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Quarkonium states in strong magnetic fields

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Summary. — Based on the constraint formalism for the Dirac equation (Crater H. W. and Van Alstine P., *Phys. Rev. D*, **30** (1984) 2585; **36** (1987) 3007) the two-body system in a strong uniform magnetic field is considered. In the framework of such an approach the ground state of a quarkonium is studied in detail. The energy of the ground state level of a quarkonium strongly depends on the quark mass and on the value of the magnetic field strength. It is shown that strong magnetic fields sufficiently decrease the quarkonium life-time compared with the zeroth magnetic field case.

1. – Introduction

Influence of a strong magnetic field on the dynamics of physical processes is important in studying the non-central high-energy collisions of heavy ions. The value of magnetic field strength is estimated to be about $B \sim 10^{18}$–$10^{19}G$ at RHIC and LHC energies [1-3] which is of the order and more than the squared pion mass. Magnetic fields play a key role in forming signals from the interior area of neutron stars [4] where the magnetic field strength turns out to be of the same order.

Quarkonium states in a magnetic field are considered [5] in detail in the non-relativistic approach in studying the bottonium and charmonium states. However, researching an annihilation process, which is a very important source of information about the strong interacting matter, demands the relativistic consideration since an annihilation takes place when a particle and anti-particle are at an extremely short distance from one another.

In the present paper we study the ground state of a quarkonium in a uniform strong magnetic field, based on the Dirac constraint dynamics [6]. Neglecting the vacuum polarization by a magnetic field the equations governing such a state are derived and studied in detail. The influence of a magnetic field on a quarkonium state is found to depend strongly on its mass that manifests itself in two aspects. They are the value of the quarkonium energy level itself and the dependence of such an energy level on a magnetic field. It is shown that a strong magnetic field leads to decreasing the quarkonium life-time due to the compression of a quarkonium state in the plane which is perpendicular to the magnetic field direction.

2. – Quarkonium states in a homogeneous magnetic field

We consider a quark-antiquark pair which is in either the $(S = 0, S_z = 0)$ or $(S = 1, S_z = 0)$ spin state, assuming that a quark and antiquark interact by means of the Cornell potential [7]

$$U(r) = -\frac{4\pi\alpha_s^2}{3r} + \sigma r + C_S, \tag{1}$$

where $\mathbf{r}$ is the radius-vector of the relative motion of a quarks and antiquark, α_s is the strong interaction coupling constant whose typical value is inside the range $0.19 \leq \alpha_s \leq 0.4$; $\sigma \simeq 0.18\,\mathrm{GeV}^2$ is the QCD string tension; $C_S \simeq \mp 0.3\,\mathrm{GeV}$ is the constant coming from the spin-spin interaction term, where the upper and down signs correspond to the spin states $(S = 0, S_z = 0)$ and $(S = 1, S_z = 0)$, respectively. Such a written spin-spin interaction term, ignoring the exponential factor $\exp(-\beta r)$, means the upper estimation of this term value.

Let a quark-antiquark pair be in a uniform magnetic field $\mathbf{B} = B\mathbf{e}_z$ directed along the OZ axis. When a quarkonium consists of a quarks-antiquark pair of the same flavor a homogeneous magnetic field does not affect the center mass motion. Therefore, to study the quarkonium ground state in a magnetic field we modify the equations of the constraint Dirac dynamics [6], having added a vector potential $A^\mu = (0, \mathbf{A}) = (0, \frac{1}{2}(\mathbf{B} \times \mathbf{r}))$ to the parapositrinium motion equation [8]. Beside that, we also change the Coulomb potential [8] which is the zeroth component of the vector interaction $A_\mu = (U, 0)$ of two particles by the Cornell potential given by eq. (1), and take into account the spin-mixing effect [5], introducing $\Psi_S(\mathbf{r})$ function which corresponds to the spin states $(S = 0, S_z = 0)$ or $(S = 1, S_z = 0)$. As a result, we obtain the closed set of equations in the center mass frame

$$\begin{aligned}&\Big(\triangle + 2U(r)E_w - U^2(r) + \frac{ie_q}{2}(\mathbf{B} \times \mathbf{r})\nabla - \frac{1}{4}e_q^2 B^2 r_\perp^2\Big)\Psi_S(\mathbf{r}) + (e_q B)\Psi_{(1-S)}(\mathbf{r}) \\ &= (m_w^2 - E_w^2)\Psi_S(\mathbf{r}); \quad S = 0, 1;\end{aligned} \tag{2}$$

where $E_w = (E_q - m_q^2/2E_q)$, $m_w = m_q^2/2E_q$; m_q and $e_q > 0$ are the mass and electric charge of a quark, whereas E_q is the energy of a quark-antiquark pair in the center mass frame. Above, we also have introduced the cylindrical coordinates for the relative motion radius-vector $\mathbf{r} = \mathbf{r}_\perp + \mathbf{e}_z z$. The function $\Psi_S(\mathbf{r})$ is assumed to be normalized by unit.

We study the ground singlet state of a quarkonium in a strong magnetic field $\mathbf{B}$ when the magnetic length $a = (e_q B)^{-1/2}$ is small, so that

$$\max(a\sigma^{1/2}; \alpha_s E_w a; |C_S|a) \ll 1. \tag{3}$$

Such an inequality occurs at the magnetic fields $B \gg 10^{16} G$ which are achieved in non-central heavy ion collisions [1-3] and in astrophysical objects [4].

The wave function of the ground state of a charged particle in a magnetic field which governs its transverse motion is [9]

$$\psi_{m=0,n=0}(\mathbf{r}_\perp) = \frac{1}{a\sqrt{2\pi}} \exp\big(-\frac{r_\perp^2}{4a^2}\big), \tag{4}$$

where m is the projection of an angular momentum onto the $\mathbf{B}$-direction, n is a radial quantum number. We look for the solution of eq. (2) in a form

$$\Psi_S(\mathbf{r}) = \psi_S(z)\ \psi_{m=0,n=0}(\mathbf{r}_\perp). \tag{5}$$

Substituting $\Psi_S(\mathbf{r})$ given by eq. (5) into eq. (2), we derive

$$\Big(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + 2U^{(1)}(\xi)\ a\varepsilon_w m_w - a^2 U^{(2)}(\xi)\Big)\psi_S(\xi) = \varepsilon_S^2 \psi_S(\xi), \tag{6}$$

where $z/a \equiv \xi$, $(m_w^2 - E_w^2)a^2 + 1 + (-1)^{S+1} \equiv \varepsilon_S^2$, $\varepsilon_w = E_w/m_w$, and

$$U^{(\lambda)}(\xi) = \left\langle \psi^*_{m=0,n=0}(\mathbf{r}_\perp) \left| \left(-\frac{4\pi\alpha_s^2}{3r} + \sigma r + C_S \right)^\lambda \right| \psi_{m=0,n=0}(\mathbf{r}_\perp) \right\rangle, \quad \lambda = 1, 2. \tag{7}$$

We study solutions of eq. (6) in the cases of light and heavy quarks.

2˙1. *Light quarks.* – Let us consider quarks whose masses are $m_q \ll 1\,\mathrm{GeV}$. They are the u-, d-, s-quarks. In this case the second term in eq. (1) dominates in order to form the particle bound state with respect to a motion along the OZ axis. Then, the equation determining the ground state of a pair is

$$\Big(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} - a^4\sigma^2\xi^2\Big)\psi_S(\xi) = \varepsilon_S^2\psi_S(\xi), \tag{8}$$

where we neglect the last term in eq. (1) which is small in this case. This is the well-known linear oscillator equation whose solution is given by the formula

$$\begin{aligned} &\psi_S(\xi) = \left(\frac{a^2\sigma}{\pi}\right)^{1/4} \exp\,(-a^2\sigma\xi^2/2), \quad \varepsilon_S^2 = -a^2\sigma + (-1)^{S+1}, \\ &E_w^2 = m_w^2 + \sigma + a^{-2}(1 + (-1)^{S+1}). \end{aligned} \tag{9}$$

This result means that the energy levels of light quarks which correspond to the quarkonium ground state lie below the main Landau level $E_{n=0,m=0} = \sqrt{m_w^2 + a^{-2}}$ because of the inequality (3).

2˙2. *Heavy quarks.* – When a quark mass is more than $m_q = 1\,\mathrm{GeV}$ the first term in eq. (1) mostly governs the quark-antiquark interaction. In this case eq. (6) takes the form

$$\begin{aligned} &\Big(\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + 2^{1/2}\sqrt{\pi}(4\pi\alpha_s/3)m_w\varepsilon_w a e^{\xi^2/2}\mathrm{erfc}(|\xi|/\sqrt{2}) \\ &\quad + \frac{(4\pi\alpha_s/3)^2 e^{\xi^2/2}}{2}E_1(\xi^2/2)\Big)\psi_S(\xi) = \varepsilon_S^2\psi_S(\xi), \end{aligned} \tag{10}$$

where $\mathrm{erfc}(x)$ is the complementary error function, $E_1(x)$ is the integral exponent [10].

Since the considered magnetic field is strong, a particle is weakly bound as for a motion along the OZ axis. Therefore, we look for solutions of eq. (10) in the form

$$\psi_S(\xi) = \sqrt{k_S}\,\exp(-k_S|\xi|), \tag{11}$$

where $k_S = \sqrt{\varepsilon_S}$ is a positive constant.

Substituting such $\psi_S(\xi)$ given by eq. (11) into eq. (10) we derive the equation to find k_S,

$$k_S = 2^{3/2}\sqrt{\pi}(4\pi\alpha_s/3)m_w\varepsilon_w a \int_0^\infty \mathrm{d}\xi \exp(\xi^2/2 - 2k_S\xi)\mathrm{erfc}(\xi/\sqrt{2}) + (4\pi\alpha_s/3)^2 \int_0^\infty \mathrm{d}\xi \exp(\xi^2/2 - 2k_S\xi)E_1(\xi^2/2). \tag{12}$$

The validity of the derived equation is restricted to small $k_S \ll 1$. At such k_S, the second integral [10] is approximately $\pi^{3/2}/\sqrt{2}$, whereas the first integral can be approximately taken to be $\ln(a_B/a)/\sqrt{\pi}$ [9] after the dimensional regularization, where $a_B = 4\pi/3m_q\alpha_s$. As a result, we find

$$k_S = 2^{3/2}(4\pi\alpha_s/3)m_w\varepsilon_w a\ln(a_B/a) + (4\pi\alpha_s)^2\frac{\pi^{3/2}}{\sqrt{2}}. \tag{13}$$

In the non-relativistic case $m_q a \gg \alpha_s$ we obtain $k_S = 2^{3/2}(4\pi\alpha_s/3)m_q a\ln(a_B/a)$, that taking into account eq. (13) results in

$$E_w = \frac{m_q}{2}\Big(1 + \frac{2(1+(-1)^{S+1})}{(m_q a)^2} - 2(4\pi\alpha_s/3)^2\ln^2(a_B/a)\Big). \tag{14}$$

The derived E_w agrees with the results obtained earlier [5] (see figs. 3, 4 at $< P_{kinetic} >= 0$), demonstrating the approximately linear growth of $(2E_w - m_q)/m_q$ with increasing a magnetic field for the triplet state $S = 1$. The growths become slower with decreasing B.

In the opposite limiting situation $m_e a \ll \alpha_s$, but provided that the vacuum polarization effects in a strong magnetic field are absent [11], we get $k_S = 9(2\pi)^{7/2}\alpha_s^2$, that gives

$$E_w \simeq \frac{1}{a^2}((1+(-1)^{S+1}) - (2\pi)^7\alpha_s^4/162). \tag{15}$$

The obtained E_w sufficiently differs from the results which have been derived above in the non-relativistic case. Equations (14), (15) show the stronger a magnetic field, the less the level depth of the ground quarkonium state under the bottom of the main Landau zone.

3. – Influence of a strong magnetic field on a decay width

A decay width Γ is proportional to [12]

$$\Gamma \simeq \frac{|\psi(0)|^2}{M^2}, \tag{16}$$

where $\psi(0)$ and M are the wave function and mass of a decaying particle, respectively.

According to eqs. (4), (9), (11) the squared wave function $|\psi(0)|^2$, depending on the quark mass, is of the order of

$$\begin{aligned} |\psi_{B=0}(0)|^2 &\sim \sigma^{3/2}, \quad |\psi_{B\neq 0}(0)|^2 \sim \frac{\sigma^{1/2}}{a^2} \quad \text{(light quarks)}, \\ |\psi_{B=0}(0)|^2 &\sim a_B^{-3}, \quad |\psi_{B\neq 0}(0)|^2 \sim \frac{1}{a_B a^2} \quad \text{(heavy quarks)}. \end{aligned} \tag{17}$$

The estimations presented by eq. (17) show that a strong magnetic field leads to sufficient increasing the decay width, as compared with the case $B = 0$, in $1/(\sigma a^2) \gg 1$ or in $(a_B/a)^2 \gg 1$ times, depending on the quarkonium mass. Such a behavior of the decay width in a magnetic field is a result of the additional compressing of the quark-antiquark state by a magnetic field in the plane which is perpendicular to the **B**-direction.

4. – Conclusion

The ground state of a quarkonium in a strong magnetic field is studied, taking into account the spin-mixing effect. The energy level of the ground state of a quarkonium is found to be always below the main Landau level, while the energy of this state depends strongly on both the quark mass and magnetic field strength. We show that a strong magnetic field essentially decreases the quarkonium life-time as compared with the case of $B = 0$.

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