The Circle's Method to investigate the Goldbach's Conjecture and the Germain primes: Mathematical connections with the p-adic strings and the zeta strings.

Michele Nardelli [1,2] **e Rosario Turco**

[1] Dipartimento di Scienze della Terra

Università degli Studi di Napoli Federico II, Largo S. Marcellino, 10

80138 Napoli, Italy

[2] Dipartimento di Matematica ed Applicazioni "R. Caccioppoli"

Università degli Studi di Napoli "Federico II" – Polo delle Scienze e delle Tecnologie

Monte S. Angelo, Via Cintia (Fuorigrotta), 80126 Napoli, Italy

Abstract

In this paper we have described in the **Section 1** some equations and theorems concerning the Circle Method applied to the Goldbach's Conjecture. In the **Section 2**, we have described some equations and theorems concerning the Circle Method to investigate Germain primes by the Major arcs. In the **Section 3**, we have described some equations concerning the equivalence between the Goldbach's Conjecture and the Generalized Riemann Hypothesis. In the **Section 4**, we have described some equations concerning the p-adic strings and the zeta strings. In conclusion, in the **Section 5**, we have described some possible mathematical connections between the arguments discussed in the various sections.

1. On some equations and theorems concerning the Circle Method applied to the Goldbach's Conjecture [1]

Generating Functions on the Circle Method

If we consider a *generating function* of a power series with $z_0=0$ and $\rho=1$ we have that:

$$f(z)=\sum_{n=0}^{\infty}a_n\cdot z^n \quad \begin{cases}\mathrm{a_n=1 \ se \ a_n}\in N\\ 0 \ \text{altrimenti}\end{cases}$$

Often are interesting only the representations at k at a time of numbers a_i, the sum of these returns another number n; for which the k-ple obtained, as part of the set of all integers N, are only a subset of the Cartesian product N^k (or of N x N if k=2).

For example, for k=2 if *n* is the number sum belonging to N (set of the integers), and a_1, a_2 the numbers belonging to N, then the representations in pairs of numbers a_i are:

$$r_2(n):=\{(a_1,a_2)\in N\times N : n=a_1+a_2\} \qquad (1)$$

In general with the Taylor's series or with the residue Theorem, we obtain that:

$$r_2(n)=\frac{1}{2\pi i}\oint\frac{f(z)}{z^{n+1}}dz \qquad (2)$$

i.e. $r_2(n)$ correspond with a_n of the series.

Having to work with a Cartesian product, it is possible also define (for the *Cauchy's product*):

$$f^2(z)=\sum_{n=0}^{\infty}c_n\cdot z^n \qquad \mathrm{c_n}=\sum_{h\geq 0}\sum_{\substack{k\leq n\\ h+k=n}}^{\infty}a_h\cdot a_k z^n \qquad (3)$$

with $a_k a_h \neq 1$ if h and k belonging to N, then c_n correspond to $r_2(n)$; thence, also here we have that:

$$r_2(n)=\frac{1}{2\pi i}\oint\frac{f^2(z)}{z^{n+1}}dz \qquad (4)$$

where the integral on the right is along a circle γ(ρ), path counterclockwise, centered in the origin and unit radius.

Is logic that in the case of Goldbach's Conjecture we replace at N the *set of primes P* and the terms a_i are belonging to P.

In general we have to deal with an *additive problem* with k>2, such as: the *Waring's problem* (any k>2 and power s); the *Vinogradov's Theorem* (for k=3 and with a_i belonging to the set of the prime numbers P); the problem of the *twin number* for n=2, if –P={-2,-3,-5,-7,-11,…} and we consider P-P, studying the pairs (p_1,p_2) with $p_1,p_2 \int P$ such that $n=p_1-p_2$.

Then in the general case for k>2 is interesting the resolution of the equation of the type:

$$n=a_1+a_2+\ldots+a_k$$

$$r_k(n):=\{(a_1,a_2,\ldots,a_k)\in N^k : n=a_1+a_2+\ldots+a_k\}$$

In this case we must choose a function

$$f^s(z) = \sum_{n=0}^{\infty} r_k(n) z^n$$

for which:

$$r_k(n) = \frac{1}{2\pi i} \oint \frac{f^s(z)dz}{z^{n+1}} \qquad (4')$$

If we choose as generating function of the series the

$$f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{(1-z)^{-1}}$$

Thence, if the power of the f is s=1, the eq. (4') can be rewritten as follow:

$$r_k(n) = \frac{1}{2\pi i} \oint \frac{dz}{(1-z)^k z^{n+1}} \qquad (5)$$

The eq. (5) of the general case is interesting because the integrand function has a easy singularity on γ(1); so it can easily integrate. In fact, for ρ<1 we can develop the k-th power of a binomial:

$$\frac{1}{(1-z)^k} = (1-z)^{-k} = \binom{-k}{0}(-z)^0 + \binom{-k}{1}(-z)^1 + \binom{-k}{2}(-z)^2 + \ldots + \binom{-k}{m}(-z)^m = \sum_{m=0}^{\infty} \binom{-k}{m}(-z)^m$$

If we replace in the eq. (5), we obtain:

$$r_k(n) = \frac{1}{2\pi i} \oint \frac{dz}{(1-z)^k z^{n+1}} = \frac{1}{2\pi i} \sum_{m=0}^{\infty} \binom{-k}{m} (-1)^m \oint z^{m-n-1} dz =$$

For m = n the integral is 2πi, while 0 in the other cases; thence the result is:

$$= (-1)^n \binom{-k}{n} = \binom{n+k-1}{k-1} \qquad (6)$$

Thence we can rewrite the eq. (6) as follows:

$$r_k(n) = \frac{1}{2\pi i} \oint \frac{dz}{(1-z)^k z^{n+1}} = \frac{1}{2\pi i} \sum_{m=0}^{\infty} \binom{-k}{m} (-1)^m \oint z^{m-n-1} dz =$$

$$= (-1)^n \binom{-k}{n} = \binom{n+k-1}{k-1} \qquad (6b)$$

The eq. (6) represent just the way of writing n as a sum of k elements of power s=1.

The Vinogradov's simplification

Vinogradov, however, made the further observation that to $r_2(n)$ can contribute only the integers m≤n; so we can introduce a different function (as opposed to one that led to eq. (6)), more useful to:

$$f_N(z) := \sum_{m=0}^{N} z^m = \frac{1-z^{N+1}}{1-z} \quad \text{true for } z\neq 1$$

For n≤N, for the Cauchy's Theorem, we can write:

$$r_k(n) = \frac{1}{2\pi i} \oint \frac{f_N(z)^k \, dz}{z^{n+1}} \qquad (7)$$

Now $f_N(z)$ is a finite sum and there are no convergence problems, so the integrand function in the equation (7) has no points of singularity. Thence, now we can permanently fix the curve on which it integrates. Indeed, we take as curve the complex exponential function $e(x) := e^{2\pi i x}$ ([1]) and making a change of variable z = e(α) the eq. (7) becomes:

$$\int_0^1 V(\alpha)^2 e(-n\alpha) d\alpha = \sum_{p_1 \le N} \sum_{p_2 \le N} e((p_1 + p_2 - n)\alpha) d\alpha = r_2(n) \qquad (8)$$

The eq. (8) is the n-th Fourier coefficient of $f_N^k \ (e(\alpha))$. If now we put $T(\alpha) = f_N \ (e(\alpha))$, we obtain:

$$T(\alpha) = T_N(\alpha) = f_N \ (e(\alpha)) = \sum_{m=0}^{N} e(m\alpha) = \begin{cases} \dfrac{1-e((N+1)\alpha)}{1-e(\alpha)} = \dfrac{e(\frac{1}{2}N\alpha)\sin(\pi(N+1)\alpha)}{\sin(\pi\alpha)} & \alpha \not\in Z \\ N+1 \quad \alpha \varepsilon Z \end{cases}$$

The function above now can be studied and is an element of elemental analysis exploitable.

Property of $T_N(\alpha)$

The function T has peaks on the integer values and decreases much on the non-integer values. It's easy to see, also as numerical computation, (see [3]) that:

$$|T_N(\alpha)| \le \min(N+1, \frac{1}{|\sin(\pi\alpha)|}) \le \min(N+1, \frac{1}{\|\alpha\|}) \quad (9)$$

---

[1] We note that this function is orthogonal in the range [0,1] indeed: $\int_1^0 \exp(az)\exp(-bz)dx = \begin{cases} 1 & se\ a=b \\ 0 & \text{altrimenti} \end{cases}$ for which is $r_k(n) = \int_1^0 f^k(\exp(z))\exp(-z)dz$

where $||\alpha||$ is the distance between two numbers, T is periodic of period 1 and $\alpha<\sin(\pi\alpha)$ for $\alpha\varepsilon(0,1/2]$. If we use the eq. (9) and $\delta = \delta(N)$ is chosen not too small, then the contribution in the range $[\delta,1-\delta]$ is negligible. For example, if $\delta>1/N$ then:

$$|\int_{\delta}^{1-\delta} T_N^k(\alpha)e(-n\alpha)d\alpha| \le \int_{\delta}^{1-\delta} |T_N^k(\alpha)|\,d\alpha \le \int_{\delta}^{1-\delta} \frac{d\alpha}{\|\alpha\|^k} \le \frac{2}{k-1}\delta^{1-k} \quad (10)$$

Taking into account of eq. (8) and eq. (10) for n = N, k=2 e $\delta^{-1} = o(N)$ we obtain that:

$$r_2(n) = \frac{1}{2\pi i}\int_0^1 \left(\frac{\sin(\pi(N+1)\alpha)}{\sin(\pi\alpha)}\right)^2 d\alpha = 2\int_{\delta}^{1-\delta} \left(\frac{\sin(\pi(N+1)\alpha)}{\sin(\pi\alpha)}\right)^2 d\alpha \quad (10b)$$

practically the Goldbach's Conjecture via the Circle Method.

The Goldbach's Conjecture, without to consider the difference between the weak and the strong conjecture, says that "given an even number greater than 4 this is always the sum of two prime numbers".

For the Goldbach's Conjecture, therefore, we are interested to the representations:

$$r_2(n) := \{(p_1, p_2) \in P \times P : n = p_1 + p_2\}$$

where p_1, p_2 are prime numbers not necessarily distinct, belonging to the set of the prime numbers P and for the moment we do not consider n as even number, but any (for example we accept also 2+3=5 at this stage of investigation). Putting:

$$V(\alpha) = V_N(\alpha) = \sum_{p \le N} e(p\alpha) \quad (11)$$

then the Goldbach's problem, with the techniques of real and complex analysis, results for $n \le N$:

$$\int_0^1 V(\alpha)^2 e(-n\alpha)d\alpha = \sum_{p_1 \le N}\sum_{p_2 \le N}\int_0^1 e((p_1+p_2-n)\alpha)d\alpha = r_2(n) \quad (12)$$

In the following, instead of consider directly the eq. (12), we can consider a weighted version with weight different from 1 (instead of consider p1+p2, we consider log(p1+p2)=logp1*logp2):

$$R_2(n) := \sum_{p_1+p_2=n} \log p_1 \log p_2 \quad (13)$$

It's clear that $r_2(n)$ is positive if $R_2(n)$ is also positive; then it is sufficient to study $R_2(n)$ for the Goldbach's conjecture.

A weighted version of eq. (11) now is:

$$S(\alpha) = S_N(\alpha) = \sum_{p \le N} \log p \; e(p\alpha) \qquad (14)$$

Bearing in mind the Dirichlet's Theorem on the arithmetic progressions, choosing q, a such that MCD(q,a)=1, we write that

$$\theta(N;q,a) = \sum_{\substack{p \le N \\ p \equiv a \bmod q}} \log p \qquad (15)$$

Theorem of Siegel - Walfisz

Let C,A>0 with q and a relatively prime, then

$$\sum_{\substack{p \le N \\ p \equiv a \bmod q}} \log p = \frac{N}{\varphi(q)} + O\left(\frac{N}{\log^C N}\right)$$

for $q \le \log^A N$

the previous constant C does not depend on N, a, q (but more depend on A: C(A)).

Thence, from the Theorem of Siegel – Walfisz (see [4]) we have that:

$$\theta(N;q,a) = \frac{N}{\varphi(q)} + O\left(\frac{N}{\log^C N}\right) \qquad (16)$$

where we have defined

$$E(N;q,a) = O\left(\frac{N}{\log^C N}\right) = O(N \exp(-C(A)\sqrt{\log N})) .$$

where φ is the Euler totient function and C must be chosen not very large. The theorem is effective when q is very small compared to N. At this point, similarly to (12) we can write that for n≤N:

$$\int_0^1 S(\alpha)^2 e(-n\alpha) d\alpha \qquad (17)$$

As preliminary operation, we see some value of S (for the exponential we recall the transformation $x \to \exp(2\pi i x)$ thence for example to ½ it comes to exp(2πi*1/2)=-1):

S(0) = θ(N,1,1)≈N

S(1/2) = - θ(N,1,1)+2 log2≈-N

S(1/3) = exp(1/3) θ(N,3,1) + exp(2/3) θ(N,3,2) + log 3 ≈ - ½ N

S(1/4) = $\exp(1/4)\,\theta(N,4,1) + \exp(3/4)\,\theta(N,4,3) + \log 2 \approx 0$

Now we see S also for some rational value a/q, when $0 \le a \le q$ and MCD(a,q)=1. In this case the eq. (14) becomes:

$$S(a/q) = \sum_{h=1}^{q} \sum_{\substack{p \le N \\ p \equiv h \bmod q}} \log p \; e(p \cdot a/q) =$$

$$= \sum_{h=1}^{q} e(h \cdot a/q) \sum_{\substack{p \le N \\ p \equiv h \bmod q}} \log p = \sum_{h=1}^{q} e(h \cdot a/q) \cdot \theta(N;q,a) = \sum_{h=1}^{q}{}^{*} e(h \cdot a/q) \cdot \theta(N;q,a) + O(\log q \log N) \quad (18)$$

the asterisk in the last summation denote the further condition that MCD(h,q)=1.

From the eq. (18) taking into account the eq. (16), we obtain:

$$S(a/q) = \frac{N}{\varphi(q)} \sum_{h=1}^{q}{}^{*} e(h \cdot a/q) + \sum_{h=1}^{q}{}^{*} e(h \cdot a/q) \cdot E(N;q,a) + O(\log q \log N)$$

$$= \frac{\mu(q)}{\varphi(q)} + \sum_{h=1}^{q}{}^{*} e(h \cdot a/q) \cdot E(N;q,a) + O(\log q \log N) \quad (19)$$

where μ is the *Moebius's function*([2]*).*

For the Moebius's function, $|S(\alpha)|$ is large when α is a rational number, in a neighborhood of a/q, and from the previous examples we have seen also that S(a/q) decreases as 1/q.

Realizing how about S, we can now try to find an expression for $R_2(n)$ and usually is used the "partial sum on the arcs".

Putting: $\alpha = \frac{a}{q} + \eta$, for $|\eta|$ small, we obtain:

$$S(a/q+\eta) = \frac{\mu(q)}{\varphi(q)} \sum_{m \le N} e(m\eta) \cdot E(N;q,a,\eta) = \frac{\mu(q)}{\varphi(q)} T(\eta) + E(N;q,a,\eta)$$

From the eqs. (16) and (19) we have that:

$$E(N;q,a,\eta) = O_A(q(1+N|\eta|)N \exp(-C(A)\sqrt{\log N}))$$

[2] $\mu(q)=0$ se q è divisibile per il quadrato di qualche numero primo, è $(-1)^k$ se $q=p_1p_2...p_k$ dove i p_i sono k numeri primi distinti.

If as in [3] we denote with M (q,a) := $(\frac{a}{q}-\xi(q,a),\frac{a}{q}+\xi'(q,a))$ the Farey's arc concerning the rational number a/q, with $\xi(q,a)$ and $\xi'(q,a)$ of order $(qQ)^{-1}$, then we define the set or the union of the *Major and Minor arcs* as follow:

$$\mathrm{M} := \bigcup_{q\le P}\bigcup_{a=1}^{q}{}^{*}\,\mathrm{M}\,(q,a) \qquad \mathrm{m} := \left[\xi(1,1), 1+\xi(1,1)\right] \setminus \mathrm{M} \quad (20)$$

Also here the asterisk indicates the additional condition that the MCD(q,a)=1. For the range of the Minor arc instead of consider [0,1] we have passed to $\left[\xi(1,1), 1+\xi(1,1)\right]$, that is possible for the periodicity 1.

It's clear that, starting again to the eq. (17), now $R_2(n)$ is the sum of two integrals, one on the Major arc and the other on the Minor arc (as we have said when we have considered the eq. (12)) and for n≤N is:

$$R_2(n) = \int_0^1 S(\alpha)^2 e(-n\alpha)d\alpha = (\int_{\mathrm{M}} + \int_{\mathrm{m}})S(\alpha)^2 e(-n\alpha)d\alpha =$$

How we have defined the Major arcs in the eq. (20), we have that

$$R_2(n) = \sum_{q\le P}\sum_{a=1}^{q}{}^{*}\int_{-\xi(q,a)}^{\xi'(q,a)} S(\frac{a}{q}+\eta)^2 e(-n(\frac{a}{q}+\eta))d\eta + \int_{\mathrm{m}} S(\alpha)^2 e(-n\alpha)d\alpha = R_{\mathrm{M}}(n) + R_m(n) \quad (21)$$

In the following with the symbol $\approx$ we denote as in [3] an asymptotic equality (to the infinity).

The eq. (21) can be rewrite also as follows:

$$R_{\mathrm{M}}(n) \approx \sum_{q\le P}\sum_{a=1}^{q}{}^{*}\int_{-\xi(q,a)}^{\xi'(q,a)} \frac{\mu(q)^2}{\varphi(q)^2} T(\eta)^2 e(-n(\frac{a}{q}+\eta))d\eta =$$

$$= \sum_{q\le P}\frac{\mu(q)^2}{\varphi(q)^2}\sum_{a=1}^{q}{}^{*} e(-n\frac{a}{q})\int_{-\xi(q,a)}^{\xi'(q,a)} T(\eta)^2 e(-n\eta)d\eta \quad (22)$$

If we extend the integral that contains T throughout the range [0,1]

$$\int_0^1 T(\eta)^2 e(-n\eta)d\eta = \sum_{m1+m2=n} 1 = n-1 \approx n \quad (23)$$

Thence, we obtain:

$$R_{\mathrm{M}}(n) \approx n\sum_{q\le P}\frac{\mu(q)^2}{\varphi(q)^2}\sum_{a=1}^{q}{}^{*} e(-n\frac{a}{q}) \qquad (24)$$

where the inner sum is called the *Ramanujan's sum* and we can show it with a Theorem that can be expressed as a function of μ and φ:

$$R_{\mathrm{M}}(n) \approx n\sum_{q\le P}\frac{\mu(q)^2}{\varphi(q)^2}\mu\left(\frac{q}{(q,n)}\right)\frac{\varphi(q)}{\varphi\left(\frac{q}{(q,n)}\right)} = n\sum_{q\le P}\frac{\mu(q)^2}{\varphi(q)}\frac{\mu\left(\frac{q}{(q,n)}\right)}{\varphi\left(\frac{q}{(q,n)}\right)}$$

If we extend the sum to q≥1 and we consider another Theorem (see [3]), we obtain:

$$R_2(n) \approx n\sum_{q\ge 1}\frac{\mu(q)^2}{\varphi(q)}\frac{\mu\left(\frac{q}{(q,n)}\right)}{\varphi\left(\frac{q}{(q,n)}\right)} = n\prod_{p}(1+f_n(p)) \quad (25)$$

The "productor" is on all the prime numbers; further we have that:

$$f_n(p) = \frac{\mu(q)^2}{\varphi(q)}\frac{\mu\left(\frac{q}{(q,n)}\right)}{\varphi\left(\frac{q}{(q,n)}\right)} = \begin{cases} \frac{1}{p-1} & \text{se p|n} \\ -\frac{1}{(p-1)^2} & \text{altrimenti} \end{cases}$$

If n is odd then $1+f_n(2)=0$ thence the eq. (23) states that there aren't Goldbach's pairs for n.

Indeed $R_2(n)=0$ if n-2 is not prime number, $R_2(n)=2\log(n-2)$ if n-2 is a prime number. If, instead, n is even, we can obtain the following expression:

$$R_2(n) \approx n\prod_{p|n}\left(1+\frac{1}{p-1}\right)\prod_{p\nmid n}\left(1-\frac{1}{(p-1)^2}\right)$$

$$R_2(n) \approx 2n\prod_{\substack{p|n\\p>2}}\left(\frac{p}{p-1}\cdot\frac{(p-1)^2}{p(p-2)}\right)\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right) = C_o n\prod_{\substack{p|n\\p>2}}\left(\frac{p-1}{p-2}\right) \quad (26)$$

where C_0 is the constant of the twin primes. The eq. (26) is the asymptotic formula for $R_2(n)$ based on the Number Theory and provides a value greater than $r_2(n)$ of a quantity $(\log n)^2$, for the weights logp1logp2.

2. On some equations concerning the Circle Method to investigate Germain Primes [2]

In this section we apply the Circle Method to investigate Germain primes. As current techniques are unable to adequately bound the Minor arc contributions, we concentrate on the Major arcs, where we perform the calculations in great detail. The methods of this section immediately generalize to other standard problems, such as investigating twin primes or prime tuples.

We remember the Siegel-Walfisz Theorem, that will be useful in the follow.

Let $C, B > 0$ and let a and q be relatively prime. Then

$$\sum_{\substack{p \le x \\ p \equiv a(q)}} \log p = \frac{x}{\phi(q)} + O\left(\frac{x}{\log^C x}\right). \quad (27)$$

Definition 1

A prime p is a Germain prime (or p and $\frac{p-1}{2}$ are a Germain prime pair) if both p and $\frac{p-1}{2}$ are prime. An alternate definition is to have p and $2p+1$ both prime.

Let B, D be positive integers with $D > 2B$. Set $Q = \log^D N$. Define the Major arc $\mathcal{M}_{a,q}$ for each pair (a,q) with a and q relatively prime and $1 \le q \le \log^B$ by

$$\mathcal{M}_{a,q} = \left\{ x \in \left(-\frac{1}{2}, \frac{1}{2}\right) : \left| x - \frac{a}{q} \right| < \frac{Q}{N} \right\} \quad (28)$$

if $\frac{a}{q} \neq \frac{1}{2}$ and

$$\mathcal{M}_{1,2} = \left[-\frac{1}{2}, -\frac{1}{2} + \frac{Q}{N}\right) \cup \left(\frac{1}{2} - \frac{Q}{N}, \frac{1}{2}\right]. \quad (29)$$

We have that the our generating function is periodic with period 1, and we can work on either $[0,1]$ or $\left[-\frac{1}{2}, \frac{1}{2}\right]$. As the Major arcs depend on N and D, we should write $\mathcal{M}_{a,q}(N,D)$ and $\mathcal{M}(N,D)$. Note we are giving ourselves a little extra flexibility by having $q \le \log^B N$ and each $\mathcal{M}_{a,q}$ of size $\frac{\log^D N}{N}$. By definition, the Minor arcs m are whatever is not in the Major arcs. Thus the Major arcs are the subset of $\left[-\frac{1}{2}, \frac{1}{2}\right]$ near rationals with small denominators, and the Minor arcs are what is left. Here near and small are relative to N. Then

$$r(1; A_{1N}, A_{2N}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} F_N(x) e(-x) dx = \int_{\mathcal{M}} F_N(x) e(-x) dx + \int_{m} F_N(x) e(-x) dx. \quad (30)$$

We chose the above definition for the Major arcs because our main tool for evaluating $F_N(x)$ is the Siegel-Walfisz formula (see eq. (27)), which states that given any $B, C > 0$, if $q \le \log^B N$ and $(r,q)=1$ then

$$\sum_{\substack{p\le N\\ p\equiv r(q)}} \log p = \frac{N}{\phi(q)} + O\left(\frac{N}{\log^C N}\right). \quad (31)$$

For C very large, the error term leads to small, manageable errors on the Major arcs.

Now we apply partial summation multiple times to show u is a good approximation to F_N on the Major arcs $\mathcal{M}_{a,q}$. Define

$$C_q(a) = \frac{c_q(a)c_q(-2a)}{\phi(q)^2}. \quad (32)$$

We show

Theorem 1

For $\alpha \in \mathcal{M}_{a,q}$,

$$F_N(\alpha) = C_q(a)u\left(\alpha - \frac{a}{q}\right) + O\left(\frac{N^2}{\log^{C-2D} N}\right). \quad (33)$$

The problem is to estimate the difference

$$S_{a,q}(\alpha) = F_N(\alpha) - C_q(a)u\left(\alpha - \frac{a}{q}\right) = F_N\left(\beta + \frac{a}{q}\right) - C_q(a)u(\beta). \quad (34)$$

To prove Theorem 1 we must show that $\left|S_{a,q}(\alpha)\right| \le \dfrac{N^2}{\log^{C-2D} N}$. It is easier to apply partial summation if we use the λ-formulation of the generating function F_N because now both F_N and u will be sums over $m_1, m_2 \le N$. Thus

$$\begin{aligned}
S_{a,q}(\alpha) &= \sum_{m_1,m_2\le N} \lambda(m_1)\lambda(m_2)e((m_1-2m_2)\beta) - C_q(a)\sum_{m_1,m_2\le N} e((m_1-2m_2)\beta) \\
&= \sum_{m_1,m_2\le N}\left[\lambda(m_1)\lambda(m_2)e\left((m_1-2m_2)\frac{a}{q}\right) - C_q(a)\right]e((m_1-2m_2)\beta) \\
&= \sum_{m_1\le N}\left[\sum_{m_2\le N}\left[\lambda(m_1)\lambda(m_2)e\left((m_1-2m_2)\frac{a}{q}\right) - C_q(a)\right]e(-2m_2\beta)\right]e(m_1\beta)
\end{aligned}$$

$$= \sum_{m_1 \le N} \left[\sum_{m_2 \le N} a_{m_2}(m_1, N) b_{m_2}(m_1, N) \right] e(m_1 \beta)$$

$$= \sum_{m_1 \le N} S_{a,q}(\alpha; m_1) e(m_1 \beta), \quad (35)$$

where

$$a_{m_2}(m_1, N) = \lambda(m_1)\lambda(m_2) e\left((m_1 - 2m_2)\frac{a}{q} \right) - C_q(a); \qquad b_{m_2}(m_1, N) = e(-2m_2\beta)$$

$$S_{a,q}(\alpha; m_1) = \sum_{m_2 \le N} a_{m_2}(m_1, N) b_{m_2}(m_1, N). \quad (36)$$

Recall the integral version of partial summation states

$$\sum_{m=1}^{N} a_m b(m) = A(N) b(N) - \int_1^N A(u) b'(u) du, \quad (37)$$

where b is a differentiable function and $A(u) = \sum_{m \le u} a_m$. We apply this to $a_{m_2}(m_1, N)$ and $b_{m_2}(m_1, N)$. As $b_{m_2} = b(m_2) = e(-2\beta m_2) = e^{-4\pi i \beta m_2}$, $b'(m_2) = -4\pi i \beta e(-2\beta m_2)$. Applying the integral version of partial summation to the m_2-sum gives

$$S_{a,q}(\alpha; m_1) = \sum_{m_2 \le N} \left[\lambda(m_1)\lambda(m_2) e\left((m_1 - 2m_2)\frac{a}{q} \right) - C_q(a) \right] e(-2m_2\beta) = \sum_{m_2 \le N} a_{m_2}(m_1, N) b_{m_2}(m_1, N) =$$

$$= \left[\sum_{m_2 \le N} a_{m_2}(m_1, N) \right] e(-2N\beta) + 4\pi i \beta \int_{u=1}^{N} \left[\sum_{m_2 \le u} a_{m_2}(m_1, N) \right] e(-u\beta) du. \quad (38)$$

The first term is called the boundary term, the second the integral term. We substitute these into (35) and find

$$S_{a,q}(\alpha) = \sum_{m_1 \le N} \left[\left[\sum_{m_2 \le N} a_{m_2}(m_1, N) \right] e(-2N\beta) \right] e(m_1\beta) +$$

$$+ \sum_{m_1 \le N} \left[4\pi i \beta \int_{u=1}^{N} \left[\sum_{m_2 \le u} a_{m_2}(m_1, N) \right] e(-u\beta) du \right] e(m_1\beta). \quad (39)$$

The proof of Theorem 1 is completed by showing $S_{a,q}(\alpha; B)$ and $S_{a,q}(\alpha; I)$, where B = Boundary and I = Integral, are small. The first deal with the boundary term from the first partial summation on $m_2, S_{a,q}(\alpha; B)$.

Lemma 1

$$S_{a,q}(\alpha;B)=O\left(\frac{N^2}{\log^{C-D}N}\right). \quad (40)$$

Proof. Recall that

$$S_{a,q}(\alpha;B)=\sum_{m_1\le N}\left[\left[\sum_{m_2\le N}a_{m_2}(m_1,N)\right]e(-2N\beta)\right]e(m_1\beta)=e(-2N\beta)\sum_{m_1\le N}\left[\sum_{m_2\le N}a_{m_2}(m_1,N)\right]e(m_1\beta). \quad (41)$$

As $|e(-2N\beta)|=1$, we can ignore it in the bounds below. We again apply the integral version of partial summation with

$$a_{m_1}=\sum_{m_2\le N}a_{m_2}(m_1,N)=\sum_{m_2\le N}\left[\lambda(m_1)\lambda(m_2)e\left((m_1-2m_2)\frac{a}{q}\right)-C_q(a)\right]; \qquad b_{m_1}=e(m_1\beta). \quad (42)$$

We find

$$e(2N\beta)S_{a,q}(\alpha;B)=\sum_{m_1\le N}\left[\sum_{m_2\le N}a_{m_2}(m_1,N)\right]e(N\beta)-2\pi i\beta\int_{t=0}^{N}\sum_{m_1\le t}\left[\sum_{m_2\le N}a_{m_2}(m_1,N)\right]e(t\beta)dt. \quad (43)$$

To prove Lemma 1, it suffices to bound the two terms in (43), which we do in Lemmas 2 and 3.

Lemma 2

$$\sum_{m_1\le N}\left[\sum_{m_2\le N}a_{m_2}(m_1,N)\right]e(N\beta)=O\left(\frac{N^2}{\log^{C}N}\right). \quad (44)$$

Proof. As $|e(N\beta)|=1$, this factor is harmless, and the m_1,m_2-sums are bounded by the Siegel-Walfisz Theorem.

$$\sum_{m_1\le N}\sum_{m_2\le N}a_{m_2}(m_1,N)=\sum_{m_1\le N}\sum_{m_2\le N}\left[\lambda(m_1)\lambda(m_2)e\left((m_1-2m_2)\frac{a}{q}\right)-C_q(a)\right]=$$

$$=\left[\sum_{m_1\le N}\lambda(m_1)e\left(m_1\frac{a}{q}\right)\right]\left[\sum_{m_2\le N}\lambda(m_2)e\left(-m_2\frac{a}{q}\right)\right]-C_q(a)N^2=$$

$$=\left[\frac{c_q(a)N}{\phi(q)}+O\left(\frac{N}{\log^{C}N}\right)\right]\cdot\left[\frac{c_q(-2a)N}{\phi(q)}+O\left(\frac{N}{\log^{C}N}\right)\right]-C_q(a)N^2=$$

$$=O\left(\frac{N^2}{\log^{C}N}\right) \quad (45)$$

as $C_q(a)=\dfrac{c_q(a)c_q(-2a)}{\phi(q)^2}$ and $|c_q(b)|\le\phi(q)$.

Lemma 3

$$2\pi i\beta\int_{t=0}^{N}\sum_{m_1\le t}\left[\sum_{m_2\le N}a_{m_2}(m_1,N)\right]e(t\beta)dt=O\left(\frac{N^2}{\log^{C-D}N}\right). \quad (46)$$

Proof. Note $|\beta|\le\frac{Q}{N}=\frac{\log^D N}{N}$, and $C_q(a)=\frac{c_q(a)}{\phi(q)^2}\frac{c_q(-2a)}{\phi(q)^2}$. For $t\le\sqrt{N}$, we trivially bound the m_2-sum by $2N$. Thus these t contribute at most

$$|\beta|\int_{t=0}^{\sqrt{N}}\sum_{m_1\le t}2Ndt=|\beta|N^2\le N\log^D N. \quad (47)$$

An identical application of Siegel-Walfisz as in the proof of Lemma 2 yields for $t\ge\sqrt{N}$,

$$\sum_{m_1\le t}\sum_{m_2\le N}a_{m_2}(m_1,N)=\left[\frac{c_q(a)t}{\phi(q)}+O\left(\frac{t}{\log^C N}\right)\right]\cdot\left[\frac{c_q(-2a)N}{\phi(q)}+O\left(\frac{N}{\log^C N}\right)\right]-C_q(a)tN=$$

$$=O\left(\frac{tN}{\log^C N}\right). \quad (48)$$

Therefore

$$|\beta|\int_{t=\sqrt{N}}^{N}\left|\sum_{m_1\le t}\sum_{m_2\le N}a_{m_2}(m_1,N)\right|dt=O\left(\frac{N^3\beta}{\log^C N}\right)=O\left(\frac{N^2}{\log^{C-D}N}\right). \quad (49)$$

We note also that:

$$2\pi i\beta\int_{t=0}^{N}\sum_{m_1\le t}\left[\sum_{m_2\le N}a_{m_2}(m_1,N)\right]e(t\beta)dt=O\left(\frac{N^2}{\log^{C-D}N}\right)\Rightarrow$$

$$\Rightarrow|\beta|\int_{t=\sqrt{N}}^{N}\left|\sum_{m_1\le t}\sum_{m_2\le N}a_{m_2}(m_1,N)\right|dt=O\left(\frac{N^3\beta}{\log^C N}\right)=O\left(\frac{N^2}{\log^{C-D}N}\right). \quad (50)$$

We now deal with the integral term from the first partial summation on $m_2, S_{a,q}(\alpha;I)$.

Lemma 4

$$S_{a,q}(\alpha;I)=O\left(\frac{N^2}{\log^{C-2D}N}\right). \quad (51)$$

Proof. Recall

$$S_{a,q}(\alpha;I)=4\pi i\beta\sum_{m_1\le N}\left[\int_{u=1}^{N}\left[\sum_{m_2\le u}a_{m_2}(m_1,N)\right]e(-u\beta)du\right]e(m_1\beta) \quad (52)$$

where

$$a_{m_2}(m_1,N)=\lambda(m_1)\lambda(m_2)e\left((m_1-2m_2)\frac{a}{q}\right)-C_q(a). \quad (53)$$

Thence, the eq. (52) can be rewrite also as follow:

$$S_{a,q}(\alpha;I)=4\pi i\beta\sum_{m_1\leq N}\left[\int_{u=1}^{N}\left[\sum_{m_2\leq u}\lambda(m_1)\lambda(m_2)e\left((m_1-2m_2)\frac{a}{q}\right)-C_q(a)\right]e(-u\beta)du\right]e(m_1\beta). \quad (53b)$$

We apply the integral version of partial summation, with

$$a_{m_1}=\int_{u=1}^{N}\left[\sum_{m_2\leq u}a_{m_2}(m_1,N)\right]e(-u\beta)du \qquad b_{m_1}=e(m_1\beta). \quad (54)$$

We find

$$S_{a,q}(\alpha;I)=4\pi i\beta\left[\sum_{m_1\leq N}\int_{u=1}^{N}\sum_{m_2\leq u}a_{m_2}(m_1,N)e(-u\beta)du\right]e(N\beta)+$$

$$+8\pi\beta^2\int_{t=1}^{N}\left[\sum_{m_1\leq t}\int_{u=1}^{N}\sum_{m_2\leq u}a_{m_2}(m_1,N)e(-u\beta)du\right]e(m_1t)dt. \quad (55)$$

For the eq. (53), we can rewrite the eq. (55) also as follow:

$$S_{a,q}(\alpha;I)=4\pi i\beta\left[\sum_{m_1\leq N}\int_{u=1}^{N}\sum_{m_2\leq u}\lambda(m_1)\lambda(m_2)e\left((m_1-2m_2)\frac{a}{q}\right)-C_q(a)e(-u\beta)du\right]e(N\beta)+$$

$$+8\pi\beta^2\int_{t=1}^{N}\left[\sum_{m_1\leq t}\int_{u=1}^{N}\sum_{m_2\leq u}\lambda(m_1)\lambda(m_2)e\left((m_1-2m_2)\frac{a}{q}\right)-C_q(a)e(-u\beta)du\right]e(m_1t)dt. \quad (55b)$$

The factor of $8\pi\beta^2=-(4\pi i\beta)\cdot(2\pi i\beta)$ and comes from the derivative of $e(m_1\beta)$. Arguing in a similar manner as above in Theorem 1 and in Lemmas 5 and 6 we show the two terms in (55) are small, which will complete the proof.

Lemma 5

$$4\pi i\beta\left[\sum_{m_1\leq N}\int_{u=1}^{N}\sum_{m_2\leq u}a_{m_2}(m_1,N)e(-u\beta)du\right]e(N\beta)=O\left(\frac{N^2}{\log^{C-D}N}\right). \quad (56)$$

Proof. Arguing along the lines of Lemma 3, one shows the contribution from $u\leq\sqrt{N}$ is bounded by $N\log^D N$. For $u\geq\sqrt{N}$ we apply the Siegel-Walfisz formula as in Lemma 3, giving a contribution bounded by

$$4|\beta|\int_{u=\sqrt{N}}^{N}\left(\left[\frac{c_q(a)u}{\phi(q)}+O\left(\frac{u}{\log^C N}\right)\right]\cdot\left[\frac{c_q(-2a)N}{\phi(q)}+O\left(\frac{N}{\log^C N}\right)\right]-C_q(a)uN\right)du <<$$

$$<< |\beta|\int_{u=\sqrt{N}}^{N}\frac{uN}{\log^C N}du << \frac{N^3|\beta|}{\log^C N}. \quad (57)$$

As $|\beta|\leq\frac{\log^B N}{N}$, the above is $O\left(\frac{N^2}{\log^{C-D} N}\right)$.

Lemma 6

$$8\pi\beta^2\int_{t=1}^{N}\left[\sum_{m_1\leq t}\int_{u=1}^{N}\sum_{m_2\leq u}a_{m_2}(m_1,N)e(-u\beta)du\right]e(m_1 t)dt = O\left(\frac{N^2}{\log^{C-2D} N}\right). \quad (58)$$

Proof. Arguing as in Lemma 3, one shows that the contribution when $t\leq\sqrt{N}$ or $u\leq\sqrt{N}$ is $O\left(\frac{N}{\log^{C-2D} N}\right)$. We then apply the Siegel-Walfisz Theorem as before, and find the contribution when $t,u\geq\sqrt{N}$ is

$$<< 8\beta^2\int_{t=\sqrt{N}}^{N}\int_{u=\sqrt{N}}^{N}\frac{ut}{\log^C N}dudt << \frac{N^4\beta^2}{\log^C N}. \quad (59)$$

As $|\beta|\leq\frac{\log^D N}{N}$, the above is $O\left(\frac{N^2}{\log^{C-2D} N}\right)$. This complete the proof of Theorem 1.

We note that, for the eq. (56) and (58), the eq. (55) can be rewritten also as follows:

$$S_{a,q}(\alpha;I)=4\pi i\beta\left[\sum_{m_1\leq N}\int_{u=1}^{N}\sum_{m_2\leq u}a_{m_2}(m_1,N)e(-u\beta)du\right]e(N\beta)+$$

$$+8\pi\beta^2\int_{t=1}^{N}\left[\sum_{m_1\leq t}\int_{u=1}^{N}\sum_{m_2\leq u}a_{m_2}(m_1,N)e(-u\beta)du\right]e(m_1 t)dt = O\left(\frac{N^2}{\log^{C-D} N}\right)+O\left(\frac{N^2}{\log^{C-2D} N}\right). \quad (59b)$$

With regard the integrals over the Major arcs, we first compute the integral of $u(x)e(-x)$ over the Major arcs and then use Theorem 1 to deduce the corresponding integral of $F_N(x)e(-x)$.

By Theorem 1 we know for $x\in\mathcal{M}_{a,q}$ that

$$\left|F_N(x)-C_q(a)u\left(x-\frac{a}{q}\right)\right| << O\left(\frac{N^2}{\log^{C-2D} N}\right). \quad (60)$$

We now evaluate the integral of $u\left(x-\frac{a}{q}\right)e(-x)$ over $\mathcal{M}_{a,q}$; by Theorem 1 we then obtain the integral of $F_N(x)e(-x)$ over $\mathcal{M}_{a,q}$. Remember that

$$u(x)=\sum_{m_1,m_2\le N} e((m_1-2m_2)x). \quad (61)$$

Theorem 2

$$\int_{\mathcal{M}_{a,q}} u\left(\alpha-\frac{a}{q}\right)\cdot e(-\alpha)d\alpha = e\left(-\frac{a}{q}\right)\frac{N}{2}+O\left(\frac{N}{\log^D N}\right). \quad (62)$$

We first determine the integral of u over all of $\left[-\frac{1}{2},\frac{1}{2}\right]$, and then show that the integral of $u(x)$ is small if $|x|>\frac{Q}{N}$.

Lemma 7

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} u(x)e(-x)dx = \frac{N}{2}+O(1). \quad (63)$$

Proof.

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} u(x)e(-x)dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{m_1\le N}\sum_{m_2\le N} e((m_1-2m_2)x)\cdot e(-x)dx = \sum_{m_1\le N}\sum_{m_2\le N}\int_{-\frac{1}{2}}^{\frac{1}{2}} e((m_1-2m_2-1)x)dx. \quad (64)$$

The integral is 1 if $m_1-2m_2-1=0$ and 0 otherwise. For $m_1,m_2\in\{1,\dots,N\}$, there are $\left[\frac{N}{2}\right]=\frac{N}{2}+O(1)$ solutions to $m_1-2m_2-1=0$, which completes the proof.

Define

$$I_1=\left[-\frac{1}{2}+\frac{Q}{N},-\frac{Q}{N}\right], \qquad I_2=\left[\frac{Q}{N},\frac{1}{2}-\frac{Q}{N}\right]. \quad (65)$$

The following bound is crucial in our investigations.

Lemma 8

For $x\in I_1$ *or* I_2, $\frac{1}{1-e(ax)} << \frac{1}{x}$ *for* $a\in\{1,-2\}$.

Lemma 9

$$\int_{x\in I_1\cup I_2} u(x)e(-x)dx = O\left(\frac{N}{\log^D N}\right). \quad (66)$$

Proof. We have

$$\int_{I_i} u(x)e(-x)dx = \int_{I_i} \sum_{m_1, m_2 \le N} e((m_1 - 2m_2 - 1)x)dx = \int_{I_i} \sum_{m_1 \le N} e(m_1 x) \sum_{m_2 \le N} e(-2m_2 x) \cdot e(-x)dx =$$

$$= \int_{I_i} \left[\frac{e(x) - e((N+1)x)}{1 - e(x)} \right] \left[\frac{e(-2x) - e(-2(N+1)x)}{1 - e(-2x)} \right] e(-x)dx \quad (67)$$

because these are geometric series. By Lemma 8, we have

$$\int_{I_i} u(x)e(-x)dx << \int_{I_i} \frac{2}{x}\frac{2}{x} dx << \frac{N}{Q} = \frac{N}{\log^D N}, \quad (68)$$

which completes the proof of Lemma 9.

Lemma 10

$$\int_{x=\frac{1}{2}-\frac{Q}{N}}^{\frac{1}{2}+\frac{Q}{N}} u(x)e(-x)dx = O\left(\log^D N\right). \quad (69)$$

Lemma 11

$$\int_{-\frac{Q}{N}}^{\frac{Q}{N}} u(x)e(-x)dx = \frac{N}{2} + O\left(\frac{N}{\log^D N}\right). \quad (70)$$

Proof of Theorem 2. We have

$$\int_{\mathcal{M}_{a,q}} u\left(\alpha - \frac{a}{q}\right) \cdot e(-\alpha)d\alpha = \int_{\frac{a}{q}-\frac{Q}{N}}^{\frac{a}{q}+\frac{Q}{N}} u\left(\alpha - \frac{a}{q}\right) \cdot e(-\alpha)d\alpha = \int_{-\frac{Q}{N}}^{\frac{Q}{N}} u(\beta) \cdot e\left(-\frac{a}{q} - \beta\right) d\beta =$$

$$= e\left(-\frac{a}{q}\right) \int_{-\frac{Q}{N}}^{\frac{Q}{N}} u(\beta)e(-\beta)d\beta = e\left(-\frac{a}{q}\right)\frac{N}{2} + O\left(\frac{N}{\log^D N}\right). \quad (71)$$

Note there are two factors in Theorem 2. The first, $e\left(-\frac{a}{q}\right)$, is an arithmetical factor which depends on which Major arc $\mathcal{M}_{a,q}$ we are in. The second factor is universal, and is the size of the contribution.

An immediate consequence of Theorem 2 is

Theorem 3

$$\int_{\mathcal{M}_{a,q}} F_N(x)e(-x)dx = C_q(a)e\left(-\frac{a}{q}\right)\frac{N}{2} + O\left(\frac{N}{\log^D N}\right) + O\left(\frac{N}{\log^{C-3D} N}\right). \quad (72)$$

From Theorem 3 we immediately obtain the integral of $F_N(x)e(-x)$ over the Major arcs $\mathcal{M}$:

Theorem 4

$$\int_{\mathcal{M}} F_N(x)e(-x)dx = \sum_{q=1}^{\log^B N} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} C_q(a)e\left(-\frac{a}{q}\right)\frac{N}{2} + O\left(\frac{N}{\log^{D-2B} N} + \frac{N}{\log^{C-3D-2B} N}\right) =$$

$$= \mathfrak{I}_N \frac{N}{2} + O\left(\frac{N}{\log^{D-2B} N} + \frac{N}{\log^{C-3D-2B} N}\right), \quad (73)$$

where

$$\mathfrak{I}_N = \sum_{q=1}^{\log^B N} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} C_q(a)e\left(-\frac{a}{q}\right) \quad (74)$$

is the truncated singular series for the Germain primes.

3. On some equations concerning the equivalence between the Goldbach's Conjecture and the Generalized Riemann Hypothesis [3]

We know the Goldbach's conjecture: "*Every even integer > 2 is the sum of two primes*". In 1922 Hardy and Littlewood guesstimated, via a heuristic based on the circle method, an asymptotic for the number of representations of an even integer as the sum of two primes: Define

$$g(2N) = \#\ \{p, q \text{ prime : } p + q = 2N\}.$$

Their conjecture is equivalent to $g(2N) \approx I(2N)$ where

$$I(2N) = C_2 \prod_{\substack{p|N \\ p>2}} \left(\frac{p-1}{p-2}\right) \int_2^{2N-2} \frac{dt}{\log t \log(2N-t)} \quad (75)$$

and C_2, the "twin prime constant", is defined by

$$C_2 = 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right) = 1.320323... \quad (76)$$

Thence, the eq. (75) can be rewritten also as follows:

$$I(2N) = 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p|N \\ p>2}} \left(\frac{p-1}{p-2}\right) \int_2^{2N-2} \frac{dt}{\log t \log(2N-t)} \quad (77)$$

and thence, we obtain:

$$g(2N) \approx 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p|N \\ p>2}} \left(\frac{p-1}{p-2}\right) \int_2^{2N-2} \frac{dt}{\log t \log(2N-t)}, \quad (78)$$

or

$$g(2N) \approx C_2 \prod_{\substack{p|N \\ p>2}} \left(\frac{p-1}{p-2} \right) \int_2^{2N-2} \frac{dt}{\log t \log(2N-t)} . \quad (78b)$$

We believe that a better guesstimate for $g(2N)$ is given by

$$I^*(2N) := I(2N) \left(1 - \frac{4}{\sqrt{2N}} \prod_{p \geq 3} \left(1 - \frac{(2N/p)}{p-2} \right) \right) . \quad (79)$$

Thence, for eq. (75) we can rewrite the eq. (79) also as follows:

$$I^*(2N) := C_2 \prod_{\substack{p|N \\ p>2}} \left(\frac{p-1}{p-2} \right) \int_2^{2N-2} \frac{dt}{\log t \log(2N-t)} \left(1 - \frac{4}{\sqrt{2N}} \prod_{p \geq 3} \left(1 - \frac{(2N/p)}{p-2} \right) \right) . \quad (79b)$$

Indeed it could well be that

$$g(2N) = I^*(2N) + O\left(\frac{\sqrt{N}}{\log N} \log\log N \right) . \quad (80)$$

Thence, for eq. (79b), we obtain the following equation:

$$g(2N) = C_2 \prod_{\substack{p|N \\ p>2}} \left(\frac{p-1}{p-2} \right) \int_2^{2N-2} \frac{dt}{\log t \log(2N-t)} \left(1 - \frac{4}{\sqrt{2N}} \prod_{p \geq 3} \left(1 - \frac{(2N/p)}{p-2} \right) \right) + O\left(\frac{\sqrt{N}}{\log N} \log\log N \right) .$$

(80b)

We introduce the function

$$G(2N) = \sum_{\substack{p+q=2N \\ p,q(prime)}} \log p \log q . \quad (81)$$

The analysis of Hardy and Littlewood suggests that $G(2N)$, plus some terms corresponding to solutions of $p^k + q^l = 2N$, should be very "well-approximated" by

$$J(2N) := C_2 \prod_{\substack{p|N \\ p>2}} \left(\frac{p-1}{p-2} \right) \cdot 2N , \quad (82)$$

and the approximation $g(2N) \approx I(2N)$ is then deduced by partial summation. (In fact we believe that $G(2N) = J(2N) + O\left(N^{1/2+o(1)}\right)$.)

Theorem 1

The Riemann Hypothesis is equivalent to estimate

$$\sum_{2N \leq x} (G(2N) - J(2N)) << x^{3/2+o(1)} . \quad (83)$$

Theorem 2

The Riemann Hypothesis for Dirichlet L-functions $L(s,\chi)$*, over all characters* $\chi \bmod m$ *which are odd squarefree divisors of* q*, is equivalent to the estimate*

$$\sum_{\substack{2N\le x\\ 2N\equiv 2(\bmod q)}}(G(2N)-J(2N)) << x^{3/2+o(1)}. \quad (84)$$

Theorem 3

The Riemann Hypothesis for Dirichlet L-functions $L(s,\chi),\chi \bmod q$ *is equivalent to the conjectured estimate*

$$\sum_{\substack{2N\le x\\ q|2N}} G(2N)=\frac{1}{\phi(q)}\sum_{2N\le x}G(2N)+O\left(x^{1+o(1)}\right). \quad (85)$$

Let

$$E(2N)=\sum_{\substack{p^k+q^l=2N\\ k,l\ge 1\\ k+l\ge 3}}\log p\log q. \quad (86)$$

First note that

$$\sum_{\substack{p^k+q^l=2N\\ k\ge 3}}\log p\log q\le \log^2 N\cdot \sum_{\substack{p^k\le 2N\\ k\ge 3}}1 << N^{1/3}\log^2 N, \quad (87)$$

and a similar argument works for $l\ge 3$. Also it is well-known that there are $N^{o(1)}$ pairs of integer p,q with $p^2+q^2=2N$. Thus

$$E(2N)=2\sum_{p+q^2=2N}\log p\log q+O\left(N^{1/3}\log^2 N\right). \quad (88)$$

Now, when we study solutions to $p+q^2=2N$ we find that l divides p if and only if $2N\equiv q^2 \bmod l$. Thus if $(2N/l)=0$ or -1 then l divides pq if and only if $q\equiv 0 \bmod l$. If $(2N/l)=1$ then there are 2 non-zero values of $q \bmod l$ for which l divides p, and we also need to count when l divides q. Therefore our factor is 2 if $l=2$, and

$$\frac{\left(1-\frac{2+(2N/l)}{l}\right)}{(1-1/l)^2} \text{ times } \begin{cases}1\\ (l-1)/(l-2)\end{cases} \quad \text{if} \quad \begin{cases}l/2n\\ l|2N, l\ge 3\end{cases}$$

Now $\#\{m,n>0: m+n^2=2N\}=\sqrt{2N}+O(1)$ so we predict that

$$\sum_{p+q^2=2N}\log p\log q\approx \prod_{l\ge 3}\left(1-\frac{(2N/l)}{l-2}\right)C_2\prod_{\substack{p|2N\\ p>2}}\left(\frac{p-1}{p-2}\right)\sqrt{2N}, \quad (89)$$

and thus, after partial summation, that

$$2\sum_{\substack{p+q^2=2N\\(p,q)\ prime}} 1 \approx 4\prod_{l\geq 3}\left(1-\frac{(2N/l)}{l-2}\right)C_2\prod_{\substack{p|2N\\p>2}}\left(\frac{p-1}{p-2}\right)\frac{\sqrt{2N}}{\log^2(2N)}. \quad (90)$$

Subtracting this from $I(2N)$, we obtain the prediction $I^*(2N)$, as in (79). We can give the more accurate prediction

$$I^*(2N)=C_2\prod_{\substack{p|N\\p>2}}\left(\frac{p-1}{p-2}\right)\int_2^{2N-2}\frac{dt}{\log t\log(2N-t)}\left(1-\prod_{p\geq 3}\left(1-\frac{(2N/p)}{p-2}\right)\left(\frac{1}{\sqrt{t}}+\frac{1}{\sqrt{2N-t}}\right)\right)dt. \quad (91)$$

The explicit version of the Prime Number Theorem gives a formula of the form

$$\sum_{p\leq x}\log p = x-\sum_{\substack{\rho\\|\mathrm{Im}\,\rho|\leq x}}\frac{x^\rho}{\rho}+O(\log^2 x), \quad (92)$$

where the sum is over zeros ρ of $\zeta(\rho)=0$ with $\mathrm{Re}(\rho)>0$. In Littlewood's famous paper he investigates the sign of $\pi(x)-Li(x)$ by a careful examination of a sum of the form $\sum_{\rho:|\mathrm{Im}\,\rho|\leq T} Li(x^\rho)$, showing that this gets bigger than $x^{1/2-\varepsilon}$ for certain values of x, and smaller than $-x^{1/2-\varepsilon}$ for other values of x. His method can easily be modified to show that the above implies that

$$\max_{y\leq x}\left|\sum_{p\leq y}\log p - y\right| = x^{B+o(1)} \quad (93)$$

where $B=\sup\{\mathrm{Re}\,\rho:\zeta(\rho)=0\}$ (note that $1\geq B\geq 1/2$). By partial summation it is not hard to show that

$$\sum_{2N\leq x}G(2N)=\sum_{p+q\leq x}\log p\log q=\frac{x^2}{2}-2\sum_{\substack{\rho\\|\mathrm{Im}\,\rho|\leq x}}\frac{x^{1+\rho}}{\rho(1+\rho)}+O\left(x^{2B+o(1)}\right) \quad (94)$$

so that, by Littlewood's method,

$$\max_{y\leq x}\left|\sum_{2N\leq y}G(2N)-\frac{y^2}{2}\right| = x^{1+B+o(1)}. \quad (95)$$

Therefore the Riemann Hypothesis $(B=1/2)$ is equivalent to the conjectured estimate

$$\sum_{2N\leq x}G(2N)=\frac{x^2}{2}+O\left(x^{3/2+o(1)}\right). \quad (96)$$

This implies Theorem 1 since

$$\sum_{2n\le x} J(2n) = C_2 \sum_{2n\le x} 2n \sum_{\substack{d|n\\ d(odd)}} \frac{\mu^2(d)}{\prod_{p|d}(p-2)} = 2C_2 \sum_{\substack{d\le x/2\\ d(odd)}} \frac{\mu^2(d)}{\prod_{p|d}(p-2)} \sum_{\substack{n\le x/2\\ d|n}} n$$

$$= 2C_2 \sum_{\substack{d\le x/2\\ d(odd)}} \frac{\mu^2(d)}{\prod_{p|d}(p-2)} \left(\frac{x^2}{8d} + O(x)\right) = \frac{x^2}{2} + O(x\log x). \quad (97)$$

Going further we note that for any coprime integers $a, q \ge 2$

$$\sum_{\substack{p\le x\\ p\equiv a(\mathrm{mod}\, q)}} \log p = \frac{1}{\phi(q)} \left(x - \sum_{\chi(\mathrm{mod}\, q)} \overline{\chi}(a) \sum_{\substack{\rho: L(\rho,\chi)=0\\ |\mathrm{Im}\,\rho|\le x}} \frac{x^\rho}{\rho} \right) + O(\log^2(qx)); \quad (98)$$

and thus

$$\max_{y\le x} \left| \sum_{\substack{p\le y\\ p\equiv a(\mathrm{mod}\, q)}} \log p - \frac{y}{\phi(q)} \right| = x^{B_q + o(1)}, \quad (99)$$

where $B_q = \sup\{\mathrm{Re}\,\rho : L(\rho,\chi) = 0\}$ for some $\chi(\mathrm{mod}\, q)$. R.C. Vaughan noted that by the same methods but now using the above formula, we get a remarkable cancellation which leads to the explicit formula

$$\sum_{\substack{2N\le x\\ q|2N}} G(2N) - \frac{1}{\phi(q)} \sum_{2N\le x} G(2N) = \frac{1}{\phi(q)} \sum_{\substack{\chi(\mathrm{mod}\, q)\\ \chi\ne\chi_0}} \chi(-1) \sum_{\substack{\rho: L(\rho,\lambda)=0\\ \sigma: L(\sigma,\overline{\chi})=0\\ |\mathrm{Im}\,\rho|, |\mathrm{Im}\,\sigma|\le x}} c_{\rho,\sigma} x^{\rho+\sigma} + O(x\log^2(qx)) \quad (100)$$

where $c_{\rho,\sigma} = \int_0^1 \frac{1}{\rho}(1-t)^\rho t^{\sigma-1} dt$ is a constant depending only on ρ and σ. Thus Theorem 3 follows since $c_{\rho,\sigma} \le (1/\rho)\int_0^1 t^{\sigma-1} dt = 1/\rho\sigma$ and has $\sum_{|\mathrm{Im}\,\sigma|\le x} 1/\rho << \log^2(qx)$. Thence, the eq. (100) can be rewritten also as follows:

$$\sum_{\substack{2N\le x\\ q|2N}} G(2N) - \frac{1}{\phi(q)} \sum_{2N\le x} G(2N) = \frac{1}{\phi(q)} \sum_{\substack{\chi(\mathrm{mod}\, q)\\ \chi\ne\chi_0}} \chi(-1) \sum_{\substack{\rho: L(\rho,\lambda)=0\\ \sigma: L(\sigma,\overline{\chi})=0\\ |\mathrm{Im}\,\rho|, |\mathrm{Im}\,\sigma|\le x}} \int_0^1 \frac{1}{\rho}(1-t)^\rho t^{\sigma-1} dt \cdot x^{\rho+\sigma} + O(x\log^2(qx))$$

(100b)

As in the proof of (97) we have

$$\sum_{\substack{2n\le x\\ q|2n}} J(2n) = \frac{x^2}{2\phi(q)} + O(x\log x). \quad (101)$$

Now, Hardy and Littlewood showed that Generalized Riemann Hypothesis implies that

$$\sum_{2n\le x}\left|G(2n)-J(2n)\right|^2 << x^{5/2+o(1)}. \quad (102)$$

We expect, as we saw in the precedent passages, that $G(2n)-J(2n)<<n^{1/2+o(1)}$ and so we believe that

$$\sum_{2n\le x}\left|G(2n)-J(2n)\right|^2 << x^{2+\delta+o(1)} \quad (103)$$

for $\delta=0$. This implies, by Cauchy's inequality, that

$$\sum_{2n\le x}G(2n)=\sum_{2n\le x}J(2n)+O\left(x^{\frac{3+\delta}{2}+o(1)}\right)=x^2/2+O\left(x^{(3+\delta)/2+o(1)}\right) \quad (104)$$

by (97), which implies the Riemann Hypothesis if $\delta=0$ (as after (96) above); and implies that $\zeta(\rho)\neq 0$ if $\operatorname{Re}\rho>3/4$ if $\delta=1/2$ (that is, assuming Hardy and Littlewood's (102)).

We find that (85) is too delicate to obtain the Riemann Hypothesis for $L(s,\chi),\chi(\bmod q)$ from (103). Instead we note that

$$\sum_{\substack{2N\le x\\ 2N\equiv 2(\bmod q)}} G(2N)=c_q\left(\frac{x^2}{2}-2\sum_{\substack{m|q\\ m(odd)}}\frac{\mu(m)}{\prod_{\substack{p|m\\ p>2}}(p-2)}\sum_{\substack{\chi \bmod m\\ \chi(primitive)}}\overline{\chi}(2)\sum_{\substack{\rho:L(\rho,\chi)=0\\ |\operatorname{Im}\rho|\le x}}\frac{x^{\rho+1}}{\rho(\rho+1)}\right) \quad (105)$$

plus an error term $O\left(x^{2B_q+o(1)}\right)$, where $c_q=\frac{(2,q)}{q}\prod_{\substack{p|q\\ p(odd)}}\frac{p(p-2)}{(p-1)^2}$. As in (97) one can show that

$$\sum_{\substack{2N\le x\\ 2N\equiv 2(\bmod q)}} J(2N)=c_q\frac{x^2}{2}+O(x\log x), \quad (106)$$

so that

$$\sum_{\substack{2N\le x\\ 2N\equiv 2(\bmod q)}}(G(2N)-J(2N))=O\left(x^{1+C_q+o(1)}\right) \quad (107)$$

where $C_q=\sup\{\operatorname{Re}\rho : L(\rho,\chi)=0\}$ for some $\chi \bmod m$, where $m|q$ and m is odd and squarefree. This implies Theorem 2. By the above we see that if (103) holds with $\delta=0$ then $C_q=1/2$ and thus the Riemann Hypothesis follows for L-functions with squarefree conductor.

4. On some equations concerning the p-adic strings and the zeta strings [4] [5] [6] [7].

Like in the ordinary string theory, the starting point of p-adic strings is a construction of the corresponding scattering amplitudes. Recall that the ordinary crossing symmetric Veneziano amplitude can be presented in the following forms:

$$A_\infty(a,b) = g^2 \int_R |x|_\infty^{a-1} |1-x|_\infty^{b-1} dx = g^2 \left[\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} + \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)} + \frac{\Gamma(c)\Gamma(a)}{\Gamma(c+a)} \right] = g^2 \frac{\zeta(1-a)}{\zeta(a)} \frac{\zeta(1-b)}{\zeta(b)} \frac{\zeta(1-c)}{\zeta(c)} =$$

$$= g^2 \int DX \exp\left(-\frac{i}{2\pi} \int d^2\sigma \partial^\alpha X_\mu \partial_\alpha X^\mu \right) \prod_{j=1}^{4} \int d^2\sigma_j \exp\left(ik_\mu^{(j)} X^\mu \right), \quad (108-111)$$

where $\hbar = 1$, $T = 1/\pi$, and $a = -\alpha(s) = -1 - \frac{s}{2}$, $b = -\alpha(t)$, $c = -\alpha(u)$ with the condition $s + t + u = -8$, i.e. $a + b + c = 1$.

The p-adic generalization of the above expression

$$A_\infty(a,b) = g^2 \int_R |x|_\infty^{a-1} |1-x|_\infty^{b-1} dx,$$

is:

$$A_p(a,b) = g_p^2 \int_{Q_p} |x|_p^{a-1} |1-x|_p^{b-1} dx, \quad (112)$$

where $|\ldots|_p$ denotes p-adic absolute value. In this case only string world-sheet parameter x is treated as p-adic variable, and all other quantities have their usual (real) valuation.

Now, we remember that the Gauss integrals satisfy adelic product formula

$$\int_R \chi_\infty(ax^2 + bx) d_\infty x \prod_{p \in P} \int_{Q_p} \chi_p(ax^2 + bx) d_p x = 1, \quad a \in Q^\times, \quad b \in Q, \quad (113)$$

what follows from

$$\int_{Q_v} \chi_v(ax^2+bx)d_v x = \lambda_v(a)|2a|_v^{-\frac{1}{2}} \chi_v\left(-\frac{b^2}{4a}\right), \quad v=\infty,2,\ldots,p\ldots . \quad (114)$$

These Gauss integrals apply in evaluation of the Feynman path integrals

$$K_v(x'',t'';x',t') = \int_{x',t'}^{x'',t''} \chi_v\left(-\frac{1}{h}\int_{t'}^{t''} L(\dot{q},q,t)dt\right)D_v q, \quad (115)$$

for kernels $K_v(x'',t'';x',t')$ of the evolution operator in adelic quantum mechanics for quadratic Lagrangians. In the case of Lagrangian

$$L(\dot{q},q) = \frac{1}{2}\left(-\frac{\dot{q}^2}{4} - \lambda q + 1\right),$$

for the de Sitter cosmological model one obtains

$$K_\infty(x'',T;x',0)\prod_{p\in P} K_p(x'',T;x',0) = 1, \quad x'',x',\lambda \in Q, \; T \in Q^\times, \quad (116)$$

where

$$K_v(x'',T;x',0) = \lambda_v(-8T)|4T|_v^{-\frac{1}{2}} \chi_v\left(-\frac{\lambda^2 T^3}{24} + [\lambda(x''+x')-2]\frac{T}{4} + \frac{(x''-x')^2}{8T}\right). \quad (117)$$

Also here we have the number 24 that correspond to the Ramanujan function that has 24 “modes”, i.e., the physical vibrations of a bosonic string. Hence, we obtain the following mathematical connection:

$$K_v(x'',T;x',0) = \lambda_v(-8T)|4T|_v^{-\frac{1}{2}} \chi_v\left(-\frac{\lambda^2 T^3}{24} + [\lambda(x''+x')-2]\frac{T}{4} + \frac{(x''-x')^2}{8T}\right) \Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \quad . \quad (117b)$$

The adelic wave function for the simplest ground state has the form

$$\psi_A(x) = \psi_\infty(x) \prod_{p \in P} \Omega(|x|_p) = \begin{cases} \psi_\infty(x), x \in Z \\ 0, x \in Q \setminus Z \end{cases} \quad , \quad (118)$$

where $\Omega(|x|_p) = 1$ if $|x|_p \leq 1$ and $\Omega(|x|_p) = 0$ if $|x|_p > 1$. Since this wave function is non-zero only in integer points it can be interpreted as discreteness of the space due to p-adic effects in adelic approach. The Gel'fand-Graev-Tate gamma and beta functions are:

$$\Gamma_\infty(a) = \int_R |x|_\infty^{a-1} \chi_\infty(x) d_\infty x = \frac{\zeta(1-a)}{\zeta(a)} \,, \quad \Gamma_p(a) = \int_{Q_p} |x|_p^{a-1} \chi_p(x) d_p x = \frac{1 - p^{a-1}}{1 - p^{-a}} \,, \quad (119)$$

$$B_\infty(a,b) = \int_R |x|_\infty^{a-1} |1-x|_\infty^{b-1} d_\infty x = \Gamma_\infty(a)\Gamma_\infty(b)\Gamma_\infty(c) \,, \quad (120)$$

$$B_p(a,b) = \int_{Q_p} |x|_p^{a-1} |1-x|_p^{b-1} d_p x = \Gamma_p(a)\Gamma_p(b)\Gamma_p(c) \,, \quad (121)$$

where $a,b,c \in C$ with condition $a+b+c=1$ and $\zeta(a)$ is the Riemann zeta function. With a regularization of the product of p-adic gamma functions one has adelic products:

$$\Gamma_\infty(u) \prod_{p \in P} \Gamma_p(u) = 1 \,, \quad B_\infty(a,b) \prod_{p \in P} B_p(a,b) = 1 \,, \quad u \neq 0,1, \quad u = a,b,c, \quad (122)$$

where $a+b+c=1$. We note that $B_\infty(a,b)$ and $B_p(a,b)$ are the crossing symmetric standard and p-adic Veneziano amplitudes for scattering of two open tachyon strings. Introducing real, p-adic and adelic zeta functions as

$$\zeta_\infty(a)=\int_R \exp\left(-\pi x^2\right)|x|_\infty^{a-1} d_\infty x=\pi^{-\frac{a}{2}}\Gamma\left(\frac{a}{2}\right), \quad (123)$$

$$\zeta_p(a)=\frac{1}{1-p^{-1}}\int_{Q_p}\Omega\left(|x|_p\right)|x|_p^{a-1} d_p x=\frac{1}{1-p^{-a}}, \quad \operatorname{Re} a>1, \quad (124)$$

$$\zeta_A(a)=\zeta_\infty(a)\prod_{p\in P}\zeta_p(a)=\zeta_\infty(a)\zeta(a), \quad (125)$$

one obtains

$$\zeta_A(1-a)=\zeta_A(a), \quad (126)$$

where $\zeta_A(a)$ can be called adelic zeta function. We have also that

$$\zeta_A(a)=\zeta_\infty(a)\prod_{p\in P}\zeta_p(a)=\zeta_\infty(a)\zeta(a)=\int_R \exp\left(-\pi x^2\right)|x|_\infty^{a-1} d_\infty x\cdot\frac{1}{1-p^{-1}}\int_{Q_p}\Omega\left(|x|_p\right)|x|_p^{a-1} d_p x. \quad (126b)$$

Let us note that $\exp\left(-\pi x^2\right)$ and $\Omega\left(|x|_p\right)$ are analogous functions in real and p-adic cases. Adelic harmonic oscillator has connection with the Riemann zeta function. The simplest vacuum state of the adelic harmonic oscillator is the following Schwartz-Bruhat function:

$$\psi_A(x)=2^{\frac{1}{4}}e^{-\pi x_\infty^2}\prod_{p\in P}\Omega\left(|x_p|_p\right), \quad (127)$$

whose the Fourier transform

$$\psi_A(k)=\int \chi_A(kx)\psi_A(x)=2^{\frac{1}{4}}e^{-\pi k_\infty^2}\prod_{p\in P}\Omega\left(\left|k_p\right|_p\right) \quad (128)$$

has the same form as $\psi_A(x)$. The Mellin transform of $\psi_A(x)$ is

$$\Phi_A(a)=\int \psi_A(x)|x|^a d_A^\times x=\int_R \psi_\infty(x)|x|^{a-1}d_\infty x\prod_{p\in P}\frac{1}{1-p^{-1}}\int_{Q_p}\Omega\left(|x|_p\right)|x|^{a-1}d_p x=\sqrt{2}\Gamma\left(\frac{a}{2}\right)\pi^{-\frac{a}{2}}\zeta(a) \quad (129)$$

and the same for $\psi_A(k)$. Then according to the Tate formula one obtains (126).

The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$\mathcal{L}_p=\frac{1}{g^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2}}\varphi+\frac{1}{p+1}\varphi^{p+1}\right], \quad (130)$$

where p is any prime number, $\Box=-\partial_t^2+\nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+\ldots+)$. Now, we want to show a model which incorporates the p-adic string Lagrangians in a restricted adelic way. Let us take the following Lagrangian

$$L=\sum_{n\geq1}C_n\mathcal{L}_n=\sum_{n\geq1}\frac{n-1}{n^2}\mathcal{L}_n=\frac{1}{g^2}\left[-\frac{1}{2}\phi\sum_{n\geq1}n^{-\frac{\Box}{2}}\phi+\sum_{n\geq1}\frac{1}{n+1}\phi^{n+1}\right]. \quad (131)$$

Recall that the Riemann zeta function is defined as

$$\zeta(s)=\sum_{n\geq1}\frac{1}{n^s}=\prod_p\frac{1}{1-p^{-s}}, \quad s=\sigma+i\tau, \quad \sigma>1. \quad (132)$$

Employing usual expansion for the logarithmic function and definition (132) we can rewrite (131) in the form

$$L = -\frac{1}{g^2}\left[\frac{1}{2}\phi\zeta\left(\frac{\Box}{2}\right)\phi + \phi + \ln(1-\phi)\right], \quad (133)$$

where $|\phi| < 1$. $\zeta\left(\frac{\Box}{2}\right)$ acts as pseudodifferential operator in the following way:

$$\zeta\left(\frac{\Box}{2}\right)\phi(x) = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk, \quad -k^2 = k_0^2 - \vec{k}^2 > 2 + \varepsilon, \quad (134)$$

where $\tilde{\phi}(k) = \int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.

Dynamics of this field ϕ is encoded in the (pseudo)differential form of the Riemann zeta function. When the d'Alambertian is an argument of the Riemann zeta function we shall call such string a "zeta string". Consequently, the above ϕ is an open scalar zeta string. The equation of motion for the zeta string ϕ is

$$\zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} \quad (135)$$

which has an evident solution $\phi = 0$.

For the case of time dependent spatially homogeneous solutions, we have the following equation of motion for the zeta string

$$\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t) = \frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon} e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0 = \frac{\phi(t)}{1-\phi(t)}. \quad (136)$$

With regard the open and closed scalar zeta strings, the equations of motion are

$$\zeta\left(\frac{\square}{2}\right)\phi = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \sum_{n\geq 1}\theta^{\frac{n(n-1)}{2}}\phi^n \quad , \quad (137)$$

$$\zeta\left(\frac{\square}{4}\right)\theta = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk = \sum_{n\geq 1}\left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right], \quad (138)$$

and one can easily see trivial solution $\phi = \theta = 0$.

The exact tree-level Lagrangian of effective scalar field φ, which describes open p-adic string tachyon, is:

$$\mathcal{L}_p = \frac{m_p^D}{g_p^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\square}{2m_p^2}}\varphi + \frac{1}{p+1}\varphi^{p+1}\right], \quad (139)$$

where p is any prime number, $\square = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+\ldots+)$, as above. Now, we want to introduce a model which incorporates all the above string Lagrangians (139) with p replaced by $n \in N$. Thence, we take the sum of all Lagrangians $\mathcal{L}_n$ in the form

$$L = \sum_{n=1}^{+\infty} C_n \mathcal{L}_n = \sum_{n=1}^{+\infty} C_n \frac{m_n^D}{g_n^2}\frac{n^2}{n-1}\left[-\frac{1}{2}\phi n^{-\frac{\square}{2m_n^2}}\phi + \frac{1}{n+1}\phi^{n+1}\right], \quad (140)$$

whose explicit realization depends on particular choice of coefficients C_n, masses m_n and coupling constants g_n.

Now, we consider the following case

$$C_n = \frac{n-1}{n^{2+h}}, \quad (141)$$

where h is a real number. The corresponding Lagrangian reads

$$L_h = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\sum_{n=1}^{+\infty} n^{-\frac{\Box}{2m^2}-h}\phi + \sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right] \quad (142)$$

and it depends on parameter h. According to the Euler product formula one can write

$$\sum_{n=1}^{+\infty} n^{-\frac{\Box}{2m^2}-h} = \prod_p \frac{1}{1-p^{-\frac{\Box}{2m^2}-h}} . \quad (143)$$

Recall that standard definition of the Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{+\infty}\frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}, \quad s=\sigma+i\tau, \quad \sigma>1, \quad (144)$$

which has analytic continuation to the entire complex s plane, excluding the point $s=1$, where it has a simple pole with residue 1. Employing definition (144) we can rewrite (142) in the form

$$L_h = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\zeta\left(\frac{\Box}{2m^2}+h\right)\phi + \sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right]. \quad (145)$$

Here $\zeta\left(\frac{\Box}{2m^2}+h\right)$ acts as a pseudodifferential operator

$$\zeta\left(\frac{\Box}{2m^2}+h\right)\phi(x)=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+h\right)\tilde{\phi}(k)dk \quad , \quad (146)$$

where $\tilde{\phi}(k)=\int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.

We consider Lagrangian (145) with analytic continuations of the zeta function and the power series $\sum\frac{n^{-h}}{n+1}\phi^{n+1}$, i.e.

$$L_h=\frac{m^D}{g^2}\left[-\frac{1}{2}\phi\zeta\left(\frac{\Box}{2m^2}+h\right)\phi+AC\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right], \quad (147)$$

where AC denotes analytic continuation.

Potential of the above zeta scalar field (147) is equal to $-L_h$ at $\Box=0$, i.e.

$$V_h(\phi)=\frac{m^D}{g^2}\left(\frac{\phi^2}{2}\zeta(h)-AC\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right), \quad (148)$$

where $h\neq 1$ since $\zeta(1)=\infty$. The term with ζ-function vanishes at $h=-2,-4,-6,\ldots$. The equation of motion in differential and integral form is

$$\zeta\left(\frac{\Box}{2m^2}+h\right)\phi=AC\sum_{n=1}^{+\infty}n^{-h}\varphi^n \quad , \quad (149)$$

$$\frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+h\right)\tilde{\phi}(k)dk=AC\sum_{n=1}^{+\infty}n^{-h}\phi^n \quad , \quad (150)$$

respectively.

Now, we consider five values of h, which seem to be the most interesting, regarding the Lagrangian (147): $h=0,\ h=\pm 1,$ and $h=\pm 2$. For $h=-2$, the corresponding equation of motion now read:

$$\zeta\left(\frac{\Box}{2m^2}-2\right)\phi=\frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}-2\right)\tilde{\phi}(k)dk=\frac{\phi(\phi+1)}{(1-\phi)^3}. \quad (151)$$

This equation has two trivial solutions: $\phi(x)=0$ and $\phi(x)=-1$. Solution $\phi(x)=-1$ can be also shown taking $\tilde{\phi}(k)=-\delta(k)(2\pi)^D$ and $\zeta(-2)=0$ in (151).

For $h=-1$, the corresponding equation of motion is:

$$\zeta\left(\frac{\Box}{2m^2}-1\right)\phi=\frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}-1\right)\tilde{\phi}(k)dk=\frac{\phi}{(1-\phi)^2}. \quad (152)$$

where $\zeta(-1)=-\frac{1}{12}$.

The equation of motion (152) has a constant trivial solution only for $\phi(x)=0$.

For $h=0$, the equation of motion is

$$\zeta\left(\frac{\Box}{2m^2}\right)\phi=\frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}. \quad (153)$$

It has two solutions: $\phi=0$ and $\phi=3$. The solution $\phi=3$ follows from the Taylor expansion of the Riemann zeta function operator

$$\zeta\left(\frac{\Box}{2m^2}\right)=\zeta(0)+\sum_{n\geq 1}\frac{\zeta^{(n)}(0)}{n!}\left(\frac{\Box}{2m^2}\right)^n, \quad (154)$$

as well as from $\tilde{\phi}(k)=(2\pi)^D 3\delta(k)$.

For $h=1$, the equation of motion is:

$$\frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+1\right)\tilde{\phi}(k)dk=-\frac{1}{2}\ln(1-\phi)^2 \quad , \quad (155)$$

where $\zeta(1)=\infty$ gives $V_1(\phi)=\infty$.

In conclusion, for $h=2$, we have the following equation of motion:

$$\frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+2\right)\tilde{\phi}(k)dk=-\int_0^{\phi}\frac{\ln(1-w)^2}{2w}dw \quad . \quad (156)$$

Since holds equality

$$-\int_0^1\frac{\ln(1-w)}{w}dw=\sum_{n=1}^{\infty}\frac{1}{n^2}=\zeta(2)$$

one has trivial solution $\phi=1$ in (156).

Now, we want to analyze the following case: $C_n=\frac{n^2-1}{n^2}$. In this case, from the Lagrangian (140), we obtain:

$$L=\frac{m^D}{g^2}\left[-\frac{1}{2}\phi\left\{\zeta\left(\frac{\Box}{2m^2}-1\right)+\zeta\left(\frac{\Box}{2m^2}\right)\right\}\phi+\frac{\phi^2}{1-\phi}\right] . \quad (157)$$

The corresponding potential is:

$$V(\phi)=-\frac{m^D}{g}\frac{31-7\phi}{24(1-\phi)}\phi^2 \quad . \quad (158)$$

We note that 7 and 31 are prime natural numbers, i.e. $6n \pm 1$ with n=1 and 5, with 1 and 5 that are Fibonacci's numbers. Furthermore, the number 24 is related to the Ramanujan function that has 24 "modes" that correspond to the physical vibrations of a bosonic string. Thence, we obtain:

$$V(\phi) = -\frac{m^D}{g}\frac{31-7\phi}{24(1-\phi)}\phi^2 \Rightarrow \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \quad . \quad (158b)$$

The equation of motion is:

$$\left[\zeta\left(\frac{\square}{2m^2}-1\right)+\zeta\left(\frac{\square}{2m^2}\right)\right]\phi = \frac{\phi\left[(\phi-1)^2+1\right]}{(\phi-1)^2} \quad . \quad (159)$$

Its weak field approximation is:

$$\left[\zeta\left(\frac{\square}{2m^2}-1\right)+\zeta\left(\frac{\square}{2m^2}\right)-2\right]\phi = 0 \quad , \quad (160)$$

which implies condition on the mass spectrum

$$\zeta\left(\frac{M^2}{2m^2}-1\right)+\zeta\left(\frac{M^2}{2m^2}\right) = 2 \quad . \quad (161)$$

From (161) it follows one solution for $M^2 > 0$ at $M^2 \approx 2.79m^2$ and many tachyon solutions when $M^2 < -38m^2$.

We note that the number 2.79 is connected with $\phi = \frac{\sqrt{5}-1}{2}$ and $\Phi = \frac{\sqrt{5}+1}{2}$, i.e. the "aurea" section and the "aurea" ratio. Indeed, we have that:

$$\left(\frac{\sqrt{5}+1}{2}\right)^2 + \frac{1}{2^2}\left(\frac{\sqrt{5}-1}{2}\right) = 2{,}772542 \cong 2{,}78 .$$

Furthermore, we have also that:

$$(\Phi)^{14/7} + (\Phi)^{-25/7} = 2{,}618033989 + 0{,}179314566 = 2{,}79734$$

With regard the extension by ordinary Lagrangian, we have the Lagrangian, potential, equation of motion and mass spectrum condition that, when $C_n = \frac{n^2-1}{n^2}$, are:

$$L = \frac{m^D}{g^2}\left[\frac{\phi}{2}\left\{\frac{\Box}{m^2} - \zeta\left(\frac{\Box}{2m^2}-1\right) - \zeta\left(\frac{\Box}{2m^2}\right) - 1\right\}\phi + \frac{\phi^2}{2}\ln\phi^2 + \frac{\phi^2}{1-\phi}\right], \quad (162)$$

$$V(\phi) = \frac{m^D}{g^2}\frac{\phi^2}{2}\left[\zeta(-1) + \zeta(0) + 1 - \ln\phi^2 - \frac{1}{1-\phi}\right], \quad (163)$$

$$\left[\zeta\left(\frac{\Box}{2m^2}-1\right) + \zeta\left(\frac{\Box}{2m^2}\right) - \frac{\Box}{m^2} + 1\right]\phi = \phi\ln\phi^2 + \phi + \frac{2\phi-\phi^2}{(1-\phi)^2}, \quad (164)$$

$$\zeta\left(\frac{M^2}{2m^2}-1\right) + \zeta\left(\frac{M^2}{2m^2}\right) = \frac{M^2}{m^2}. \quad (165)$$

In addition to many tachyon solutions, equation (165) has two solutions with positive mass: $M^2 \approx 2.67m^2$ and $M^2 \approx 4.66m^2$.

We note also here, that the numbers 2.67 and 4.66 are related to the "aureo" numbers. Indeed, we have that:

$$\left(\frac{\sqrt{5}+1}{2}\right)^2+\frac{1}{2\cdot 5}\left(\frac{\sqrt{5}-1}{2}\right)\cong 2.6798$$
,

$$\left(\frac{\sqrt{5}+1}{2}\right)^2+\left(\frac{\sqrt{5}+1}{2}\right)+\frac{1}{2^2}\left(\frac{\sqrt{5}+1}{2}\right)\cong 4.64057$$
.

Furthermore, we have also that:

$$(\Phi)^{14/7}+(\Phi)^{-41/7}=2{,}618033989+0{,}059693843=2{,}6777278;$$

$$(\Phi)^{22/7}+(\Phi)^{-30/7}=4{,}537517342+0{,}1271565635=4{,}6646738.$$

Now, we describe the case of $C_n=\mu(n)\frac{n-1}{n^2}$. Here $\mu(n)$ is the Mobius function, which is defined for all positive integers and has values 1, 0, – 1 depending on factorization of n into prime numbers p. It is defined as follows:

$$\mu(n)=\begin{cases}0, & n=p^2m\\ (-1)^k, & n=p_1p_2\ldots p_k,\ p_i\neq p_j\\ 1, & n=1,(k=0).\end{cases} \quad (166)$$

The corresponding Lagrangian is

$$L_\mu=C_0\mathcal{L}_0+\frac{m^D}{g^2}\left[-\frac{1}{2}\phi\sum_{n=1}^{+\infty}\frac{\mu(n)}{n^{\frac{\square}{2m^2}}}\phi+\sum_{n=1}^{+\infty}\frac{\mu(n)}{n+1}\phi^{n+1}\right] \quad (167)$$

Recall that the inverse Riemann zeta function can be defined by

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s}, \quad s = \sigma + it, \quad \sigma > 1. \quad (168)$$

Now (167) can be rewritten as

$$L_\mu = C_0 \mathcal{L}_0 + \frac{m^D}{g^2} \left[-\frac{1}{2} \phi \frac{1}{\zeta\left(\frac{\Box}{2m^2}\right)} \phi + \int_0^\infty \mathcal{M}(\phi) d\phi \right], \quad (169)$$

where $\mathcal{M}(\phi) = \sum_{n=1}^{+\infty} \mu(n) \phi^n = \phi - \phi^2 - \phi^3 - \phi^5 + \phi^6 - \phi^7 + \phi^{10} - \phi^{11} - \ldots$ The corresponding potential, equation of motion and mass spectrum formula, respectively, are:

$$V_\mu(\phi) = -L_\mu(\Box = 0) = \frac{m^D}{g^2} \left[\frac{C_0}{2} \phi^2 \left(1 - \ln \phi^2\right) - \phi^2 - \int_0^\phi \mathcal{M}(\phi) d\phi \right], \quad (170)$$

$$\frac{1}{\zeta\left(\frac{\Box}{2m^2}\right)} \phi - \mathcal{M}(\phi) - C_0 \frac{\Box}{m^2} \phi - 2C_0 \phi \ln \phi = 0, \quad (171)$$

$$\frac{1}{\zeta\left(\frac{M^2}{2m^2}\right)} - C_0 \frac{M^2}{m^2} + 2C_0 - 1 = 0, \quad |\phi| << 1, \quad (172)$$

where usual relativistic kinematic relation $k^2 = -k_0^2 + \vec{k}^2 = -M^2$ is used.

Now, we take the pure numbers concerning the eqs. (161) and (165). They are: 2.79, 2.67 and 4.66. We note that all the numbers are related with $\Phi = \frac{\sqrt{5}+1}{2}$, thence with the aurea ratio, by the following expressions:

$$2{,}79 \cong (\Phi)^{15/7}; \quad 2{,}67 \cong (\Phi)^{13/7} + (\Phi)^{-21/7}; \quad 4{,}66 \cong (\Phi)^{22/7} + (\Phi)^{-30/7}. \quad (173)$$

5. Mathematical connections

We take the eqs. (7) and (10b). We have the following expression:

$$r_k(n) = \frac{1}{2\pi i}\oint \frac{f_N(z)^k\, dz}{z^{n+1}} \Rightarrow r_2(n) = \frac{1}{2\pi i}\int_0^1 \left(\frac{\sin(\pi(N+1)\alpha)}{\sin(\pi\alpha)}\right)^2 d\alpha = 2\int_{\delta}^{1-\delta}\left(\frac{\sin(\pi(N+1)\alpha)}{\sin(\pi\alpha)}\right)^2 d\alpha . \quad (174)$$

We have the following possible mathematical connection with the eq. (126b) concerning the adelic strings:

$$r_k(n) = \frac{1}{2\pi i}\oint \frac{f_N(z)^k\, dz}{z^{n+1}} \Rightarrow r_2(n) = \frac{1}{2\pi i}\int_0^1 \left(\frac{\sin(\pi(N+1)\alpha)}{\sin(\pi\alpha)}\right)^2 d\alpha = 2\int_{\delta}^{1-\delta}\left(\frac{\sin(\pi(N+1)\alpha)}{\sin(\pi\alpha)}\right)^2 d\alpha \Rightarrow$$

$$\Rightarrow \zeta_A(a) = \zeta_\infty(a)\prod_{p\in P}\zeta_p(a) = \zeta_\infty(a)\zeta(a) = \int_R \exp(-\pi x^2)|x|_\infty^{a-1} d_\infty x \cdot \frac{1}{1-p^{-1}}\int_{Q_p}\Omega(|x|_p)|x|_p^{a-1} d_p x . \quad (175)$$

We note that also the eqs. (22), (39) and (50) can be connected with the (126b), as follows:

$$R_M(n) \approx \sum_{q\le P}\sum_{a=1}^{q}{}^{*}\int_{-\xi(q,a)}^{\xi'(q,a)} \frac{\mu(q)^2}{\varphi(q)^2} T(\eta)^2 e(-n(\frac{a}{q}+\eta))d\eta =$$

$$= \sum_{q\le P}\frac{\mu(q)^2}{\varphi(q)^2}\sum_{a=1}^{q}{}^{*} e(-n\frac{a}{q})\int_{-\xi(q,a)}^{\xi'(q,a)} T(\eta)^2 e(-n\eta)d\eta \Rightarrow$$

$$\Rightarrow \zeta_A(a) = \zeta_\infty(a)\prod_{p\in P}\zeta_p(a) = \zeta_\infty(a)\zeta(a) = \int_R \exp(-\pi x^2)|x|_\infty^{a-1} d_\infty x \cdot \frac{1}{1-p^{-1}}\int_{Q_p}\Omega(|x|_p)|x|_p^{a-1} d_p x , \quad (176)$$

$$S_{a,q}(\alpha) = \sum_{m_1\le N}\left[\left[\sum_{m_2\le N} a_{m_2}(m_1,N)\right] e(-2N\beta)\right] e(m_1\beta) +$$

$$+ \sum_{m_1\le N}\left[4\pi i\beta\int_{u=1}^{N}\left[\sum_{m_2\le u} a_{m_2}(m_1,N)\right] e(-u\beta)du\right] e(m_1\beta) \Rightarrow$$

$$\Rightarrow \zeta_A(a) = \zeta_\infty(a)\prod_{p\in P}\zeta_p(a) = \zeta_\infty(a)\zeta(a) = \int_R \exp(-\pi x^2)|x|_\infty^{a-1} d_\infty x \cdot \frac{1}{1-p^{-1}}\int_{Q_p}\Omega(|x|_p)|x|_p^{a-1} d_p x , \quad (177)$$

$$2\pi i\beta\int_{t=0}^{N}\sum_{m_1\le t}\left[\sum_{m_2\le N}a_{m_2}(m_1,N)\right]e(t\beta)dt=O\left(\frac{N^2}{\log^{C-D}N}\right)\Rightarrow$$

$$\Rightarrow|\beta|\int_{t=\sqrt{N}}^{N}\left|\sum_{m_1\le t}\sum_{m_2\le N}a_{m_2}(m_1,N)\right|dt=O\left(\frac{N^3\beta}{\log^{C}N}\right)=O\left(\frac{N^2}{\log^{C-D}N}\right)\Rightarrow$$

$$\Rightarrow\zeta_A(a)=\zeta_\infty(a)\prod_{p\in P}\zeta_p(a)=\zeta_\infty(a)\zeta(a)=\int_R\exp(-\pi x^2)|x|_\infty^{a-1}d_\infty x\cdot\frac{1}{1-p^{-1}}\int_{Q_p}\Omega(|x|_p)|x|_p^{a-1}d_px \quad (178)$$

While the eq. (59b) can be related further that with the eq. (126b) also with the Ramanujan modular identity concerning the physical vibrations of the superstrings, i.e. the number 8, that is also a Fibonacci's number. Thence, we have that

$$\zeta_A(a)=\zeta_\infty(a)\prod_{p\in P}\zeta_p(a)=\zeta_\infty(a)\zeta(a)=\int_R\exp(-\pi x^2)|x|_\infty^{a-1}d_\infty x\cdot\frac{1}{1-p^{-1}}\int_{Q_p}\Omega(|x|_p)|x|_p^{a-1}d_px\Rightarrow$$

$$\Rightarrow S_{a,q}(\alpha;I)=4\pi i\beta\left[\sum_{m_1\le N}\int_{u=1}^{N}\sum_{m_2\le u}a_{m_2}(m_1,N)e(-u\beta)du\right]e(N\beta)+$$

$$+8\pi\beta^2\int_{t=1}^{N}\left[\sum_{m_1\le t}\int_{u=1}^{N}\sum_{m_2\le u}a_{m_2}(m_1,N)e(-u\beta)du\right]e(m_1t)dt=O\left(\frac{N^2}{\log^{C-D}N}\right)+O\left(\frac{N^2}{\log^{C-2D}N}\right)\Rightarrow$$

$$\Rightarrow\frac{1}{3}\times\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (179)$$

Also the eq. (73) can be related with the eq. (126b), thence we obtain the following expression:

$$\int_{\mathcal{M}}F_N(x)e(-x)dx=\sum_{q=1}^{\log^B N}\sum_{\substack{a=1\\(a,q)=1}}^{q}C_q(a)e\left(-\frac{a}{q}\right)\frac{N}{2}+O\left(\frac{N}{\log^{D-2B}N}+\frac{N}{\log^{C-3D-2B}N}\right)=$$

$$=\Im_N\frac{N}{2}+O\left(\frac{N}{\log^{D-2B}N}+\frac{N}{\log^{C-3D-2B}N}\right)\Rightarrow$$

$$\Rightarrow\zeta_A(a)=\zeta_\infty(a)\prod_{p\in P}\zeta_p(a)=\zeta_\infty(a)\zeta(a)=\int_R\exp(-\pi x^2)|x|_\infty^{a-1}d_\infty x\cdot\frac{1}{1-p^{-1}}\int_{Q_p}\Omega(|x|_p)|x|_p^{a-1}d_px \quad (180)$$

Now, with regards the mathematical connections with the zeta strings, we have that the eq. (78b) can be related with the eq. (136), that is the equation of motion for the zeta string concerning the case of time dependent spatially homogeneous solutions. Thence, we obtain:

$$g(2N) \approx C_2 \prod_{\substack{p|N \\ p>2}} \left(\frac{p-1}{p-2}\right) \int_2^{2N-2} \frac{dt}{\log t \log(2N-t)} \Rightarrow$$

$$\Rightarrow \zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t) = \frac{1}{(2\pi)} \int_{|k_0|>\sqrt{2}+\varepsilon} e^{-ik_0 t} \zeta\left(\frac{k_0^2}{2}\right) \tilde{\phi}(k_0) dk_0 = \frac{\phi(t)}{1-\phi(t)} . \quad (181)$$

With regard the eq. (80b), it can be related with the eq. (138) i.e. the equation of motion concerning the closed scalar zeta strings:

$$g(2N) = C_2 \prod_{\substack{p|N \\ p>2}} \left(\frac{p-1}{p-2}\right) \int_2^{2N-2} \frac{dt}{\log t \log(2N-t)} \left(1 - \frac{4}{\sqrt{2N}} \prod_{p\geq 3} \left(1 - \frac{(2N/p)}{p-2}\right)\right) + O\left(\frac{\sqrt{N}}{\log N} \log\log N\right) \Rightarrow$$

$$\Rightarrow \zeta\left(\frac{\Box}{4}\right)\theta = \frac{1}{(2\pi)^D} \int e^{ixk} \zeta\left(-\frac{k^2}{4}\right) \tilde{\theta}(k) dk = \sum_{n\geq 1} \left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)} \theta^{\frac{n(n-1)}{2}-1} \left(\phi^{n+1} - 1\right)\right] . \quad (182)$$

With regard the eq. (91), it can be related with the eq. (138) and with the eq. (156), thence we obtain the following expressions:

$$I^*(2N) = C_2 \prod_{\substack{p|N \\ p>2}} \left(\frac{p-1}{p-2}\right) \int_2^{2N-2} \frac{dt}{\log t \log(2N-t)} \left(1 - \prod_{p\geq 3} \left(1 - \frac{(2N/p)}{p-2}\right)\left(\frac{1}{\sqrt{t}} + \frac{1}{\sqrt{2N-t}}\right)\right) dt \Rightarrow$$

$$\Rightarrow \zeta\left(\frac{\Box}{4}\right)\theta = \frac{1}{(2\pi)^D} \int e^{ixk} \zeta\left(-\frac{k^2}{4}\right) \tilde{\theta}(k) dk = \sum_{n\geq 1} \left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)} \theta^{\frac{n(n-1)}{2}-1} \left(\phi^{n+1} - 1\right)\right] , \quad (183)$$

$$I^*(2N) = C_2 \prod_{\substack{p|N \\ p>2}} \left(\frac{p-1}{p-2}\right) \int_2^{2N-2} \frac{dt}{\log t \log(2N-t)} \left(1 - \prod_{p\geq 3} \left(1 - \frac{(2N/p)}{p-2}\right)\left(\frac{1}{\sqrt{t}} + \frac{1}{\sqrt{2N-t}}\right)\right) dt \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D} \int_{R^D} e^{ixk} \zeta\left(-\frac{k^2}{2m^2} + 2\right) \tilde{\phi}(k) dk = -\int_0^{\phi} \frac{\ln(1-w)^2}{2w} dw . \quad (184)$$

In conclusion, the eq. (100b) can be related with the eq. (156) and we obtain the following mathematical connection:

$$\sum_{\substack{2N\le x\\ q|2N}} G(2N)-\frac{1}{\phi(q)}\sum_{2N\le x} G(2N)=\frac{1}{\phi(q)}\sum_{\substack{\chi(\mathrm{mod}\,q)\\ \chi\ne\chi_0}}\chi(-1)\sum_{\substack{\rho:L(\rho,\lambda)=0\\ \sigma:L(\sigma,\overline{\chi})=0\\ |\mathrm{Im}\,\rho|,|\mathrm{Im}\,\sigma|\le x}}\int_0^1\frac{1}{\rho}(1-t)^{\rho}t^{\sigma-1}dt\cdot x^{\rho+\sigma}+O\left(x\log^2(qx)\right)\Rightarrow$$

$$\Rightarrow\frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+2\right)\tilde{\phi}(k)dk=-\int_0^{\phi}\frac{\ln(1-w)^2}{2w}dw. \quad (185)$$

We want to evidence, also in this paper, the fundamental connection between π and $\phi=\frac{\sqrt{5}-1}{2}$, i.e. the Aurea ratio by the simple formula

$$\arccos\phi=0{,}2879\pi. \quad (186)$$

Acknowledgments

The co-author Nardelli Michele would like to thank Prof. **Branko Dragovich** of Institute of Physics of Belgrade (Serbia) for his availability and friendship.

References

[1] Rosario Turco: *" Il Metodo del Cerchio"* – Gruppo Eratostene – http://www.gruppoeratostene.com/articoli/Cerchio.pdf.

[2] Steven J. Miller and Ramin Takloo-Bighash: "*The Circle Method*" – pg 1- 73 ;July 14, 2004 – Chapter of the book: "*An Invitation to Modern Number Theory*" Princeton University Press ISBN: 9780691120607 - March 2006.

[3] Andrew Granville: "*Refinements of Goldbach's Conjecture, and the Generalized Riemann Hypothesis*" – Functiones et Approximatio XXXVII (2007), 7 – 21.

[4] Branko Dragovich: "*Adeles in Mathematical Physics*" – arXiv:0707.3876v1 [hep-th]– 26 Jul 2007.

[5] Branko Dragovich: "*Zeta Strings*" – arXiv:hep-th/0703008v1 – 1 Mar 2007.

[6] Branko Dragovich: "*Zeta Nonlocal Scalar Fields*" – arXiv:0804.4114v1 – [hep-th] – 25 Apr 2008.

[7] Branko Dragovich: "*Some Lagrangians with Zeta Function Nonlocality*" – arXiv:0805.0403 v1 – [hep-th] – 4 May 2008.