus recovering the standard definition of a QW. Notice that this relation does not require one to restrict to unweighted graphs, *i.e.*, each link can be weighted differently by the corresponding matrix element of $\mathbf{L}$. Moreover, the previous equation tells us that the diagonal elements $\mathbf{L}_{jj}$ are equal to the fluctuation of the Hamiltonian on the state j, $\mathbf{L}_{jj} = \langle j|\mathbf{H}^2|j\rangle - \langle j|\mathbf{H}|j\rangle^2$ which, together with the stochasticity constraint([2]) $[\mathbf{L}]_{jj} = -\sum_{s\neq j}[\mathbf{L}]_{js} = \sum_{s\neq j}|[\mathbf{H}]_{js}|^2$ implies that these are in turn equal to the sum of the square moduli of the transition amplitude rates between the localized state j and all other sites connected to it, which can be interpreted as the total probability rate for the quantum walker to leave site j: this is in perfect analogy with the standard interpretation of the diagonal elements of the Laplacian for the corresponding classical random walk.

To understand what kind of generalization of the quantum evolution is implied by eq. (4), let us imagine to fix a particular (unweighted) graph G and let $\mathbf{L}$ denote its unique Laplacian matrix. Then for $j \neq k$, the above relation implies $[\mathbf{L}]_{jk} = -|[\mathbf{H}]_{jk}|^2$: this means that $\mathbf{H}_{jk} \neq 0$ if and only if j and k are connected by an edge in G, and, when this holds, the matrix element is a pure phase. Additionally, as discussed above, even when $j = k$ eq. (4) does not involve the diagonal elements of $\mathbf{H}$. Thus we may conclude that, for a given graph, we have in principle a large number of new degrees of freedom for the hermitian generator of a quantum walk on it: we may choose a complex phase for each link and an arbitrary real weight for each diagonal entry of $\mathbf{H}$.

Let us first comment on the role played by the diagonal elements. From the point of view of $\mathbf{H}$, if we attribute a spatial meaning to the localized basis, the diagonal elements naturally take the meaning of a potential energy which can vary from site to site. On the classical side, it is true that one can modify the Laplacian by marking so-called *absorbing* vertices, which can be represented as self-loops in the graph, but this comes at the cost of having a non-symmetric adjacency matrix; otherwise, it is not possible to alter just the diagonal elements of $\mathbf{L}$ without breaking the conservation of probability for the classical evolution. On the quantum side, instead, we just saw that they can be adjusted at will, independently of the Laplacian of the graph. The complex phases that appear in the off-diagonal entries of a generic hermitian $\mathbf{H}$ that solves eq. (4) for a given Laplacian $\mathbf{L}$, instead, are truly new degrees of freedom that appear to be inherently quantum. Continuous-time quantum walks with complex Hamiltonians have been studied in previous works under the name of *chiral quantum walks* [18, 19]. As a matter of fact, these complex phases can break the time-reversal symmetry of CTQWs with real Hamiltonians and also the corresponding chiral symmetry on reflection-symmetric graphs. In the next sections we shall explore how these phases arise, how they affect the evolution of the amplitude for the chiral quantum walker and how they can be harnessed to improve the performance of some CTQWs at certain tasks.

4. – Phases and quasi-gauge transformation

The first crucial observation to understand how the phases in $\mathbf{H}$ can affect a quantum walk comes from the study of tree graphs, *i.e.*, graphs with no loops. Since the off-diagonal elements of the Hamiltonian governing a CTQW identify edges of the graph, we can graphically imagine a phase ϕ_{jk} attached to the edge connecting vertex j to vertex k

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([2]) Here $|\cdot|$ denotes the modulus of a complex number.

such that $[\mathbf{H}]_{jk} = e^{i\phi_{jk}}$ and, since $\mathbf{H}$ must be hermitian, $\phi_{kj} = -\phi_{jk}$: traversing the edge in the opposite direction changes the sign of the corresponding phase. For a tree graph, it is straightforward to show [18] that a generic hermitian $\mathbf{H}$ respecting the graph topology, there will exist a *diagonal, unitary* transformation $\mathbf{U} = \text{diag}\left(e^{i\varphi_1}, e^{i\varphi_2}, ..., e^{i\varphi_N}\right)$ such that

$$\mathbf{U} \cdot \mathbf{H} \cdot \mathbf{U}^\dagger \ = \ \mathbf{D} + \mathbf{A}, \tag{5}$$

where $\mathbf{A}$ is the adjacency matrix of the graph and $\mathbf{D}$ is a diagonal matrix with the same diagonal elements as the original $\mathbf{H}$; from now on, we shall disregard the diagonal elements in our treatment. Equation (5) means that, on a tree graph, all the complex phases can be eliminated by a unitary transformation that acts diagonally in the localized basis. By analogy with the local phase invariance of electromagnetism, we will call them *quasi-gauge transformations*. In counting the number of phases that can be cancelled this way, we see that the phase of each element $|j\rangle$ of the localized basis can be changed independently by the diagonal action of $\mathbf{U}$, but an overall phase in $\mathbf{U}$ does not affect eq. (5); therefore, if G has N vertices, we can cancel $N-1$ phases by such transformations. Indeed, a tree graph with N vertices has precisely $N-1$ edges, so that all their phases can be transformed away with this method. On the other hand, any graph with loops must have more edges than the corresponding tree graph with the same number of vertices, because cutting one edge per loop results in a tree graph; this implies that a generic hermitian matrix $\mathbf{H}$ describing a chiral quantum walk on a graph with loops will have some phases that cannot be cancelled by a quasi-gauge transformation, although they can typically be moved along a loop by such transformations. It is easy to see that these transformations form a quasi-gauge group which is isomorphic to $U(1)^{\otimes(N-1)}$.

The reason to consider quasi-gauge transformations and equivalence classes of hermitian Hamiltonians with respect to them has to do with the central characters in the study of quantum walks. Typically, we are interested in transition probabilities between an initial state $|\psi_0\rangle$ and a final state $|\psi_1\rangle$ after a time t. Considering eq. (5) with $\mathbf{D} = 0$, we have

$$|\langle\psi_1|\exp(-i\mathbf{H}t)|\psi_0\rangle|^2 = |\langle\psi_1|\mathbf{U}^\dagger\exp(-i\mathbf{A}t)\mathbf{U}|\psi_0\rangle|^2 = |\langle\psi'_1|\exp(-i\mathbf{A}t)|\psi'_0\rangle|^2. \tag{6}$$

Therefore, if $\mathbf{H}$ can be brought to the adjacency matrix of the graph by a quasi-gauge transformation, the transition probabilities between generic states for the evolution generated by $\mathbf{H}$ and $\mathbf{A}$ can be obtained one from the other by simply transforming the initial and final states with the same quasi-gauge unitary operation $\mathbf{U}$. Since these act diagonally on the localized basis, transition probabilities between localized states are precisely the same for all Hamiltonians in the same orbit of the quasi-gauge group,

$$\mathbf{H'} = \mathbf{U} \cdot \mathbf{H} \cdot \mathbf{U}^\dagger \Rightarrow P_{j\to k}(t) = |\langle k|e^{-i\mathbf{H}t}|j\rangle|^2 = P'_{j\to k}(t) = |\langle k|e^{-i\mathbf{H'}t}|j\rangle|^2. \tag{7}$$

To sum it up, given a generic graph, only a small number of phases can be attached to its edges in order to have a nontrivial effect on the chiral QW evolution; let N be the number of vertices and $|E|$ the number of edges, then there will be $L = |E| - N + 1$ relevant phases degrees of freedom. If the graph is planar, one can exploit the definition of the Euler characteristic to conclude that L is precisely the number of independent loops, while for non-planar graphs it is less clear how to define independent loops. It has also been shown that two phase configurations on the same graph G, which we may equivalently define as two distinct hermitian solutions of eq. (4) where **L** is fixed to the Laplacian of G and the diagonal elements of the Hamiltonians are assumed to be zero, are related by a quasi-gauge transformation if and only if the sum of the phases along any loop([3]) of the graph has the same value in both configurations [20].

4˙1. *Aharonov-Bohm phases.* – The relation with the electromagnetic field actually goes beyond the quasi-gauge transformations. Indeed, to some extent and at least on certain graphs, it is possible to interpret the chiral quantum walks as an extension of the tight binding model to a quantum particle with charge that is interacting with an external, classical electromagnetic field. The phases attached to hopping terms are known as *Peierl's phases* in condensed matter physics [21], where they are widely employed as a tool to introduce the effect of a static magnetic field. A tidy explanation on how they arise is present in one of *Feynman's Lectures on Quantum Mechanics* [22], where Peierl's phases are derived by discretizing the Schrödinger equation for a charged particle in a classical electromagnetic field. Feynman's example is based on a one-dimensional lattice which cannot support nonzero magnetic fields. However, it can be extended to cubic lattices in 2 and 3 dimensions and also to show the converse, *i.e.*, that the *only* well-defined continuum limit for a chiral quantum walk on a cubic lattice in 2 and 3 dimensions reproduces the Schrödinger equation with the minimal coupling prescription [17,23]. Formally speaking, the argument of these complex phase factors is the integral of the gauge potential along the path joining the two vertices of an edge and therefore, when computed along a closed loop, it bears the meaning of an Aharonov-Bohm phase (ABP) [24]. According to the standard treatment of the ABP, only the line integral of the vector potential along a closed loop is physically well-defined, being gauge-invariant and observable. This makes it clearer why graphs with loops are essential in order to have truly chiral quantum walks. A renowned example that is encompassed by chiral quantum walks is Hofstadter's butterfly [25]. This fractal, depicted in the right panel of fig. 2, shows the spectrum of the Hamiltonian for the tight-binding model on an infinite square lattice immersed in a perpendicular, homogeneous magnetic field (left panel of fig. 2). The flux of the magnetic field per cell of the square lattice is plotted along the horizontal axis of the right panel of fig. 2, while the vertical axis shows the energy levels.

5. – Examples of chiral quantum walks

One of the major differences of chiral quantum walks with respect to CTQWs generated by the Laplacian (or by the adjacency matrix) is that the second part of eq. (3) is no longer valid, because **H** is not real anymore. This means that, when an overall nonzero phase is present along the cycle in fig. 1, the probability for the quantum walker starting at vertex 1 to be found at vertex 4 can be different from the one at vertex 9 and thus they can exceed $\frac{1}{2}$ at some times thanks to chiral effects. However, the addition of nontrivial phases along loops does not necessarily imply time-reversal symmetry breaking nor chiral symmetry breaking. In fact, on even cycles, although the dynamics is still strongly affected by the introduction of a phase along the loop, it will remain non-chiral. In general, all bipartite graphs will remain time-symmetric (*i.e.*, respecting the second part of eq. (3)) with any phases configuration attached to their edges [18].

([3]) By considering all possible loops, we are not burdened by the difficulty in defining which of them are *independent* when the graph is non-planar.

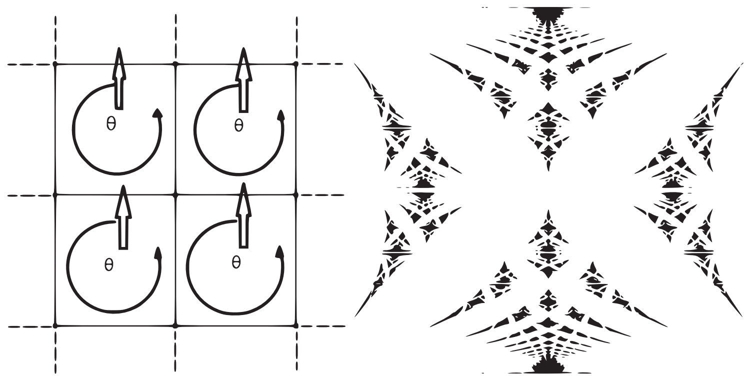

Fig. 2. – On the left, a square lattice in two dimensions, with an outgoing magnetic flux θ per cell. On the right, the celebrated Hofstadter's butterfly fractal, representing the energy spectrum (vertical direction) for a single quantum walker on such a graph, as a function of the magnetic flux (horizontal direction).

One can wonder whether these differences can be of any use at improving the performance of quantum walks for certain tasks. A full characterization of the effect of phases on spectra, quantum transport properties and quantum search is still missing, although some general algebraic results have been derived (see, *e.g.*, [26]). Here we will focus just on two examples which clearly illustrate the potential benefits of exploiting phases.

5˙1. *Quantum switches.* – Considering the graph on the left panel of fig. 3, when the phase on the central loop is zero, the transition probabilities $P_{1\to 3}(t)$ and $P_{1\to 5}(t)$ must be equal at all times, therefore they are again upper bounded by $\frac{1}{2}$. On the other hand, for an optimal value of the overall phase $\varphi = \frac{\pi}{2}$ as indicated by the curly arrow, directional quantum transport is possible and the first peak of $P_{1\to 3}(t)$ can grow as large as $\simeq 0.88$ for $t \simeq 2.63$, while $P_{1\to 5}$ drops to nearly zero at the same time.

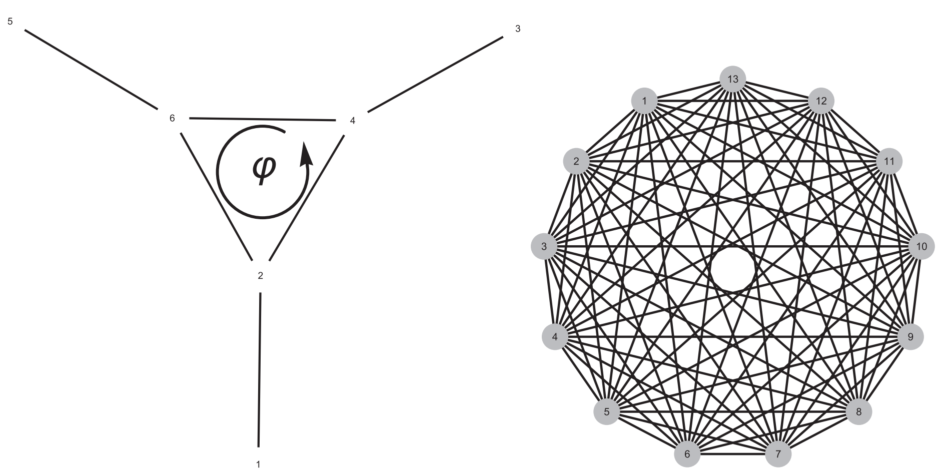

Fig. 3. – On the left, the quantum switch with phase φ. On the right, the complete graph with 13 vertices.

5˙2. *Quantum search at the quantum speed limit.* – A celebrated result in the theory of quantum walks is a version of Grover's algorithm to search for a marked entry in an unstructured database of dimension N. The database is represented by a complete graph with N vertices (second panel of fig. 3) in the CTQW implementation, and Grover's Hamiltonian equals the Laplacian plus a term proportional to the projector on the target site, the so-called *oracle*. By optimizing the weight in front of the projector, the flat state $|f\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} |j\rangle$, which is the coherent superposition of all localized states, will evolve into a state localized in the target vertex in a time $t_G = \frac{\pi}{2\sqrt{N}}$. As such, Grover's algorithm is not chiral; in [27] a chiral CTQW on a complete graph without an oracle is presented, and it is shown that it searches for the target vertex in a time $t_L = \frac{1}{\sqrt{N-1}} \arccos \frac{1}{\sqrt{N}} \leq t_G$ and moreover it equals the ultimate quantum speed limit for quantum walks on complete graphs. The absence of an oracle is compensated by a structured configuration of phases, which breaks the symmetry of the complete graph in favour of the target vertex.

5˙3. *The quantum-classical distance.* – In order to investigate the phase configurations when the space of parameters is large, it is worth comparing the chiral evolutions with some reference evolution on the same graph. Having in sight a possible quantum advantage as a goal, a candidate for the evolution to which one should compare the chiral CTQWs is the classical one. Heuristically, if anything outstanding happens during the quantum evolution, it should manifest itself through a large departure from the behaviour of the classically evolving random walk. A convenient figure of merit to quantify this difference was introduced in [28] under the name of *quantum-classical distance* (DQC). In [27] the maximization of the DQC at short times over all possible chiral evolutions is carried out for the examples cited above. For the quantum switch, it yields the optimal phase value of $\frac{\pi}{2}$, while for the complete graph it recovers the phases configuration that achieves quantum search at the quantum speed limit.

6. – Conclusion and outlook

We have seen that chiral quantum walks establish a simple and natural generalization of continuous-time quantum walks to generic hermitian Hamiltonian generators compatible with the topology of the underlying graph. The new effects allowed by the phases appended to the edges of a chiral CTQW can be harnessed for directional quantum transport, improved quantum search and fast uniform mixing. We also reviewed the connection between these phases and the Aharonov-Bohm phases that arise in the context of quantum particles interacting with classical magnetic fields. However, a more thorough analysis of this comparison should be carried out for non-planar and irregular graphs, for which there exists no concept of a continuum limit, and the possibility of implementing these additional degrees of freedom with the current experimental protocols to simulate quantum walks should be further studied in future developments on this line of research. On the theoretical side, the design of optimal graph topologies on which phases can be exploited to achieve better quantum transport is still an open problem. It would be helpful to characterize what kind of tasks, *e.g.*, quantum search or targeting, that are unattainable with ordinary CTQWs on particular families of graphs become achievable with chiral quantum walks. Another very prominent issue yet to be addressed is the robustness to noise, which could affect the couplings between sites as well as the on-site potential, encoded in the diagonal elements of $\mathbf{H}$, and the magnetic fluxes through the loops, generating the phases of $\mathbf{H}$. Further research will clarify if the chiral symmetry breaking effects can withstand this kind of noises and could thus actually be observed in practical implementations.

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