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The modular S_3 group: A simpler model for neutrinos(*)

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Summary. — A minimalistic model for leptons based on the modular group Γ_N of lowest level $N = 2$ is presented. As opposed to the only existing model of $\Gamma_2 \cong S_3$ formulated in a SUSY framework, the only non-SM field is the modulus τ, and a generalised CP symmetry is implemented. A normal ordering for the neutrino masses is predicted. Predictions for the CP violating phases are also given: the Dirac CP phase is predicted around 1.6π, the Majorana phases lie in narrow regions near $\pm\pi$. Given the reduced number of free input parameters as compared to the existing literature on modular S_3, this work renews interest for a unified predictive model of quark-lepton sectors based on $\Gamma_2 \cong S_3$.

1. – Description

In the recent past, a substantial effort went into the understanding of lepton mixing and masses through flavour symmetries. A suitable framework has been provided by non-Abelian discrete symmetries [1]. These symmetries act linearly on the fields, which are supposed to belong to irreducible representations (irreps) of the group. The spurion fields that break the flavour symmetry are called flavons, and their specific vacuum expectation value (VEV) in flavour space helps in shaping the mass matrices. The approach has been quite successful at reproducing (at leading order), approximate forms of the Pontecorvo-Maki-Nakagawa-Sakata mixing matrix U_{PMNS}, which can then be made compatible with experiments through small perturbative corrections. On the other hand, due to these necessary corrections, typical drawbacks of traditional flavour models are related to the increased number of free parameters, and to the complicated scalar sector needed to correctly align the flavons in the flavour space.

In [2] a new promising direction to address the flavour problem was suggested, a "bottom-up" approach based on modular invariance: the Yukawa couplings of the Standard Model (SM) become modular forms of level N, and functions of a complex scalar field τ, (called *modulus*), which acquires a VEV at some high-energy scale. These are supposed to transform in irreps under the action of the finite modular group Γ_N. In some

(*) IFAE 2024 - "Energy Frontier" session of its minimal realizations, no flavons other than the τ are needed, and the VEV of the modulus is the only source of flavour symmetry breaking. As opposed to models based on non-Abelian discrete symmetries, the Modular Symmetry is non-linearly realized. An interesting feature of Γ_N is that, if we limit ourselves to study modular forms of level $N \leq 5$, we benefit from the fact that the finite modular group $\Gamma_{N\leq 5}$ is isomorphic to the non-Abelian discrete groups S_3, A_4, S_4 and A_5 considered in past flavour models. A comprehensive list of examples in this direction are given in [3]. A direct consequence of holomorphicity and modular invariance is the remarkably limited number of free parameters. One can additionally require the theory to be CP-symmetric, as was done in ref. [4]. In this approach, the only source of CP-violation is the VEV of the modulus τ, and the CP-symmetry forces the parameters of the theory to be real.

In our work we wanted to address the problem of building a modular lepton flavour model based on the $\Gamma_2 \cong S_3$ group. This possibility had not been scrutinized in detail as for other groups and, for that, it deserved special attention. We went beyond the existing literature, proposing a modular model based on S_3 which employs the least number of free parameters and uses no flavons other than the modulus, and additionally with less severe fine tuning of its free parameters [5].

2. – The model

We work in a $\mathcal{N} = 1$ SUSY framework. Under the modular group $\bar{\Gamma} \equiv \mathrm{SL}(2,\mathbb{Z})/\{\pm\mathbb{I}\}$, the chiral superfields transform as

$$
\begin{cases} \tau \to \gamma(\tau) = \dfrac{a\tau + b}{c\tau + d} \\ \\ \varphi^{(I)} \to (c\tau + d)^{-k_I} \rho^{(I)}(\gamma)\varphi^{(I)} \end{cases} \quad , \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \bar{\Gamma} \,, \tag{1}
$$

where the letter I indicates a specific chiral superfield sector. To counterbalance the transformations of the chiral superfields, the Yukawa modular forms $Y_{I_1 \ldots I_n}(\tau)$ of weight k_Y and level N transform under the modular group as:

$$
Y_{I_1 \ldots I_n}(\tau) \to (c\tau + d)^{k_Y} \rho(\gamma) Y_{I_1 \ldots I_n}(\tau) \,, \tag{2}
$$

such that the superpotential is invariant if and only if the following conditions are met:

$$
\begin{cases} \rho \otimes \rho_{I_1} \otimes \rho_{I_2} \ldots \otimes \rho_{I_n} \supset \mathbf{1} \\ k_Y = k_{I_1} + k_{I_2} + \ldots + k_{I_n} \end{cases} \quad , \tag{3}
$$

i.e., the modular weights of the Yukawa couplings counterbalance the modular charges of the chiral superfields.

The symbol $\rho^{(I)}(\gamma)$ stands for a unitary irreducible representation of $\Gamma_2 = \Gamma/\Gamma(2)$, where $\Gamma(2)$ is the principal congruence subgroup of level 2([1]). The group Γ_2 is finite and isomorphic to S_3, the symmetry group of the equilateral triangle. The S_3 group is equipped with three irreducible representations: the doublet $\mathbf{2}$, the pseudo-singlet $\mathbf{1'}$ and the singlet $\mathbf{1}$, with the following composition rules for two doublets: $\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{1'} \oplus \mathbf{2}$.

([1]) Its elements satisfy $a, d \equiv 1 \,(\mathrm{mod}\, 2)$ and $c, b \equiv 0 (\mathrm{mod}\, 2)$.

TABLE I. – *Chiral supermultiplets, transformation properties under $\Gamma_2 \cong S_3$ and their modular charges.*

	E_1^c	E_2^c	E_3^c	D_ℓ	ℓ_3	$H_{d,u}$
$SU(2)_L \times U(1)_Y$	$(\mathbf{1}, +1)$	$(\mathbf{1}, +1)$	$(\mathbf{1}, +1)$	$(\mathbf{2}, -1/2)$	$(\mathbf{2}, -1/2)$	$(\mathbf{2}, \mp 1/2)$
$\Gamma_2 \cong S_3$	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{1}'$	$\mathbf{2}$	$\mathbf{1}'$	$\mathbf{1}$
k_I	4	0	-2	2	2	0

Modular forms of level 2 and (even) weight k span a linear space of finite dimension $k/2+1$. The two modular forms of lowest weight transform in a doublet and can be expanded as:

$$\begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}_{\mathbf{2}} = \begin{pmatrix} \dfrac{7}{100} + \dfrac{42}{25}q + \dfrac{42}{25}q^2 + \dfrac{168}{25}q^3 + ... \\[2ex] \dfrac{14\sqrt{3}}{25}q^{1/2}(1 + 4q + 6q^2 + ...) \end{pmatrix}, \tag{4}$$

where $q \equiv e^{2\pi i \tau}$, and $\tau = \mathrm{Re}\,\tau + i\,\mathrm{Im}\tau$, in the fundamental domain where $-1/2 \le \mathrm{Re}\,\tau \le 1/2$, $\mathrm{Im}\,\tau > 0$ and $|\tau| > 1$. Their structure allows us to define a useful quantity:

$$\frac{Y_2(\tau)}{Y_1(\tau)} \equiv \zeta = |\zeta|\, e^{i\,g}\,, \tag{5}$$

where $g = g(\mathrm{Re}\,\tau)$ is a real function depending on $\mathrm{Re}\tau$. The absolute value $|\zeta|$ satisfies $|\zeta| \lesssim 1$, suppressed exponentially by $\mathrm{Im}\tau > 0$.

2·1. *Charged leptons sector*. – We were able to single out the most suitable superfield assignments, summarised in table I.

The superfields E_i^c correspond to the three flavours of right-handed charged leptons (respectively $\{i = 1, 2, 3\} \equiv \{e, \mu, \tau\}$). The left-handed $SU(2)_{\mathrm{L}}$ doublets $\ell_i = \begin{pmatrix} \nu_i \\ E_i \end{pmatrix}$ are grouped into a doublet and a pseudo-singlet of S_3: $D_\ell \equiv \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} \sim \mathbf{2}\,,\ \ell_3 \sim \mathbf{1}'\,$,. Their modular charges are listed in table I. The Higgs doublets are completely sterile with respect to the modular symmetry: they transform as invariant singlets and $k_u = k_d = 0$. The resulting mass matrix (in the right-left basis) after electroweak symmetry breaking reads(²):

$$M_\ell = \begin{pmatrix} \alpha(Y^{(3)})_1 & \alpha(Y^{(3)})_2 & \alpha_D Y_3^{(3)} \\ \beta Y_2 & -\beta Y_1 & 0 \\ 0 & 0 & \gamma \end{pmatrix}_{\mathrm{RL}} v_d\,, \tag{6}$$

where v_d is the VEV of H_d and the modular forms of weight 6 are given by $(Y_1^{(3)}, Y_2^{(3)})^T = (Y_1(Y_1^2+Y_2^2), Y_2(Y_1^2+Y_2^2))^T_{\mathbf{2}}$ and $Y_3^{(3)} = (Y_2^3 - 3Y_1^2 Y_2)_{\mathbf{1}'}$. Expanding the eigenvalues in

(²) In the parentheses notation $(...)_{1,2}$ we denote the two components of the corresponding doublet $Y_{\mathbf{2}}^{(a)}$.

terms of the expansion parameter ζ defined in (5), we obtain:

$$m_e = v_d\alpha\left(|Y_1^3| + \frac{3}{2}|Y_1^3||\zeta|^2 + \mathcal{O}(\zeta^3)\right) \tag{7}$$

$$m_\mu = v_d\alpha\left(|Y_1| + \frac{1}{2}|Y_1||\zeta|^2 + \mathcal{O}(\zeta^3)\right) \tag{8}$$

$$m_\tau = v_d\alpha\left(1 + \frac{9A^2}{2}|Y_1^6||\zeta|^2 + \mathcal{O}(\zeta^3)\right). \tag{9}$$

The hierarchy $(m_\tau, m_\mu, m_e) \sim m_\tau(1, |Y_1|, |Y_1|^3)$ naturally arises considering that $|Y_1| \approx 7/100$.

2·2. *Neutrino sector*. – With the assignments of table I, the neutrino sector is built through the dimension five Weinberg operator([3]), and the mass matrix is:

$$\begin{aligned} m_\nu = \frac{2gv_u^2}{\Lambda}\Bigg[&\begin{pmatrix} -(Y_2^2 - Y_1^2) & 2Y_1Y_2 & \frac{g'}{2g}2Y_1Y_2 \\ 2Y_1Y_2 & (Y_2^2 - Y_1^2) & -\frac{g'}{2g}(Y_2^2 - Y_1^2) \\ \frac{g'}{2g}2Y_1Y_2 & -\frac{g'}{2g}(Y_2^2 - Y_1^2) & 0 \end{pmatrix} + \\ &+ \begin{pmatrix} \frac{g''}{g}(Y_1^2 + Y_2^2) & 0 & 0 \\ 0 & \frac{g''}{g}(Y_1^2 + Y_2^2) & 0 \\ 0 & 0 & \frac{g_p}{g}(Y_1^2 + Y_2^2) \end{pmatrix}\Bigg] \end{aligned} \tag{10}$$

where g'/g, g''/g and g_p/g are dimensionless real free parameters. Here Λ is the scale of new physics associated with the non-renormaizable Weinberg operator, accounting for the smallness of neutrino masses.

3. – Results

We denote with $q_j(p_i)$ the observables obtained from the model with the p_i set of parameters taken as input, and with $q_j^{\text{b-f}}$ the corresponding best-fit values of neutrino observables. To properly explore the viable parameter regions which in modular models are typically characterised by peculiar shapes, a "figure of merit" $l(p_i) \equiv \sqrt{\chi^2(p_i)}$ can be introduced by making use of the Gaussian approximation:

$$\chi^2(p_i) = \sum_{j=1}^{6}\left(\frac{q_j(p_i) - q_j^{\text{b-f}}}{\sigma_j}\right)^2, \tag{11}$$

where we compare the predictions of the model to the following set of six dimensionless observables: $\{\sin^2\theta_{12}, \sin^2\theta_{13}, \sin^2\theta_{23}, m_e/m_\mu, m_\mu/m_\tau, r\}$ where $r \equiv \Delta m^2_{\text{sol}}/|\Delta m^2_{\text{atm}}|$, which are extracted from the recent global analysis of ref. [7]. It should be emphasised that the Dirac CP phase δ_{CP} was not included in the fit, hence its value is a prediction of the model.

([3]) A renormalizable realization with $\Gamma_2 \cong S_3$ can be found in [6].

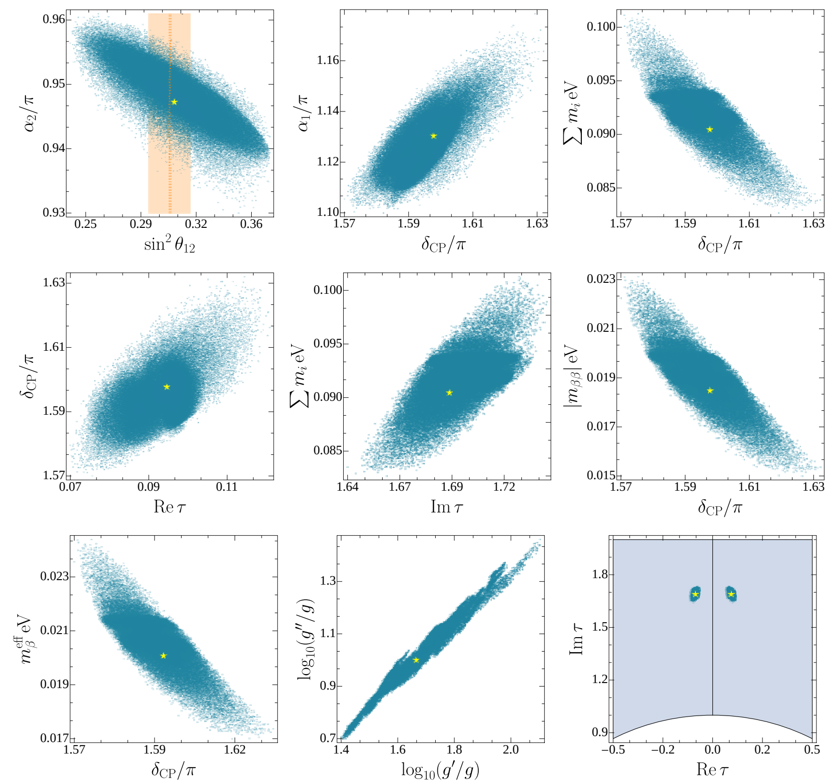

Fig. 1. – Correlations between pair of observables and parameters. The plotted points satisfy $\sqrt{\Delta\chi^2} < 5$. Here, $|m_{\beta\beta}|$ is the Majorana effective mass for the neutrinoless double-beta decay experiments, and m_β^{eff} is the effective neutrino mass in beta decay experiments.

The fit is in excellent agreement with the data, with $\chi^2_{\text{min}} = 0.074$. As a result of the numerical analysis, a set of interesting correlations among model parameters and observables has been reported in fig. 1. The model also predicts a normal ordering of neutrino masses. On the other hand, the inverted ordering case is strongly disfavored. The other predictions are:

- a CP-violating phase $\delta_{\text{CP}} \sim \pm 1.6\pi$, which in general lies in a pretty narrow interval $\delta_{\text{CP}}/\pi \sim \pm[1.57, 1.63]$ as can be seen from the plot in fig. 1.
- values of the sum of neutrino masses $\sum_i m_i$ around $0.090\,\text{eV}$, which is compatible with the present upper bound of $0.115\,\text{eV}$ ($95\,\%$ C.L.), see [8];
- the Majorana effective mass $|m_{\beta\beta}|$ for the neutrinoless double-beta decay to lie around $\sim 20\,\text{meV}$, not too far from the recent KamLAND-Zen upper bound $|m_{\beta\beta}| < (36 - 156)\,\text{meV}$ [9];
- both Majorana phases α_1, α_2 in narrow regions around $\pm 1.13\pi, \pm 0.95\pi$, respectively.

Our predictions for δ_{CP} can be tested in forthcoming oscillation experiments, whereas the Majorana effective mass $|m_{\beta\beta}|$ for the neutrinoless double-beta decay can be probed by future ton-scale experiments.

4. – Conclusions

For the first time, $\Gamma_2 \cong S_3$ lepton constructions have been realised without the aid of beyond Standard Model fields besides the modulus, and with the fewest number of free parameters. The charged-leptons mass hierarchy is accounted for by symmetry arguments with the careful assignments of irreps and modular charges of the superfields. In the model, the charged-leptons mass pattern is $\sim m_\tau(1, |Y_1|, |Y_1|^3)$ with the small parameter $|Y_1| \approx 7/100$. Neutrino masses are then generated through Weinberg operators following the assignments made for the charged-leptons sector. We performed a numerical scan and found an excellent fit to neutrino mixing data, with the prediction of a normal ordered spectrum, narrow ranges for the Dirac CP violation phase δ_{CP} and for the Majorana phases α_1, α_2. All the predictions for the sum of neutrino masses, neutrinoless double-beta decay and tritium decay effective masses are compatible with present experimental bounds. Some of these predictions can be tested by forthcoming neutrino experiments, in particular the ordering, the CP-violating phase and the effective mass for the neutrinoless double-beta decay.

Our results reopened the interest for a unification of quark-lepton flavour theories under the $\Gamma_2 \cong S_3$ group. A successful example of a $\Gamma_2 \cong S_3$ quark model can be found in [10], where a solution to the strong CP problem was also provided.

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