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Inducing the construction of formal axioms of Quantum Mechanics and fostering their comprehension by high school students: The effectiveness of a conceptual approach

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Summary. — Taking as a paradigmatic example an educational reconstruction of Quantum Mechanics (QM), we discuss the approach and general research lines followed by the Physics Education Research Group of the Universities of Milan and Rome. Our choice is prompted by the fact that through QM students can lay out the structure of a new grammar, which is necessary for the presentation of any quantum theory (in particular, Quantum Field Theory) in high school. In particular, we discuss the results of a 15-hour pilot experimentation made in A.Y. 2021–2022, with high school students and teachers. The results obtained are highly encouraging since they appear to indicate that the introduction of formal aspects of QM in Italian high school is possible with more than satisfying learning outcomes.

1. – Introduction

The aim of this work is to present a paradigmatic example of an educational reconstruction of Quantum Mechanics (QM) for high school [1,2] within the general framework and research approach context in Quantum Physics (QP) of the Physics Education Research Group (PERG) of the Universities of Milan and Rome.

QP (meaning the general complex of any formulation of a quantum theory) constitutes the theoretical paradigm for the description of the world and, together with Relativity, poses the basis for the most powerful physics theory we have: the Standard Model. For this reason, it is fundamental in creating not only the theoretical framework of our understanding of physics, but also of our *weltanschauung*. Furthermore, it influences

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practically every aspect of our life (technologies, biology, medicine...) and is, therefore, a socially relevant theme.

The importance of understanding QP for the education of future citizens [3] has prompted, from the early 2000s, deep reforms within most school systems, making the teaching of QP essentially compulsory in Europe, America and Asia [4, 5]. These reforms led to a fast increase in research and publications: between 2000 and 2021, more than 1500 papers were published [6]. Various projects were launched. In Europe, the "Quantum Technologies Flagship" (QFlag) [7] and the QTEdu (Coordination and Support Action for Quantum Technology Education) were created [8]. The purpose of the QTEdu is to assist QFlag with the creation of the learning ecosystem necessary to inform and educate society about quantum technologies. In the USA [9], the National Quantum Initiative was proposed, to increase the development of quantum technologies and teaching methodologies useful for making QP accessible to students. Furthermore, in 2019, a community on teaching and learning QP in high school was created and became a GIREP (Groupe International de Recherche sur l'Enseignement de la Physique) thematic group [10, 11].

Despite such great work, there is no general consensus about the contents to be presented and the approaches to be adopted [12]. Therefore, further efforts are needed to obtain a didactic reconstruction effective in leading students to an adequate conceptual change —*i.e.*, a gradual replacement of what they think with the explanatory framework linked to the structure of QP through the establishment of new ontological categories. For a conceptual change to be possible, coherence in the path is mandatory. Therefore, it is necessary to take a precise physical theory as a reference paradigm; in fact, concepts find their meaning only within the conceptual network that is proposed by a given theory; otherwise, they are rather interpretative schemes or, in the worst case, confused ideas [13, 14].

Precisely to foster conceptual change, any educational reconstruction of QP has to involve many aspects, and not only the inductive-experimental and logical-rational ones used in physics courses. In general, disciplinary structured knowledge presents topics in an already conceptually "cleaned" and "simplified" way. In an active-learning approach to the physical theme and to its modeling, these cleanliness and simplicity are the result of a modeling work [15, 16]. The choice of what must be eliminated in order to obtain simplification is a difficult and long process, that must be slowly learned. In this sense, simplification is the result of a long learning process and not the starting point. Result from which to start again to achieve an adequate formalization of the problem.

It follows that various levels of interventions are needed: in formal contexts (essentially at school), but also in non-formal ones (such as with exhibits and open labs for secondary students), and in informal contexts in society (such as with scientific theatre) [17]. In fact, taking care of the aspect and of the image that the subject has in society is of fundamental importance: proposing stories for children, scientific theatre plays, videos, science exhibits, games, technical-educational-informative books, comics, cartoons, social activities, is overall useful in creating the cultural environment conductive to the appropriation of a theory. Moreover, concerning formal approaches, we believe that disciplinary teaching should take advantage of the introduction of elements drawn from the history of physics, and also by other disciplines: readings of suitably chosen works by poets and philosophers, and discussions of pictorial and musical works are, in fact, important for the appropriation of physics concepts themselves, especially when teaching is conceived for a cultural understanding of QP.

2. – Physics Education and Quantum Mechanics: the international educational context

From a gnoseological point of view, it is important to observe that there are many different formulations of QM [18] which bring different ontological images with them. This fact has important educational consequences, as different approaches might have different impacts on students thinking. When interpretations of experiments and concepts are not sufficiently discussed, students develop their personal images; these images are often scientifically unreliable, and also not completely related to the formulation presented [12, 19, 20].

While, in general, the various PERGs work for an introduction of QM in schools (albeit with very different approaches), what is essentially proposed in class, and in textbooks [5] is a "standard" pseudo-historical treatment, which presents the "crisis of classical physics" [5] and the attempts to overcome it through well-known *ad hoc* recipes, producing deep misunderstandings and making QP confusing, obscure and somewhat incomprehensible [13, 21, 22]. Teachers are not of great help, since they are often unaware of the methodological and epistemological shift which is necessary to pass from Old Quantum Theory (OQT) to QM. Indeed, they lack the tools and resources to face a coherent path of QM; hence the importance and the urgency for the PER community to prepare research-based projects and paths. As regards literature, we can consider three main types of approaches to teaching QM in high school [12, 23]:

1) Historical approaches, based on the rational reconstruction of the historical development of ideas through the presentation of particularly important experiments and their first interpretation leading to a reconstruction of the genesis of OQT;

2) Formal approaches, substantially based on wave mechanics (Schrödinger's approach or Feynman's sum over paths formalism) or on QFT [24, 25];

3) Conceptual approaches, which try to make plausible the description of formal aspects; most of them follow the Dirac formulation of QM [12].

Within these approaches, the strategies proposed by the various PERGs are very different and, probably, there is not an ideal one that addresses all learning problems in the simplest way. For instance, some are based on the Feynman path-integral formulation [20, 26, 27], others, instead, on a matrix formulation, starting from the study of spin ("spin-first" approach) [28, 29] or from polarization [30-32], still others are based on linear systems [33-35]. We note that, at least at European level, there is a widespread diffusion of didactic proposals essentially focused (and limited) to two-level systems [36-38].

Furthermore, even informal aspects (for example, outreach) are receiving more and more attention from the research point of view. In fact, QM can produce important changes for every citizen's life; therefore, skill-oriented presentations are important also in informal and non-formal approaches to QP education [39].

3. – Physics Education and Quantum Mechanics: the educational context of Milan PERG

To better contextualize the work we will deal with in this paper, *i.e.*, the understanding by high school students of some aspects of the formal axiomatization of QM, we deem it useful to have a general picture of the research work we are carrying out in various contexts.

As far as the informal/non-formal contexts are concerned, our PERG has designed and implemented a lesson-show entitled "QM", written and performed by one of the authors (M.G.) and actress Elisabetta Raimondi Lucchetti, and aimed at everyone. It is a reworking of a so-called augmented lecture entitled "There are no things inside things" [40, 41]. "QM" consists of a dialogic but rigorous presentation, set in the context of a fairy tale, of some crucial QM experiments. Furthermore, for its use in the classroom, a commented version of the script has also been prepared, which explains the theatrical choices and allows reflections on many aspects of QP. In addition to this, a collaboration is planned for an exhibition on entanglement in Milan within the "Italian Quantum Weeks" events [42].

Concerning the formal context, research work has been conducted on five lines:

L1) Classical physics: a) a partial redesign of the approach to classical physics [43] aimed at leading students to QP in a more natural way [44, 45]; b) an educational reconstruction of oscillations and normal modes [46,47] (fundamental for the later introduction of Quantum Field Theory (QFT)); c) an educational reconstruction of electromagnetic induction with the aid of the magnetic vector potential [48, 49] (indispensable to truly introduce the concept of the photon).

L2) Description, high school-oriented adaptation and explanation of some experiments particularly related to QP: d) scattering experiments [50]; e) visualization of tracks in a cloud chamber; f) phenomenology of superconductivity [51, 52]; g) the Higgs mechanism [53, 54].

L3) Educational reconstruction of Old Quantum Theory (OQT): h) a historical-pedagogical reconstruction of OQT and early QM aimed at providing a wide and coherent conceptual framework.

L4) Educational reconstruction of QM: i) an educational path for high school [55, 56].

L5) Conceptual developments towards a QFT approach (the aim of all our work about QP education [14, 57]): l) a path, that starts from the concept of field, for the formal-conceptual introduction of QFT in high school, prepared in two different forms [2, 25, 58], already partially tested.

3˙1. *Some details about lines 3 and 4.* – The fact observed in sect. 2 that at school level and in textbooks there is a widespread presentation only of a pseudo-historical point of view concerning the OQT has important consequences for teaching. In fact, since we need to start from where people are (and not where we would have liked them to be), and since high school teachers and textbooks usually deal only with OQT, we believe that a critical analysis of the historical path that led to the birth of OQT and QM is more than ever appropriate.

Therefore, in Milan, we conducted a research based on more than 800 primary sources and enriched with several explanatory comments and notes, specially designed for teachers. It led to a path experimented with teachers and students [59] which, while discussing OQT, highlighted the importance of relying on a precise theory (QM) to teach QP, and not only on the introduction of mere *ad hoc* models. This historical reconstruction is aimed both at presenting OQP in a "correct" and effective way and at showing its structural and coherence limits from a didactic-educational point of view.

In parallel with this research, we are working on an educational reconstruction of QM. This step is somewhat necessary: in fact, jumping directly from OQT to QFT is too a big leap (at least for now), and is more reassuring for teachers to make the intermediate step of QM to acquire the language needed to tackle QFT (which is our goal). This reconstruction is aimed at presenting the formal and conceptual aspects of QM in a plausible and understandable way. This path was also offered to high school teachers, within the IDIFO21 Second Level Master [55]. The 20-hour course (held in Autumn

2022) was supplemented by readings of poetry, literature and philosophy by actor Flavio Albanese.

In the following, we will focus only on some aspects concerning our research on QM education.

4. – Designing principles and considerations

As a starting point, we believe that, as for classical theories, a primary objective of the introduction of QP in school is to provide a unifying perspective of disciplinary knowledge [29] to avoid the fragmentation [60] of models and concepts that even university students show [22]. The choice of the PERG of Milan and Rome is to build this unity of treatment through the conceptual and formal introduction of the axiomatic structure of the theory. The aim is to construct a formal axiomatic framework, accessible to students, starting from some crucial experiments, which highlight the fundamental and specific characteristics of QM in the simplest and most direct possible way.

The path (summarized in Appendix), thus focuses on elucidating the motivations that lead to induce and then introduce, one after the other, some principles from which inducing axioms which are mathematically well-structured.

At this point, it is natural to express some considerations on the modeling and formalization process that we are carrying out. In fact, QM is full of principles that are widely used in educational presentations. In particular, we mention the correspondence principle, Heisenberg's (uncertainty) principle, and the superposition principle. The word "principles" means statements that underlie the theory and which, in some way, are indispensable for understanding it and the physical world. The correspondence principle is a heuristic way of requiring that our formulations merge with the classical ones in a suitable limit. It is a very useful request, but it concerns essentially all physical theories. Heisenberg's principle is certainly the basis of QM as regards its cognitive content, but it is essentially a theorem. The superposition principle is the fundamental observation that the state space of our system must be a vector one. However, our teaching experience leads us to believe that the words "superposition principle" can easily lead to misunderstandings by attributing (in an exaggerate way) a privileged *status* to some states. In the presentation proposed in this work, the idea is not so much that of inducing principles from reality, as that of enunciating "reasonably" axioms induced by suitable experiments. In classical mechanics, the laws on which our description of the world is based are called "principles": *Principia* are those of Newton, but also the much more formal principles of stationary action or, in general, of analytical mechanics. Those of thermodynamics are also principles, but, curiously, electromagnetism has no principle or even an axiom, having only Maxwell's equations. In relativity, postulates are often stated, but in QM we have both principles and axioms, and they have a slightly different epistemological *status*.

Mathematization is inherent in doing physics [61], and physics without formalization is simply not physics. To understand QM, it is necessary (like in classical physics) to present its formal structure and to provide the mathematics necessary to formulate problems in a form suitable for students. For this purpose, we rely on visualization with different tools. In fact, only the interplay of different ways of representing a physical phenomenon can help give a somewhat complete picture of a given topic [61]. Mathematical representations, in general, complement each other, thus contributing to the creation of meaning. Naturally, learning different methods of representation can be a very hard task for students, sometimes so hard as to be counterproductive [62]. Therefore, the effectiveness of using different representations has to be monitored.

The meaning of mathematical representations should be reflected in a greater understanding of the physics described, but physical concepts acquire their meaning only within a formal theory (*e.g.*, the concept of force takes on its physical meaning within classical mechanics with its three principles of dynamics). In this sense, the word "electron" will become a meaningful concept only once we have established the theory of reference (and it will be different in QM and QFT); however, when dealing with photons in the Dirac formalism of QM, we are surely able to interpret interferometry experiments and discuss entanglement, but we have still few clues about the "nature" of the photon and the quantization of the electromagnetic field, and of how this interacts with matter (for which we need quantum electrodynamics).

In addition to this, if presentations dealt mostly with qubits or qubit-like systems, and their representations in Hilbert spaces with self-adjoint operators, etc., the physical understanding of such systems could only be limited to their linear and statistical quantum behaviour, and to how this quantum behaviour is distant from that of a classical system. That is why we believe that the theoretical description of any relevant phenomenology should be part of the process (*e.g.*, discretization of energy levels in bound systems, explanation of conduction, superconduction...).

However, as far as our pilot experimentation is concerned, besides the introduction of formal axioms and their justification with examples, only an interaction Hamiltonian for two-level systems has been introduced. We could not thus pretend that students grasped the physical meaning of basic concepts like "electron", "photon", and "discretization of energy levels in atoms".

When we formally introduce the axioms of QM, we are taking a fundamental step forward for the physical description of the world; in fact, we are providing the grammar rules necessary to write our descriptions. But only when the descriptions are actually given, we may wonder what students' understanding of physics will be. Before that, only general (although fundamental) properties of systems (preparation – *i.e.*, a procedure that outputs a system with specified physical properties –, superposition, measurement, probabilistic description) may be conceptualized.

5. – "The Elegance of Quantum Mechanics"

A pilot experimentation entitled "The Elegance of Quantum Mechanics", was organized in A.Y. 2021–2022 (from October 2021 to January 2022) by the University of Milan [56]. The activity was proposed jointly to 90 students of the last three years of high school (grades 11th–13th) and 30 teachers, through 10 weekly appointments of an hour and a half each (15 hours overall). This means that all the students, whatever their grades, and all the teachers attended the meetings together. The course was the first occasion in which students approached QM, since they had not previously attended any other QP classes taught by their physics teachers. Given the pandemic and the large number of participants, the course was held online, integrating the lessons with slides, questions with *Kahoot!* and interactive graphic examples created with *GeoGebra*.

As seen so far, our research work is also aimed at studying methods for the actual presentation of QM at school. Therefore, an assessment of the validity of our path in this respect is important. This consideration leads to the first two research questions.

RQ1) Within Italian high school, is it possible to propose such a formalized and structured path in a sustainable way (*i.e.*, the path provides learning outcomes that, once assessed with the same standard school evaluating metrics as other common topics, provides similar outcomes)?

RQ2) What is teachers' opinions concerning the concrete possibility of presenting the path at school, and about mathematics being used?

From what emerges in the research literature and briefly summarized above, it is of particular interest to know what is the understanding of the presented formalism and its use by students. This leads to our third research question.

RQ3) Are students able to use the formalism in cases similar, but different, from those presented during the path, and to explain and to motivate their use?

6. – Data collection and data sources

During the experimentation, we collected data from different kinds of sources: a) ongoing tests by means of 9 Google Forms; b) individual interviews, with 13 students (all attending a scientific high school); c) individual interviews, with 6 teachers; d) anonymous satisfaction surveys (given at the end of the course) to all participants.

6˙1. *Ongoing Google Forms.* – 9 Google Forms were administered during the course (one at the end of each of the first 9 meetings) containing a total of 38 open questions and 24 exercises. Questions and exercises were written by two authors of this paper (M.G. and L.L.); before being given to students, they were shared and discussed with some university professors and high school teachers to reach agreement on the meaning of the questions and to verify that the questions were sufficiently well posed. Subsequently, they were administered to a sample of 6 university students (enrolled in the course of Preparation of Educational Experiences 2, within the master's degree in Physics), to check their understanding of the questions' meaning. The necessary changes were then made. To conduct an analysis based on significant data, for each module, we discarded the answers provided by students who did not attend and/or answer continuously to the previous lessons and modules, or did not attend the lesson to which the module was related. Overall, 3018 answers were provided and analysed, but only 2004 were considered for our research according to the above criteria (as in table I).

6˙2. *Individual interviews with students.* – 13 students from different schools were chosen for individual interviews of about 30 minutes each. They were selected by their teachers (who had also joined the course), on the basis of their attendance (at least 9 lessons out of 10) and their active participation during the lessons. Students interviewed were not necessarily the ones with the highest marks in physics, the greatest interest in the subject, or the best results in Google Forms: for example, one student confessed to having always obtained scarce results in physics and to have never particularly appreciated the subject, but to have chosen the course just to see whether a different approach would have allowed her to become interested about the topics covered (as actually happened).

Interviews were carried out by two authors of this paper (M.G. and L.L.), with questions aimed at analysing students' comprehension about contents presented during the course (see RQ3) and at understanding their reasoning and strategies used in approaching even problems not addressed during the lessons, both of a mathematical and physical nature.

Interviews were composed of three main issues, substantially equal for all students:

I1) Students had to determine the state of a system (an electron in a hypothetical hydrogen atom with 3 energy levels) from numerical data, to speak about the energy levels, and to write them with the correct formalism: 1a) *"Let us take a hydrogen atom and suppose, for simplicity, that it is characterized by three energy levels:* $E_1 = -13.6\,eV$*,* $E_2 = -3.4\,eV$*, and* $E_3 = -1.5\,eV$*. If the state of an electron is such that it has a probability 1/3 of having energy* E_1*, 1/2 of having energy* E_2 *and 1/6 of having energy* E_3*, how do we represent its state in QM? What can we say about the energy?"*.

After that, a new situation (never encountered during the course) was presented: 1b) *"Let us now suppose that a measurement is performed, whose result is that the energy is not* E_2*. What is the state of the electron after the measurement?"*. These two first problems were aimed at testing students' reasoning about superposition, probability and measurements, their ability to link these aspects together, whether they were able to use a correct formalism, and to what extent they could move from one representation to another.

I2) Students were asked to determine whether a given 3×3 matrix (in $\mathbb{C}$) represented or not a self-adjoint operator, motivating the answer. Students were also asked to fill in a partially empty 4×4 matrix to obtain a self-adjoint operator. Moreover, the definition of self-adjoint operators was asked. These questions were intended to verify the knowledge of mathematical aspects discussed during the course (but usually not presented at school).

I3) Students were asked to explain the essence of QM and what makes it different from classical mechanics. This was an open question, on the general comprehension of physics. After two very formal issues, we wanted to leave students free to express their ideas in their own words.

6˙3. *Individual interviews with teachers*. – 6 teachers were interviewed at the end of the course, to have general feedback on the course (effectiveness, feasibility, adequacy, etc.). Moreover, teachers who had their own students among the participants were asked how students had reacted, whether the work done during the course had repercussions

TABLE I. – *Answered given in Google Forms and answers considered.*

Module	Questions	Forms filled	Forms considered	Answ. given	Answ. considered
1	6	65	65	390	390
2	3	60	0	180	0
3	5	64	48	320	240
4	4	52	39	208	156
5	6	61	56	366	336
6	8	55	36	440	288
7	8	47	26	376	208
8	14	39	19	546	266
9	8	24	15	192	120
				Total=3018	Total=2004

in class activities, and which of the mathematical and physical aspects addressed proved to be the most difficult.

7. – Data analysis

We asked 4 teachers among those who had attended the course to help us preparing an evaluation grid similar to those traditionally used in school. We reached a consensus upon a grid based on 4 categories (each divided into 10 levels of marks, from 1 to 10): a) knowledge of the contents; b) logical development and technical skills; c) correctness, clarity of procedures, use of specific language; 4) completeness and originality of the resolution. The 15 students who had completed all the 9 Google Forms were evaluated by two of the researchers (M.G. and L.L.) using the grid but without teachers' support to be more homogeneous and impartial in the evaluation. Questions and answers were made anonymous, only keeping track of the school grade attended.

Furthermore, as regards the 13 students interviewed, for each of the questions addressed in the interview, we followed students' reasonings and compared them with the answers given in Google Forms related to the same topic. Questions and answers were transcribed, but made anonymous, only keeping track of the school grade attended.

We exploited the Knowledge Integration Construct (KIC) —provided in table I of [63]— to assess the students' knowledge integration, giving an evaluation from 0 to 5 to the following parameters: no answer, off task, no link, partial link, full link, complex link. KIC evaluates students' ability in connecting different ideas in a given context, developing a more coherent and consistent view of scientific phenomena. For what concerns KIC evaluation, answers were evaluated individually by two of the authors (M.G. and L.L.); a comparison of the evaluations was then carried out and the score we provide is the one agreed upon after comparison.

To better understand the work made, we report some of the exercises assigned in the Google Forms (with the answers provided) and some extracts from the interview of one of the 13 selected students, whom we will call Filippo (pseudonym).

8. – The example of Filippo

Filippo was a student of the 12th grade (the second-last one) of a scientific high school, with average marks in physics; his previous knowledge on matrices concerned the calculation of determinants in the 2×2 case, but had never studied complex numbers before the course. Moreover, at the time of the course and the interview, he had just started to study electrostatics. He attended all the 10 meetings and filled in all the 9 Google Forms.

8·1. *Google Forms.* – During the 7th meeting, measurements and operators were presented. In the related Google Form, students were asked some open questions and exercises (Q). Let us see some of the answers (literally translated) provided by Filippo (F).

Q: *"In your opinion, is it possible to associate an operator also with the double slit, as was done during the meeting with the beam-splitter? Justify your answer."*

F: *"Yes, I think so. If a photon encounters a double slit, its initial state should change into a superposition of states that contemplates the two possibilities. Therefore, it should be possible to find an operator that allows this transformation."*

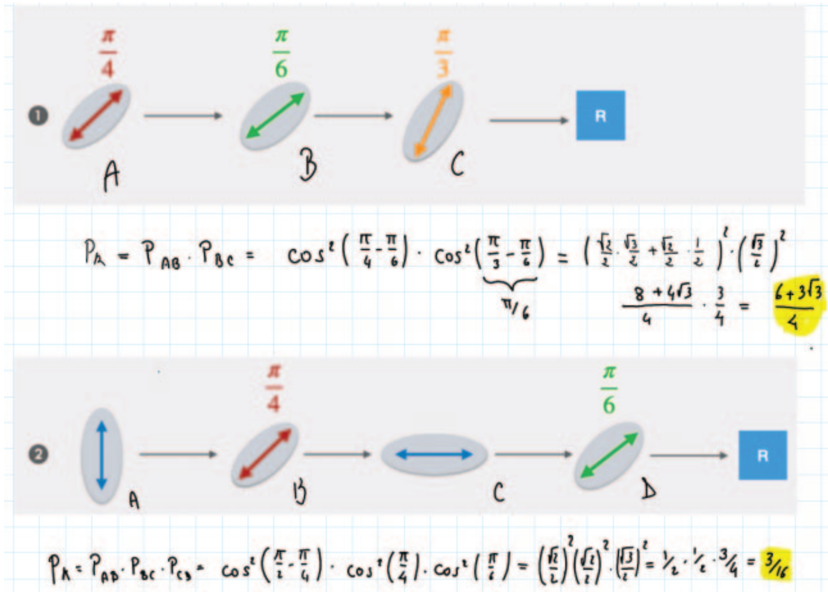

Fig. 1. – Calculations made by Filippo (excerpted from the document uploaded by the student).

Q: *"In your opinion, is it possible to associate an operator (as we defined the operators during the meeting) to the measurement process? Justify your answer."*

F: *"The measurement process determines a change of state, therefore, if the operator is the tool that allows passing from one state to another, it seems legitimate to me to be able to define an operator for this operation as well."*

Q: *"Calculate the probability that the photon has, after passing the first filter, of being detected by the detector R."* The calculations made by Filippo are shown in fig. 1.

As we can see from the answers given, Filippo was able to independently understand (since this aspect had not yet been highlighted at that point of the course) how the beam-splitter and the double slit are substantially comparable in terms of what concerns their action on the state. The evaluation of the KIC is therefore equal to "Complex link" (score 5) (understanding how at least two science concepts interact in a given context).

During the 8th meeting, observables and operators were presented. In the related Google Form, students were asked to solve a new exercise, with questions never seen before (Q). Let us see the answers provided by Filippo (F).

Q: *"Suppose you want to know the energy of an electron in an atom. You know that:*

1) *the state it is in, before carrying out the measurement, is:* $\widehat{A} = \frac{2}{3}|e_1\rangle + \frac{1}{3}|e_2\rangle + \frac{2}{3}|e_3\rangle$

2) *the operator associated with the energy variable is*

$$\widehat{H} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

Determine: a) how many and what are the possible results of this measure; b) with what probability you get one of the possible outcomes; c) the average energy value $\langle E \rangle$*"*.

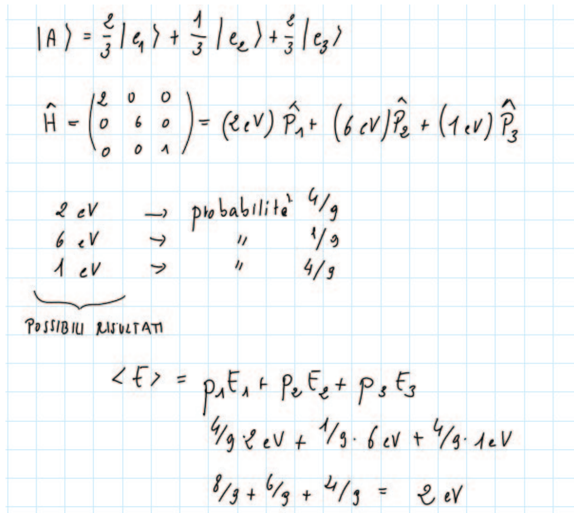

Fig. 2. – Calculations made by Filippo (excerpted from the document uploaded by the student).

The answer provided by Filippo is reported in fig. 2. As we can see, Filippo was able to answer the theoretical question about the measurement process and the action of an operator. He solved the problems with clarity and using the correct formalism. He was able to move from matrix notation to the one with projector operators; he correctly calculated probabilities and he defined the mean value in a correct way. The evaluation we gave with the KIC is, therefore, "Complex link" (score 5).

8·2. *Individual interview.* – We now report some excerpts from the interview, with the questions asked by the interviewers (Q) and the answers by Filippo (F).

Q: *"Let us take a hydrogen atom and suppose, for simplicity, that it is characterized by three energy levels:* $E_1 = -13.6\,eV$*,* $E_2 = -3.4\,eV$*, and* $E_3 = -1.5\,eV$*. If the state of an electron is such that it has a probability 1/3 of having energy* E_1*, 1/2 of having energy* E_2 *and 1/6 of having energy* E_3*, how do we represent its state in QM?"*

F: *"The state of the electron is a superposition of several states because we have three possibilities. So I can write it as a combination: each state is multiplied by a constant, and the square of the coefficients that determine the combination represents the probability of having that state. More formally:* $|e\rangle = \sqrt{p_1}|E_1\rangle + \sqrt{p_2}|E_2\rangle + \sqrt{p_3}|E_3\rangle$ *or, in a different way,*

$$|e\rangle = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \text{''}.$$

Note: Filippo uniquely associated the state, starting from probabilities. The problem of the non-bijective relationship between state and probability was addressed during the course, but not adequately. An overall multiplication by a phase factor does not change the probabilities nor the normalisation but it acquires meaning when the states involved are at least 2 (as in the discussion of entanglement, which, however, was not dealt with in this experimentation).

Q: *"What can we say about the energy?"*

F: *"As far as energy is concerned, energy is an observable. Thus, the energy observable can be associated with an operator that can be written as a linear combination in which the possible results are the coefficients (i.e.,* E_1*,* E_2 *and* E_3*), multiplied by the 3 orthogonal projectors:* $\widehat{E} = E_1\widehat{P_1} + E_2\widehat{P_2} + E_3\widehat{P_3}$."

Q: *"What if we want to represent energy in matrix form?"*

F: *"In matrix form? Well, I would write a* 3×3 *matrix, because I have three energy levels, with the energy values on the diagonal and then 0 everywhere."*

Q: *"If I measure* E_3*, what state do I get?"*

F: *"We have seen that the state collapses into the state that represents the result of the measurement, and therefore* $|e\rangle = |E_3\rangle$*."*

Q: *"And what coefficient do you put before* $|E_3\rangle$*?"*

F: *"I do not write anything, because it is 1."*

Q: *"And why is it 1?"*

F: *"Because the state collapses into* E_3*, so there is only that energy level, and the probability of finding* E_3 *is 1."*

Q: *"Now, let us go back to the initial system, and let us suppose that a measurement is performed, whose result is that the energy is not* E_2*. What is the state of the electron after the measurement?"*

F: *"Well... the state does not have that energy, but I took a measurement to say it. So the state collapses anyway and will still be a linear combination of* E_1 *and* E_3*. But the coefficients must be different now, because, if I consider* p_1 *and* p_3 *as they were originally given, the vector is no longer unitary. So the probability must be rethought... but I must have some connection with the starting probabilities. So... one is twice the other, and in total they must give 1."*

For reasons of space, we omit the part of the interview related to self-adjoint operators, simply observing that Filippo was able to answer all the questions correctly, giving reasons for what he said. We thus move on question 3.

Q: *"What is the essence of QM and what makes it different from classical mechanics?"*

F: *"I think that the essence of QM is contained in the probabilistic nature of the theory and in the superposition principle: they seem to suggest a vision of reality that is significantly different from the classical interpretation. If in classical mechanics, the probability remains confined to the theory of errors, therefore exclusively in the interaction of the knowing subject with the object to be known, in QM the probability is introduced in the very definition of the state of the object."*

Note: in this regard, it is useful to highlight the fact that, during the course, we did not discuss issues concerning error theory: this reflection and this comparison are therefore the result of a personal reworking by Filippo, who was able to use personal knowledge to answer the question.

Q: *"What does it mean that a quantum physical system is in a superposition state? Can you give some examples?"*

F: *"A quantum physical system is in a superposition state when it can be written as a sum (superposition) of other distinct states. A photon interacting with a beam-splitter or with a double slit or with a calcite mineral are examples of superposition because the state of the photon can be written as a combination of the possible states, according to the probability that they occur."*

KIC evaluation was done answer by answer for each student, obviously where the answers allowed it (in fact, some questions did not foresee links between different con-

Table II. – *KIC evaluation obtained by Filippo in answers provided in Google Forms (modules 3-9) and in the individual interview.*

GF3	GF4	GF5	GF6	GF7	GF8	GF9	I
5-3-5	5	5-4	5-3-5-5	5-5-5	5-3-5-4-5-5	5-5-4-5	5-5-5-5-5

cepts or ideas), and separately by each of the researchers (M.G. and L.L.). Subsequently, in the case of an assessment discrepancy, a discussion was held, which led to a shared mediation. The KIC assessment of a student was then done by looking at the KIC level of their various answers starting from the average of the values obtained and: 1) bearing in mind whether there was a correspondence between the KIC scores obtained in the modules and those in the interview (attributing greater weight to the answers of the interview than to those of the individual modules), 2) giving a greater weight to the most frequent score obtained by the individual student, 3) looking at whether the scores obtained have been increasing or decreasing, and rewarding growth over time. The answers given in the Google Forms (excluding modules 1 and 2) for each student were 53, of which approximately (depending on the student) 23 could be evaluated with the KIC. In the interview, questions evaluated with the KIC were 5 (out of 8 questions given).

In the case of Filippo, the evaluations obtained were as reported in table II.

From the answers provided by Filippo in the interview, it can be seen that he was able to make correct reasoning, also using different mathematical formalisms (column vectors and matrices, bra-ket notation, linear combinations...). He was able to argue and explain (in a good way) what has been said, and was able to answer questions not previously seen, neither during lessons nor during homework assignments. To this respect, we must note that students had to rely simply on their notes, since neither slides nor lecture notes were provided. For what concerns KIC, Filippo saw similarities between a beam-splitter, a double slit and the calcite. He was able to make deep reasoning about probabilities (*"the probability must be rethought"*); he interpreted the postulate of state collapse as the action of a projector on the state and he poses the problem of the new normalization of the projected state. He used specific language, in a clear and correct way. Therefore, answers given in Google Forms and in interviews confirm a level of KIC equal to "Complex link" (score 5) (understanding how more than two science concepts interact in a given context: probability, effects of a measurement, normalization of the state, etc.). Concerning RQ3, Filippo was able to use the formalism in cases similar, but different, from those presented during the path, and to explain and to motivate their use.

9. – Results

9·1. *RQ1.* – As already written at the beginning of sect. 7 , for what concerns the answer to RQ1, we did not use KIC scores, but a standard evaluation grid used in Italian high school. We considered the 15 students who had completed all the 9 modules (1 in the 11th grade, 10 in the 12th grade, and 4 in the 13th grade), and we evaluated which were the marks obtained, comparing them with those achieved on average in classical standard topics in high school curricula, such as Newtonian mechanics, thermodynamics, electromagnetism... The marks obtained by these 15 students in each individual Google

TABLE III. – *Marks obtained by the 15 students who completed all the 9 Google Forms.*

Student	GF3	GF4	GF5	GF6	GF7	GF8	GF9	Average grade
1	5.0	5.0	7.0	5.5	7.0	5.5	6.5	6.0
2	4.5	5.0	8.0	6.5	7.0	4.5	6.5	6.1
3	6.0	7.5	7.5	7.0	7.0	6.5	7.5	7.1
4	4.5	7.5	6.0	6.5	4.5	4.5	4.5	5.3
5	5.0	7.5	7.5	5.5	7.5	5.5	6.5	6.5
6	5.0	7.5	7.5	7.0	7.5	5.5	7.5	6.9
7	4.0	7.5	8.0	5.5	6.0	5.5	6.0	6.1
8	5.0	6.0	6.5	5.0	7.5	4.5	5.5	5.7
9	4.5	5.5	5.5	6.0	6.5	5.5	7.5	6.1
10	6.0	7.5	8.0	7.0	7.5	6.5	7.5	7.2
11	5.0	6.0	7.0	5.5	7.5	5.0	5.0	5.8
12	4.0	7.5	7.5	5.5	7.5	5.5	6.0	6.2
13	4.5	5.0	8.0	6.5	7.0	5.0	6.5	6.1
14	4.5	7.5	7.5	5.5	5.0	5.0	5.5	5.8
15	4.0	6.5	7.0	7.0	5.5	5.0	6.0	5.9

Form (which in Italy range from 1 to 10, which is the top score, with the sufficiency threshold corresponding to 6/10) revealed to be comprised between 4.0 and 8.0 for each student per module, with an overall (all forms and all students) average of 6.2. This result is in line with the general trend in physics at school level (regardless of the topic). Without any claim to absoluteness from a statistical point of view (being the sample of students too small to obtain any statistical considerations), we also observe that the 12th year students obtained a slightly higher mark on average (6.4), with respect to 11th year student (6.1) and 13th year students (6.0). This fact may indicate that, in order to face our QM course, the necessary mathematical knowledge (matrices and complex numbers) are within the reach of students of the 12th year.

In table III we show the evaluations obtained by the 15 students, in each Google Form. As can be seen, module 1 and module 2 are missing: in fact, the first concerned questions regarding how students imagined "quantum" objects about which it is practically impossible not to have information, even though they had never been taught quantum issues (atom, electron, photon, etc.) in high school physics classes. The second one was instead not taken into consideration, since one of the three questions given, despite having been read and discussed together with other colleagues and students, was misinterpreted by almost all of the students. Since the other two questions were related to it, we decided to discard the entire module.

Actually, the analysis made was more extensive. In fact, we evaluated module by module not only the 15 students, but also all the other ones; grades obtained in each Google Form by the other students were comprised between 2.0 and 8.5, with an average grade of 6.1 (see table IV). As we can see, the sample of 15 students considered has provided answers in Google Forms which were in line to those given by the other participants.

9·2. *RQ2.* – In order to answer RQ2, we analysed the 6 individual interviews and the anonymous satisfaction questionnaires administered to teachers (of whom only 10 filled in the module). The fact that the average marks obtained by students was above the sufficiency threshold, helped a lot in making teachers perceive the path as actually feasible in high school.

In questionnaires, 7 teachers considered the duration of the course as appropriate (corresponding to the usual amount of time dedicated to QP at school), while 3 as "not sufficient", since 2 hours were devoted to mathematical aspects (deemed necessary in any case), thus limiting the time dedicated to physical issues. All 10 teachers would recommend the course, in the case it was held again. From interviews and questionnaires, it emerged that teachers particularly appreciated the decision to start from experiments to build and justify the necessary mathematics. In their opinion, this made students and teachers "active" authors in the construction of the theory behind QM. The fact that among the participants in the course there were more teachers belonging to the same schools, allowed them to discuss and compare notes, even outside the hours of the course, in some cases also involving colleagues not enrolled in the experimentation. Furthermore, the use of software such as *GeoGebra* has facilitated the visualization of aspects (such as matrix calculation and the search for eigenvalues) that are more difficult to understand if seen only from an algebraic point of view, without any graphic support.

Finally, 4 teachers, who had their own students among the participants, reported how the students, in the lessons following each meeting, spontaneously asked to take back what was discussed in class. This made it possible to present to the whole class some aspects addressed during the course, providing an initial verification of the understanding of the topics covered (both by the teacher and by the students) and allowing to test the effectiveness of this approach with a greater number of young people, in a traditional didactic context.

9·3. *RQ3*. – Concerning QR3, we analysed the answers provided by the 13 students in individual interviews and in Google Forms, in the same way as done with Filippo. Two authors of the paper (M.G. and L.L.) individually read the answers given by the students and assigned evaluations, then compared, and negotiated in two discordant cases.

Regarding the KIC evaluation (in an overall rating), 2 students obtained score 2 (scientifically invalid links): *e.g.*, confusing the writing of a state based on energy with the sum of energies (score 2). 3 students obtained score 3 (partial links), being able to indicate a link between important concepts, but not to fully elaborate it: *e.g.*, knowing the connection between matrix representation and formal representation through operators of the energy operator, but inserting the energy eigenvalues in wrong positions (score 3). 2 students obtained score 4, elaborating a scientifically valid link between two relevant

TABLE IV. – *Average grades and standard deviations for each Google Form related to the 15 students considered and to the other participants.*

15 students	GF3	GF4	GF5	GF6	GF7	GF8	GF9
Average grade	4.8	6.6	7.2	6.1	6.7	5.3	6.3
Standard deviation	0.6	1.0	0.8	0.7	1.0	0.6	0.9
Other participants	GF3	GF4	GF5	GF6	GF7	GF8	GF9
Average grade	4.7	6.7	7.0	6.1	6.5	5.4	6.1
Standard deviation	0.7	1.1	0.8	0.8	0.7	0.5	1.0

ideas in a given context: *e.g.*, being able to link probability and coefficients in the expansion of the state in a given base, or between a representation in Dirac's notation or in vector-column notation (score 4). 6 students (among whom also Filippo) obtained score 5, having elaborated two or more scientifically valid links between different ideas in a given context (see sect. 8).

Concerning the contents (see subsect. 6.2):

I1) Students were generally able to determine the state of a system starting from numerical data, to correctly speak about the energy observable as an operator, and to write them with the correct formalism. About question 1a), 10 students were able to write the state of the electron in a correct way (and, among them, 6 were able to write it with two different formalisms); 2 students forgot to use the square root of probabilities in Dirac's notation, and 1 was not able to answer.

For what concerns question 1b), 12 students were aware that they needed a 3×3 matrix. 9 students correctly answered the question; 1 student wrote all the energies in the same column of the matrix (rather than in the diagonal); another student correctly wrote the energies on the diagonal, but was not able to fill all the other spaces with zeros.

I2) 10 students were able to determine whether a given 3×3 matrix (in $\mathbb{C}$) represented a self-adjoint operator, motivating the answer (being able also to give the definition of self-adjoint operator). Therefore, they were asked to fill a partially empty 4×4 matrix to obtain a self-adjoint operator: 6 of them were able to complete the exercise. Concerning the remaining 3 students, who failed in answering the first question, 2 correctly provided the definition of self-adjoint operator, but made some mistakes in dealing with complex numbers; 1 failed also in giving the definition of this concept (being not able to write it with the correct bra-ket formalism).

I3) About the essence of QM, and what makes it different from classical mechanics, 12 out of 13 students highlighted that in QM probabilistic aspects are intrinsic in the theory; *e.g.*: *"If the single-quantum double-slit experiment is repeated twice under the same conditions, the first quantum of the first and second experiments are not located in the same point, but the interference figures obtained are equal"*.

8 of them also recalled that, in QM, probabilities are calculated in a different way with respect to classical mechanics, as emerged, for example, from the double-slit experiment. 10 students also added that it is the measurement process that establishes the state of a system, which, otherwise, remains "undetermined" (an expression used by 2 students), being a superposition of different states. According to these 10 students, the superposition is another element peculiar of QM, which does not exist in the classical counterpart. Furthermore, 3 students focused also on the concept of operators, which have a fundamental role in QM.

9˙4. *Students' satisfaction questionnaires.* – In the satisfaction questionnaire (based on a five-point scale: very satisfied, satisfied, neither, dissatisfied, very dissatisfied), 27% of students declared "very satisfied" and 45% "satisfied" by the course; 29% "very satisfied" by the methodology and the approach used, and 51% "satisfied" by them. Mathematics not only did not appear to be an obstacle (83%) in conceptual understanding, but rather it was seen as a help, a support and a reassuring aspect (56%). Typical students' comments were: *"I really liked the course because we also tackled it from a mathematical* *point of view, and not only in a popular way."*; *"Simple language was used in explanations, never assuming that we already knew things before, but introducing everything that we needed, step by step."*. Therefore, also students seem to have appreciated the course and the approach proposed.

10. – Comments

Concerning the concrete feasibility at school of the path here discussed (RQ1, RQ2), the response seems clearly positive. In fact, the average evaluation of students' exercises and problems is in line with mean evaluations obtained in other physics topics. Moreover, teachers expressed a very favorable judgment on the effectiveness of the proposed path, to such an extent that two of them have decided to experiment it in their school (Liceo Palli, Casale Monferrato, AL). In A.Y. 2022-2023 a teacher led a 30-hour extracurricular experimentation with 20 students of the 13th grade; the colleague is going to experiment the path in her class in A.Y. 2023-2024.

About KIC analysis, 8 students out of 13 were able to scientifically elaborate (at least) two valid links between relevant ideas in a given context. Moreover, 9 out of 13 were able to solve new problems, in situations never seen before. The measurement problem given in I1 was by no means trivial: establishing the state of a system knowing that a measurement did not provide a specific result is certainly a non-standard question. One thing is understanding that, if the result of a measurement is a certain value, then the state is projected onto the eigenvector relating to the eigenvalue found; another thing, however, is understanding that, with a negative result (*"the measurement does not give* E_2*"*), the measurement projects the state onto the eigenstate orthogonal to the one related to the negative result (and the new state must also be re-normalized). The fact that 9 students were able to understand this aspect surprised us, as they showed an ability to make connections that goes beyond elementary aspects of QM.

Other aspects concerning further research questions emerged from the analysis of Google Forms of all participants. They have been already presented during 2022 GIREP Conference [56], and deepened in the related paper sent for the proceedings.

11. – Conclusions and discussion

The work presented in this paper has the dual purpose of 1) studying an approach to QM which, instead of being substantially superficial or oversimplified, is coherent, and, moreover, 2) demonstrating to teachers that this work is substantially proposable and feasible in today's schools, with few variations to the usual teaching plan.

As regards aspect 1 (coherence), we believe that it must be explicit on at least two levels. First of all, from a conceptual point of view, in the sense that the inductive connections between the phenomenological and formal aspects are clear and well placed, and that the construction of the theory can logically follow from the axioms in a complete way. Second, from an educational point of view, for the conceptual nodes developed by the course being consistent with students' learning difficulties. In this respect, we believe our path coherent since the phenomenological aspects we start from are coherently related to the axiomatic construction (a fact which was also highlighted by high school teachers and students, during the interviews).

These axioms constitute the core (perhaps, part of the nucleus, in the terminology of [64]) of our theory. From these axioms it is possible to derive all fundamental quantum properties, predicting the result of some experiments and explaining the behavior of quantum systems. The theory we build is linear (the state space is vectorial, and, therefore, each state is always conceivable as a linear combination of other states). Moreover, the theory is probabilistic, but the description of the probabilities is different from that given in classical physics. Finally, the theory is necessarily set in a complex space (a complex Hilbert space). Most formulations of QM are based on the concept of state as a vector in a complex Hilbert space [18, 61] and on the concept of operator as a mathematical representation of observable physical quantities. Our path is based precisely on these concepts, bearing in mind that literature [61] indicates the importance of different mathematical representations in understanding concepts. A little bit more precise description can be found in the appendix.

In many cases, the teaching of QP at school is cursory and oversimplified, often emphasizing phenomena and making large use of semi-classical analogies, while barely touching on principles and theoretical aspects [5,21,64]. In the literature, on the contrary, it is clearly put in evidence [64-70] that teaching should avoid oversimplification, and highlight the departure from classical analogies that are often misleading; in fact, as it is done in educational presentations of classical mechanics, of thermodynamics, of electromagnetism, etc., also QP has to be presented with a specific set of logical laws.

In addition to this fact, the non-quantitative approach, often used in teaching QP, is completely in contrast with the one used in presenting classical issues, in which several computational exercises are exploited and presented. Although we want to use visualizations as an aid to the description, we believe that computational exercises must be tackled as it is done in classical physics and electromagnetism: this fact also falls into the "coherence" aspect, since education coherence means also a uniform approach for the building of physics meaning.

As for the essential aspects to be presented in high school, many studies have been done and continue to be done [64, 71, 72]. Many of these studies are based on questionnaires addressed to professional physicists. It seems to us that, although they certainly are experts in physics, most of them have neither QM nor Physics Education as their research field. Therefore, we cannot consider them specialists in QM education and their statements should be taken *cum grano salis*. In this sense, we do not consider the results of these studies particularly significant for the purposes of didactic research. We believe that they are more interesting to know what the opinions of physicists are, than to have indications on what are the most important didactic aspects to be taken into account.

For example, the study of Winkler *et al.* [72] concerns what the expression "Quantum Phsyics" brings to mind to different categories of physicists and how it can be categorised. In general, the concepts that first come to mind in an open question represent the most striking ones. Nonetheless, it does not mean that people would place these concepts with the same order of importance in the teaching of QM. In fact, the answers given also depend on the research one is doing and how some concepts struck her/him while studying the topic at school and/or university. Therefore, we do not believe that the answers found should be interpreted as a sign of a greater or lesser importance attributed to the various aspects. However, this fact certainly highlights that, depending on the field of research, the aspects that catch the eye are different. In any case, we agree with the authors that *"This uncertainty is also a chance: if there is no standard of quantum physics essentials, new curricula can put more emphasis on how well a new concept is learned rather than how well it represents a teaching tradition."* ([72] p. 9).

As far as our opinion is concerned, we believe that in a didactic reconstruction, however, the importance of the formally well-established principles of a theory is an essential element for understanding. It would be very difficult to carry out significant didactic courses in classical mechanics without having the three principles of dynamics as conceptual pillars. For this reason, it is usual to start from a series of experiments and from these to induce the significant principles. The same happens for electromagnetism, where Maxwell's equations (perhaps in integral form) are there to guide our teaching. However, Newtonian mechanics can be presented at school even without reading Newton's *Principia* and without knowing how to solve Cauchy problems, but by handling second degree equations. Electromagnetism can be treated without using complicated differential equations, as in Maxwell's treatise, but by passing through integrals and derivatives. In the same way, an appropriate educational reconstruction is also necessary for QM. Obviously taking into account for the reconstruction not only the conceptual nodes, but also those of students' learning.

Another example is given by Krijtenburg-Lewerissa *et al.* [71], who describe *"a Delphi study aiming to investigate which quantum mechanics topics experts consider to be important to teach at the secondary level, and what arguments these experts give"* ([71] p. 349). Our opinion on the concept of experts is also in this case the same as expressed above. Moreover, at the end of the study, in the mean ranking, the most important 6 concepts, examples and applications were the following (we limited to the first 6, since applications were 6 overall): Concepts: wave-particle duality, particle behavior of light, wave function, de Broglie wavelength, probability; Examples: double slit experiment, spectral line, photoelectric effect, atomic structure, 1D infinite potential well, hydrogen atom; Applications: solar cells, STM, LEDs, lasers, quantum information, quantum computers. We observe a clear lack of structure. What we can comment is that, wanting to give credit to the opinions expressed, what would be needed is a conceptual structure that "contains" what is highlighted here. In any case, what seems decisive to us is that it is indispensable to present a theory for QM, with a very precise axiomatic and conceptual structure.

We conclude our analysis on the opinions of the experts by also quoting the work of Weissman *et al.* [64] which highlights how, in the founding nucleus of the discipline, these experts would put states and eigenstates, the principle of the superposition of states, the wave function, the wavity of matter and superposition, the probabilistic interpretation and the measurement process, Heisenberg's uncertainties, the complementarity principle, the entanglement, quantum indistinguishability, bosons and fermions. As can be seen, each study, using a different sample of experts, highlights slightly different things. We think we have already commented enough on this point. However, it seems interesting to us to comment how the principle of superposition (inevitable in QM, if we see it in terms of linearity) is well present (in the founding nucleus) in [64], completely absent in [72] and present only in 8th place in [71]. Obviously, the small number of samples, their in-homogeneity and the different formulation of the questions largely explain the diversity in the answers obtained. As far as our path is concerned, the superposition principle is certainly present in the need for the state space to be linear, but we believe that the "superposition principle" terminology is inherently misleading. In fact, it easily leads to think that there are somehow privileged states, of which the considered state is the superposition, when instead its meaning is to make explicit that any (state) vector of a vector space is a linear combination of the vectors of any basis of the space.

As regards feasibility of the approach proposed in this paper in the Italian school, we believe that, if the learning outcomes are measured with methodologies similar to those used in the Italian school today (tables III and IV), the average grades are completely in line with those obtained in the other traditional topics addressed in high school physics courses. Furthermore, KIC assessments appear to indicate a non-superficial degree of linking topics by students. The fact that this experimentation was carried out by a teacher and that, starting from next year, this topic will become a curricular experimentation within a high school, seems to show that the educational reconstruction carried out, although still in progress of development, is directed on the right path.

* * *

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Appendix: the Path ("The elegance of quantum mechanics")

The path (summarised here), starts from some clear experimental situations, and focuses on elucidating the motivations that lead to induce and to introduce the axioms of QM, giving meaning to terms such as "Hilbert space", "state of a system", "quantum probability" and "observables". In summary, the questions we will try to answer will be:

Q1) Why and how are quantum aspects linked to a specific probability theory?

Q2) Why is a Hilbert space associated with each physical system and why is the state of the system represented by a unitary vector?

Q3) What does the measurement process involve and how is it described?

Q4) Why do we use self-adjoint operators to represent observables?

From experiments to a linear, complex and probabilistic quantum theory. – We start from analyzing behavior analogies between suitable prepared beams of matter (electrons, neutrons, fullerenes), mechanical waves and electromagnetic beams. From interference experiments we can infer that the theory we want to build must be linear, being linearity the fundamental characteristic for interpreting interference phenomena. Unlike what happens for sound or electromagnetic radiation, in matter beams physical quantities do not show an oscillating trend (charge density, mass density, energy… are in fact constant). In interference experiments, on the other hand, wave aspects emerge clearly. Therefore, wave aspects of matter beams are less evident than those of electromagnetic beams, and we can think that they are somehow hidden in a complex description. Since the main physical quantities related to the propagation of general mechanical or electromagnetic waves are obtained starting from the squares of the fields, we will consider "objects" through which constructing quadratic quantities that are independent of time but which also allow describing interference aspects: we will thus pass from expressions of the type $Asen(kx-\omega t)$, usual for mechanical or electromagnetic waves, to "complex" expressions of the type $Ae^{i(kx-\omega t)}$ whose square modulus remains constant, but which allow for a description of interference phenomena.

Experiments at very low intensity show the "granularity" of the radiation detected and introduce the need for a quantum description; moreover, the analysis of the distributions of *quanta* highlights the probabilistic nature of the theory. From the analysis of single quantum interference experiments (typically "which path" experiments, such as double slit, Mach-Zehnder interferometer, experiments with birefringent crystals…) linear and probabilistic aspects are confirmed.

Space, state and probability. – Now, we can proceed by constructing a theory of waves interacting through quanta (going towards the QFT) or, instead, by elaborating a quantum theory showing wave-like behaviour (arriving at QM). Once chosen the second way, our system will thus be conceived as a set of quanta (in a non-relativistic description), whose state must comply with linear and complex aspects. This leads us to represent the state of a quantum by means of a vector in a linear complex space. But, how many dimensions does this space have? What kind of mathematical structure, if any, does it have? What meaning do bases have in this space? What role do the components of the vector play with respect to a fixed basis?

To answer, we have to specify the way in which to calculate probabilities. In fact, "which path" experiments lead us to think that the (perhaps) most natural way of calculating probability is not adequate (a fact that we can immediately realise by taking, for example, the double slit experiment). We must therefore look for a new way to calculate probabilities. Since probabilities must be positive numbers which add up to 1, a fairly natural way to introduce them is to consider the projections on orthogonal axes of a segment of unit length. Therefore, we can consider a space whose dimension is equal to the number of possible events, and take an orthogonal basis inside it: the state of the system will be given by a unitary vector, whose projections on the axes (taken in square modulus) will correspond to the probabilities that each of the possible outcomes will occur. Independent events will thus correspond to orthogonal segments (Q1).

Dot product. – The need to consider orthogonality between vectors leads to the introduction of a dot product. Since our space is complex, the only possibility is to introduce a sesquilinear form. Leaving aside issues of space completeness (unnecessary in this summary), a linear, complex space, endowed with an inner product is called Hilbert space (Q2).

Operations on states and measurement. – We now consider situations in which the state of the quantum system is changed, as the operations connected to the measurement in single quantum polarization experiments.

The use of a polarizer causes a photon to pass or not with a certain probability; once passed, the photon is polarized in the direction of the polarizer: its state has changed, "precipitated", becoming the one given by the polarizer (Q3). Of course, the state of a photon can also be changed in other ways, for example by making it pass through a crystal of calcite. In fact, measuring, using beam-splitters, etc., are all procedures which, in general, change the state of a system: hence the need to provide a mathematical representation of this fact. Therefore, the concept of (linear) operator is introduced as an object that associates a vector with another vector (respecting the linearity of the structure).

Measure and projection operator. – We can now move on to the formalization of the measurement process. We consider a physical system and an "observable" quantity G which can provide results $g_1, \ldots, g_n$. For what has been said about probabilities, there must be an orthogonal basis identified by the axes corresponding to $g_1, \ldots, g_n$, and measuring G consists in obtaining one of the possible values g_i. Given the state of our system, the probability with which this result will be found will be given by the square modulus of the projection of the state vector on the i-th axis. Furthermore, immediately after the measurement, the state of the system will be given by a unit vector along the i-th axis. In fact, once the measurement has been carried out, we know with certainty (that is with probability equal to 1) the value of the quantity G; which means that the state of the system will have to be given by a vector parallel to the i-th axis. This introduces the idea of projection operator.

The fact that the measurement provides a result expressed by a real number leads us to consider linear combinations with real coefficients of projectors. In fact, a linear combination of orthogonal projectors is the operator that projects onto the subspaces identified by the various possibilities given by the measurement. It is now possible to identify the coefficients of the linear combination with the possible outcomes of the measurement: we thus obtain an operator which, starting from the state of the system, allows us to have both the probability and the result of the measurement "together".

Projection operators and self-adjoint operators. – We observe that orthogonal projectors are idempotent and self-adjoint, *i.e.*, they can be moved from one part of a scalar product to the other without altering the result. Therefore, also a linear combination of them enjoys the same property. At this point, the eigenvalues and eigenvectors of the self-adjoint operators are connected to fundamental physical concepts: the eigenvalues provide the possible results of a measurement, while the eigenvectors provide the axes of the basis associated with the observable in question. We can thus associate a self-adjoint operator to each observable of the system (Q4).

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