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A pseudo-spectral numerical approach to solve the Einstein field equations

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Summary. — In this paper, we present a numerical study of the Einstein field equations, based on the $3+1$ foliation of the spacetime. A pseudo-spectral technique has been employed for simulations in vacuum conditions, within the formalism of Baumgarte-Shapiro-Shibata-Nakamura (BSSN). We use the Spectral-FIltered Numerical Gravity codE (`SFINGE`), a numerical code based on the Fourier decomposition, accompanied by different filtering techniques. The accuracy of the model has been validated through standard testbeds, revealing that the filtered pseudo-spectral technique is incredibly accurate. We evolved black hole dynamics in vacuum conditions, in small domains, making use of hyperviscous dissipation that suppresses spurious boundary problems. This simple algorithm can be applied to a variety of gravitational problems, including those related to massive objects dynamics.

1. – Introduction

Because of the intrinsic nonlinearity and complexity of the Einstein field equations, different forms of the governing equations have been proposed and different numerical strategies have been developed, in a variety of initial conditions known as *gravitational testbeds* [1-3]. The common approach to the Einstein equations relies on slicing the four-dimensional spacetime into three-dimensional spacelike hypersurfaces. From a practical point of view, it reduces to a decomposition of the spacetime into "space" + "time", so that one manipulates only time-varying tensor fields in the three-dimensional space.

Here we use the Baumgarte-Shapiro-Shibata-Nakamura (BSSN) decomposition [4-7]. Our numerical code is based upon a Fourier pseudo-spectral method, in concert with filtering techniques. The main advantage of this approach is the accuracy of the solutions due to the spectral projection, while its limitation relies on the imposition of periodicity at the boundaries. For singular spacetimes when the periodicity is violated, we propose a new strategy based on hyper-viscous dissipation in order to suppress spurious boundary problems.

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2. – The numerical model

The BSSN decomposition is a formalism based on the work by Baumgarte and Shapiro [4] and by Shibata and Nakamura [6]. We follow the approach of Campanelli *et al.* [8], and we start from a conformal rescaling of the physical metric, namely $\widetilde{\gamma}_{ij} = \chi\gamma_{ij}$, where χ is a conformal factor. In the BSSN approach, the extrinsic curvature K_{ij} is divided in two independents variables, namely the trace K and its trace-free parts A_{ij}. The last one is subjected to the same conformal transformation of the metric, *i.e.*, $\widetilde{A}_{ij} = \chi\big(K_{ij} - \frac{1}{3}\gamma_{ij}K\big)$. Finally, a new field is introduced, namely the contracted Christoffel symbols associated with the conformal metric, namely $\widetilde{\Gamma}^i = \widetilde{\gamma}^{jk}\widetilde{\Gamma}^i_{jk}$.

With the above change of variables, the system of BSSN equations reads

$$(\partial_t - \mathcal{L}_\beta)\widetilde{\gamma}_{ij} = -2\alpha\widetilde{A}_{ij}, \tag{1}$$

$$\partial_t\chi = \frac{2}{3}\chi(\alpha K - \partial_k\beta^k) + \beta^k\partial_k\chi, \tag{2}$$

$$(\partial_t - \mathcal{L}_\beta)\widetilde{A}_{ij} = \chi\Big[-D_iD_j\alpha + \alpha R_{ij}\Big]^{TF} + \alpha\Big(K\widetilde{A}_{ij} - 2\widetilde{A}_{ik}\widetilde{A}^k_j\Big), \tag{3}$$

$$(\partial_t - \mathcal{L}_\beta)K = -D^kD_k\alpha + \alpha\Big(\widetilde{A}_{lm}\widetilde{A}^{lm} + \frac{1}{3}K^2\Big), \tag{4}$$

$$\begin{aligned}\partial_t\widetilde{\Gamma}^i = {}& \widetilde{\gamma}^{lm}\partial_l\partial_m\beta^i + \frac{1}{3}\widetilde{\gamma}^{il}\partial_l\partial_m\beta^m + \beta^k\partial_k\widetilde{\Gamma}^i - \widetilde{\Gamma}^k\partial_k\beta^i + \frac{2}{3}\widetilde{\Gamma}^i\partial_k\beta^k \\ & -2\widetilde{A}^{ik}\partial_k\alpha + \alpha\Big(2\widetilde{\Gamma}^i_{lm}\widetilde{A}^{lm} - \frac{3}{\chi}\widetilde{A}^{ik}\partial_k\chi - \frac{4}{3}\widetilde{\gamma}^{ik}\partial_kK\Big).\end{aligned} \tag{5}$$

In addition to the evolution equations, the BSSN variables must also obey to the constraints equations, that reads

$$0 = \mathcal{H} = R - \widetilde{A}_{lm}\widetilde{A}^{lm} + \frac{2}{3}K^2,$$

$$0 = \mathcal{M}^i = \partial_k\widetilde{A}^{ik} + \widetilde{\Gamma}^i_{lm}\widetilde{A}^{lm} - \frac{3}{2\chi}\widetilde{A}^{ik}\partial_k\chi - \frac{2}{3}\widetilde{\gamma}^{ik}\partial_kK.$$

The specific choice of evolution variables introduces five additional constraints, namely

$$1 = \det\widetilde{\gamma}_{ij}, \tag{6}$$

$$0 = \mathrm{tr}\widetilde{A}_{ij}, \tag{7}$$

$$0 = \mathcal{G}^i = \widetilde{\Gamma}^i + \partial_j\widetilde{\gamma}^{ij}. \tag{8}$$

3. – Numerical testbeds

The `SFINGE` code makes use of standard Cartesian Fast Fourier Transforms (FFTs) and has been associated to a Runge-Kutta (RK) method for the time integration. We performed direct numerical simulations via standard "apples with apples" tests [9]. In particular, we tested the Gauge wave, the robust stability test, the linearized gravitational wave, and the Gowdy spacetimes [10, 11]. For each initial data, we monitored the accuracy of the code by inspecting the conservation of several quantities. Regarding the spatial integration, we showed how to stabilize the pseudo-spectral code by varying our spectral filter by changing both the filter shape and the k^*, namely the k-vector cut-off in the Fourier space. For the time integration, in each test, we applied the Running Stability Check (RSC) method, namely an adaptive time refinement in order to choose, in a self-consistently way, the best time-step of integration during the simulation. In the singular spacetimes, we introduce a new method in order to treat the boundary conditions, based on the use of absorbing hyperviscous boundaries, namely the Implicit Hyperviscous Boundary method (IHB). For details of the code see [12]. In this paper, we show a more stressful condition, *i.e.*, the Head-on collision of two equal masses black holes.

3˙1. *Head-on collision.* – We present here results about the head-on collision of two equal-mass Misner-Wheeler-Brill-Lindquist (MWBL) black holes using the puncture approach [8]. Both the ADM linear and angular momentum are set to zero, this means that the black holes start without boost and spin. The extrinsic curvature is set initially to zero, and the conformal factor that is initialized as

$$\chi = \left[1 + \frac{m_1}{2|\mathbf{r} - \mathbf{r}_1|} + \frac{m_2}{2|\mathbf{r} - \mathbf{r}_2|}\right]^{-4}. \tag{9}$$

In this case, the 3D computational domain extends over $x, y, z \in [0, 25]$. We set the initial location of the j-th puncture and the mass parameters respectively $\mathbf{C}_j = \{12.5, 12.5 \pm 2, 12.5\}$ and $m_j = 0.5$. We use the harmonic slicing to evolve the lapse, and the "Gamma

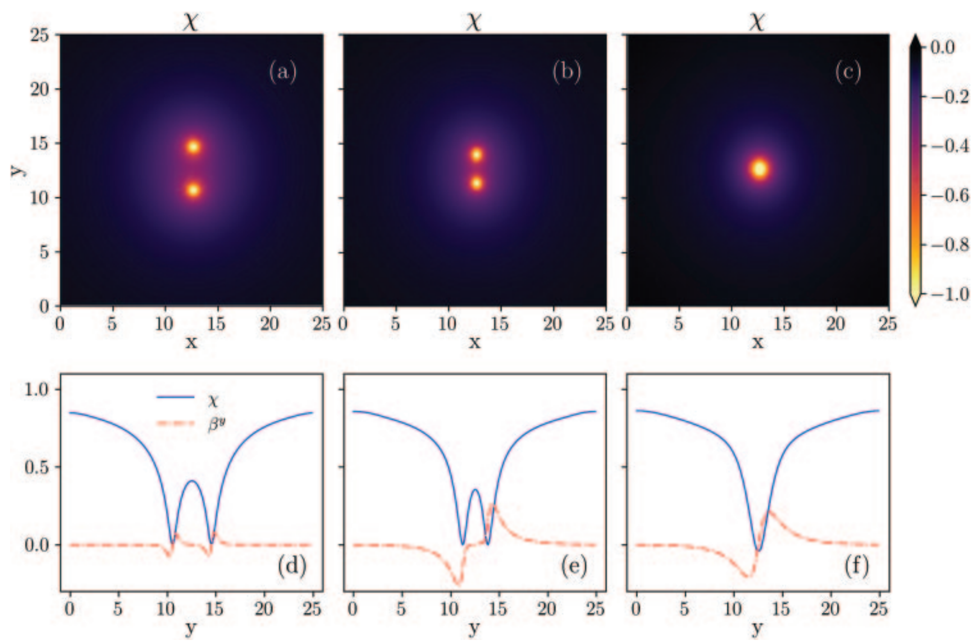

Fig. 1. – Head-on collision with a stabilizing procedure. Top row: xy-plane at time $t = 1M, t = 13M$ and $t = 26M$ for the conformal factor χ. Bottom row: 1D cut in the y-direction for χ and β^y, at same times. The field remains well-behaved at the boundaries. The merger occurs at $t{\sim}21M$ and the simulation is carried out until $t = 50M$. (Figure taken from Meringolo *et al.* [11].)

driver" condition to evolve the shift [13]. We have shown in [12] that with the spectral filter and the IHB condition, the code is able to evolve for long times singular spacetimes. The goodness of our approach is confirmed from a small violation of constraints, even for many crossing times after the merging of the black holes (not shown here). In the top row of fig. 1 is reported a section in the xy-plane (at $N_z/2$) of the conformal factor χ, at three times, namely $t/M = 1, 13$ and 26. Note that the merging occurs at $t \sim 21M$, even though the simulation is carried out until $t = 50M$. In the bottom row, we show, at the same times, 1D cuts in the y-direction (at the middle of the lattice) of the conformal factor χ and the shift β^y. For completeness, we monitor the evolution in time of all the BSSN constraints during the simulation (see [11] for details).

The `SFINGE` code, with the above filtering and boundary treatments, is then able to handle such strong gravitational dynamics. A similar strategy can be used for a variety of studies, including the inspiraling binaries and the multiple black holes systems [14], which will be presented in future works.

4. – Conclusions

We have studied the evolution of gravitational fields by solving numerically the Einstein field equations, in vacuum conditions using the `SFINGE` code. Our numerical method is based on a pseudo-spectral technique, where we compute spatial derivatives via accurate Fast Fourier Transforms. Via these typical initial data, we have validated the goodness and the robustness of our code. For inhomogeneous metrics, as in the case of the head-on collision of two black holes, the code relies on a novel technique, namely the IHB method. This strategy is able to suppress both spurious boundary effects and aliasing phenomena. Future works will concentrate on the approach of the present algorithm to the dynamics of in-spiraling binary systems, as well as to the dynamics of many-body problems. Finally, we plan to export the method to the solution of the gravitational fields in presence of matter as well its coupling with electromagnetic fields, in the framework of general relativistic magnetohydrodynamics.

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