**New mathematical connections concerning string theory: II**

**Michele Nardelli**

Dipartimento di Matematica ed Applicazioni "R.Caccioppoli"
Università degli Studi di Napoli "Federico II"- Polo delle Scienze e delle Tecnologie
Monte S.Angelo, via Cintia (Fuorigrotta), 80126 Napoli, Italy

**Riassunto.**

Nella presente tesi, vengono evidenziate ulteriori connessioni trovate tra alcuni settori della teoria di stringa ed il modello di Palumbo.

Ricordiamo che tale modello è sintetizzato dalla relazione $F = \int_0^\infty F_i dF_i$, dove $F$ rappresenta l'energia iniziale del Big Bang, ossia, l'esplosione del buco nero dal quale si originò l'universo, (correlata all'azione di stringa bosonica) costituita a sua volta da insiemi parziali di onde, definite $F_i$ (correlate all'azione di superstringa). Vengono evidenziate le connessioni trovate tra il modello di Palumbo e: 1) le D-stringhe, 2) la corrispondenza gauge/gravità e la dualità stringa aperta/chiusa, 3) la connessione trovata tra alcune equazioni della tesi di Durr "On a Gauge and Conformal Invariant Nonlinear Spinor Theory" e le azioni Dirac-Born-Infeld per una D3-brana e quelle che sono alla base della congettura di dualità $Het/T^4 - IIA/K3$.
Vengono inoltre descritte ulteriori connessioni trovate tra altre formule legate alla funzione zeta di Riemann ed alcune soluzioni in cosmologia di stringa e teoria di campo di stringa.
Infine, vengono studiate alcune equazioni differenziali che descrivono configurazioni con singolarità nude e le connessioni matematiche trovate tra singolarità nude ed alcuni teoremi applicati a soluzioni di problemi al contorno per equazioni differenziali riguardanti insiemi aperti. Di tali equazioni differenziali, definite in insiemi aperti, sono state studiate anche le condizioni al contorno alla frontiera di tali insiemi.

1.<u>Mathematical connections between Palumbo's model and some equations concerning D-term strings.[1]-[2]</u>

It is known that string theories admit various BPS-saturated string-like objects in the effective 4d theory. These are $D_{1+q}$-branes wrapped on some q-cycle. We shall refer to these objects as effective $D_1$-strings, or D-strings for short. Thus, we conjecture that the string theory D-strings (that is, wrapped $D_{1+q}$-branes) are seen as D-terms strings in 4d supergravity. Since according to the conjecture $D_{1+q}$ branes are D-term strings, it immediately follows that the energy of the $D_{3+q} - \overline{D}_{3+q}$-system must be seen from the point of view of the 4d supergravity as D-term energy.

The supergravity model is defined by one scalar field ϕ, charged under U(1), with K=$\phi^*\phi$ and superpotential W=0, so that we reproduce the supergravity version of the cosmic string in the critical Einstein-Higgs-Abelian gauge field model. This model can be also viewed as a D-term inflation model. In such case, the bosonic part of the supergravity action is reduced to

$e^{-1}L_{bos} = -\frac{1}{2}M_P^2 R - \hat{\partial}_\mu \phi \hat{\partial}^\mu \phi^* - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V^D$, (1.1) where D-term potential is defined by

$V^D = \frac{1}{2}D^2 \qquad D = g\xi - g\phi^*\phi$. (1.2) Here $W_\mu$ is an abelian gauge field,

$F_{\mu\nu} \equiv \partial_\mu W_\nu - \partial_\nu W_\mu \,, \qquad \hat{\partial}_\mu \phi \equiv (\partial_\mu - igW_\mu)\phi$. (1.3)

The energy of the string is:

$$\mu_{string}=\int\sqrt{\det g}\,drd\theta\left[\left(\hat{\partial}_\mu\phi^*\right)\left(\hat{\partial}^\mu\phi\right)+\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}D^2+\frac{M_P^2}{2}R\right]+$$
$$+M_P^2\left(\int d\theta\sqrt{\det h}K\Big|_{r=\infty}-\int d\theta\sqrt{\det h}K\Big|_{r=0}\right),\ (1.4)$$

where K is the Gaussian curvature at the boundaries (on which the metric is h). These boundaries are at $r=\infty$ and $r=0$. Further, for the metric $ds^2=-dt^2+dz^2+dr^2+C^2(r)d\theta^2$, (1.5) we have $\sqrt{\det g}=C(r)$, $\sqrt{\det g}R=2C''$, $\sqrt{\det h}K=-C'$ (1.6)

Eq.(1.4) can be rewritten by using the Bogomol'nyi method as follows

$$\mu_{string}=\int drd\theta C(r)\left\{\left|\left(\hat{\partial}_r\phi\pm iC^{-1}\hat{\partial}_\theta\right)\phi\right|^2+\frac{1}{2}\left[F_{12}\mp D\right]^2\right\}+$$
$$+M_P^2\int drd\theta\left[\partial_r(C'\pm A_\theta)^B\mp\partial_\theta A_r^B\right]-M_P^2\int d\theta C'\Big|_{r=\infty}+M_P^2\int d\theta C'\Big|_{r=0},\ (1.7)$$

Where we have used the explicit form of the metric (1.5).
The energy of the string, can be also defined as:

$\mu_{string}=\int drd\theta\sqrt{\det g}T_0^0=2\pi|n|\xi$, (1.8) where

$$T_0^0=\left\{\left|\left(\hat{\partial}_r\phi\pm iC^{-1}\hat{\partial}_\theta\right)\phi\right|^2+\frac{1}{2}\left[F_{12}\mp D\right]^2\right\}\pm M_P^2\left[\partial_r A_\theta^B-\partial_\theta A_r^B\right].\ (1.9)$$

The definition of the energy of the string that we are using in (1.4), which is valid for time independent configurations, is

$$E=\int_M\sqrt{\det g}\left(\frac{M_P^2}{2}R-L_{matter}\right)+M_P^2\int_{\partial M}\sqrt{\det h}K.\ (1.10)$$

Now we see that the term $\left(\frac{M_P^2}{2}R-L_{matter}\right)$ produced in addition to two BPS bounds in (1.9) also a term $\left[\partial_r(C'\pm A_\theta)^B\mp\partial_\theta A_r^B\right]$. (Note that the BPS state is a state that is invariant under a nontrivial subalgebra of the full supersymmetry algebra. Such states always carry conserved charges, and the supersymmetry algebra determines the mass of the state exactly in terms of its charges). Due to the gravitino BPS bound $1-C'(r)=\pm A_\theta^B$, the surface term $\partial_r A_\theta^B$ in $T_0^0$ is cancelled by the Einstein term $\sqrt{g}R$. This is not surprising since the Einstein equation of motion must be satisfied due to vanishing gravitino transformations. The remaining term in the energy, the Gibbons-Hawking K surface term, give the non-vanishing contribution to the energy of the string which is directly related to the deficit angle $\Delta$, where $M_P^2\Delta=\mu_{string}$.

The "SuperSwirl" is a static, supersymmetric, codimension-two configuration for a nonlinear sigma model, in the context of six dimensional gauged supergravity.

The energy per unit four dimensional volume of the superswirl turns out to diverge, due to the contributions from the boundaries. This energy can be computed from

$$\varepsilon=\int drd\theta\sqrt{g}\left[\frac{1}{4}R+\frac{D_m\phi D^m\phi^*}{\left(1-|\phi|^2\right)^2}+\frac{1}{4}e^{\varphi_0}F_{mn}F^{mn}+\frac{1}{8}\frac{g'^2e^{-\varphi_0}}{\left(1-|\phi|^2\right)^2}\right]$$
$$+\frac{1}{2}\left(\int d\theta\sqrt{h}K\Big|_{r=r_+}-\int d\theta\sqrt{h}K\Big|_{r=r_-}\right),\ (1.11)$$

where K is the extrinsic curvature of the surfaces r=constant, whose metric is h. In this case these surfaces are the "boundaries" at $r_\pm$. This energy can be expressed in a Bogomol'nyi type form as follows:

$$\varepsilon=\frac{1}{2}\int drd\theta\frac{1}{r}\left[\frac{\left|rD_r\phi+iD_\theta\phi\right|^2}{\left(1-|\phi|^2\right)^2}+e^{\varphi_0}\left(f+\frac{g'e^{\varphi_0}}{2\left(1-|\phi|^2\right)}\right)^2\right]$$

$$+\frac{1}{2}\left(\int d\theta rB'\Big|_{r_+} - \int d\theta rB'\Big|_{r_-}\right). \text{ (1.12)}$$

From this expression is clear that the supersymmetry constraints $f = -\frac{g'}{2}\frac{e^{-\varphi_0}}{\left(1-|\phi|^2\right)}$ and $D_{\bar{z}}\phi = 0$, $D_z\phi^* = 0$ in terms of the $(r,\theta)$ coordinates, imply the vanishing of the first two terms of the energy. Thus the energy is given entirely by the last two terms. These are given by

$$\varepsilon = -\pi\left(\frac{r\psi'}{1-\psi} + \frac{r\psi'}{\psi}\right)\Big|_{r_+} + \pi\left(\frac{r\psi'}{1-\psi} + \frac{r\psi'}{\psi}\right)\Big|_{r_-}. \text{ (1.13)}$$

Hence, we have that the energy (per unit volume) is infinite, since it is proportional to the boundary terms computed at the singular points. This system should have boundary source terms that cover the singularities. These should regularise the latter, rendering the total energy finite. This new solution constitutes a new class of supersymmetric vacua for 6D chiral gauged supergravity, with possible implications for a deeper understanding of the theory itself, in particular its origin from higher dimensional supergravities or string theories.

We note that the equations (1.11) and (1.12) are related at the equations (1.4) and (1.7), above mentioned.

Further, these equations can be related to Palumbo's model, precisely at the D-brane actions, thus with $F_i$.

We take the equation of coupling of a D-brane to NS-NS closed string fields and the equation of the Born-Infeld form for the gauge action applies by T-duality to the type I theory. For parallelism Palumbo's model → string theory, we have:

$$-\mu_{25}\int d^{26}\xi Tr\left\{e^{-\Phi}\left[-\det\left(G_{ab}+B_{ab}+2\pi\alpha' F_{ab}\right)\right]^{1/2}\right\} =$$

$$= \int_0^\infty -\frac{1}{(2\pi\alpha')^2 g_{YM}^2}\int d^{10}x Tr\left\{\left[-\det\left(\eta_{\mu\nu}+2\pi\alpha' F_{\mu\nu}\right)\right]^{1/2}\right\} \Rightarrow$$

$$\Rightarrow \int_0^\infty \int\sqrt{\det g}\, dr d\theta\left[\left(\hat{\partial}_\mu\phi^*\right)\left(\hat{\partial}^\mu\phi\right) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D^2 + \frac{M_P^2}{2}R\right] +$$

$$+ M_P^2\left(\int d\theta\sqrt{\det h}K\Big|_{r=\infty} - \int d\theta\sqrt{\det h}K\Big|_{r=0}\right) \Rightarrow$$

$$\Rightarrow \int_0^\infty \int dr d\theta\sqrt{g}\left[\frac{1}{4}R + \frac{D_m\phi D^m\phi^*}{\left(1-|\varphi|^2\right)^2} + \frac{1}{4}e^{\varphi_0}F_{mn}F^{mn} + \frac{1}{8}\frac{g'^2 e^{-\varphi_0}}{\left(1-|\varphi|^2\right)^2}\right] +$$

$$+ \frac{1}{2}\left(\int d\theta\sqrt{h}K\Big|_{r=r_+} - \int d\theta\sqrt{h}K\Big|_{r=r_-}\right). \text{ (1.14)}$$

Here, we see that also the energy of the D-strings can be related at the Palumbo's model.

## 2. Mathematical connections between Palumbo's model and some equations concerning gauge/gravity correspondence and open/closed string duality.[3]

With regard to gauge/gravity relations for the gauge theory living on fractional D3 and wrapped D5 branes using supergravity calculations, we have that since also the fractional D3 branes are D5 branes wrapped on a vanishing 2-cycle located at the orbifold fixed point, we can start from the world-volume action of a D5 brane, that is given by:

$S = S_{BI} + S_{WZW}$ , (2.1) where the Born-Infeld action $S_{BI}$ reads as:

$S_{BI} = -\frac{1}{g_s\sqrt{\alpha'}\left(2\pi\sqrt{\alpha'}\right)^5}\int d^6\xi e^{-\varphi}\sqrt{-\det\left(G_{IJ}+B_{IJ}+2\pi\alpha' F_{IJ}\right)}$, (2.2) while the Wess-Zumino-Witten action $S_{WZW}$ is given by: $S_{WZW} = \frac{1}{g_s\sqrt{\alpha'}\left(2\pi\sqrt{\alpha'}\right)^5}\int_{V_6}\left[\sum_n C_n \wedge e^{2\pi\alpha' F + B_2}\right]$. (2.3) Hence, we have:

$$S = \frac{1}{g_s \sqrt{\alpha'}\left(2\pi\sqrt{\alpha'}\right)^5}\left[-\int d^6\xi e^{-\phi}\sqrt{-\det\left(G_{IJ}+B_{IJ}+2\pi\alpha' F_{IJ}\right)}+\int_{V_6}\left(\sum_n C_n \wedge e^{2\pi\alpha' F+B_2}\right)\right]. \quad (2.4)$$

We divide the six-dimensional world-volume into four flat directions in which the gauge theory lives and two directions on which the brane is wrapped. Let us denote them with the indices I,J=(α,β;A,B) where α and β denote the flat four-dimensional ones and A e B the wrapped ones. We assume the supergravity fields to be independent from the coordinates α, β. We also assume that the determinant in eq.(2.2) factorizes into a product of two determinants, one corresponding to the four-dimensional flat directions where the gauge theory lives and the other one corresponding to the wrapped ones where we have only the metric and the NS-NS two-form field. By expanding the first determinant and keeping only the quadratic term in the gauge field we obtain:

$$\left(S_{BI}\right)_2 = -\frac{1}{g_s\sqrt{\alpha'}\left(2\pi\sqrt{\alpha'}\right)^5}\frac{(2\pi\alpha')^2}{8}\int d^6\xi e^{-\phi}\sqrt{-\det G_{\alpha\beta}}G^{\alpha\gamma}G^{\beta\delta}F^a_{\alpha\beta}F^a_{\gamma\delta}\sqrt{\det\left(G_{AB}+B_{AB}\right)}, \quad (2.5)$$

where we have included a factor 1/2 coming from the normalization of the gauge group generators

$$Tr\left[T^a T^b\right] = \frac{\delta^{ab}}{2}.$$

Now we compute the one-loop vacuum amplitude of an open string stretching between a fractional D3 brane of the orbifold $C^2/Z_2$ dressed with a background SU(N) gauge field on its world-volume and a stack of N ordinary fractional D3 branes. The free energy of an open string stretched between a dressed D3 brane and a stack of N D3 branes located at a distance y in the plane $\left(x^4, x^5\right)$ that is orthogonal to both the world-volume of the D3 branes and the four-dimensional space on which the orbifolds acts, is given by:

$$Z = N\int_0^\infty \frac{d\tau}{\tau} Tr_{NS-R}\left[\left(\frac{e+h}{2}\right)(-1)^{F_s}(-1)^{G_{bc}} P_{GSO} e^{-2\pi\tau L_0}\right] \equiv Z_e^o + Z_h^o, \quad (2.6)$$

where $F_s$ is the space-time fermion number, $G_{bc}$ is the ghost number and the GSO projector is given by:

$P_{GSO} = \frac{(-1)^{G_{\beta\gamma}}+(-1)^F}{2}$, (2.7) with $G_{\beta\gamma}$ being the superghost number:

$G_{\beta\gamma} = -\sum_{m=1/2}^\infty\left(\gamma_{-m}\beta_m + \beta_{-m}\gamma_m\right)$ , $G_{\beta\gamma} = -\gamma_0\beta_0 - \sum_{m=1}^\infty\left(\gamma_{-m}\beta_m + \beta_{-m}\gamma_m\right)$ (2.8) respectively in the NS and in the R sector. F is the world-sheet fermion number defined by

$F = \sum_{t=1/2}^\infty \psi_{-t}\cdot\psi_t - 1$ (2.9) in the NS sector and by

$(-1)^F = \Gamma^{11}(-1)^{F_R}$, $\Gamma^{11} \equiv \Gamma^0\Gamma^1 ... \Gamma^9$, $F_R = \sum_{n=1}^\infty \psi_{-n}\cdot\psi_n$ (2.10) in the R sector. The superscript o in Eq.(2.6) stands for "open" because we are computing the annulus diagram in the open string channel. We have:

$$Z_e^o = -\frac{N}{\left(8\pi^2\alpha'\right)^2}\int d^4x\sqrt{-\det\left(\eta+\hat{F}\right)}\times\int_0^\infty\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}\frac{\sin\pi\nu_f\sin\pi\nu_g}{f_1^4\left(e^{-\pi\tau}\right)\Theta_1\left(i\nu_f\tau|i\tau\right)\Theta_1\left(i\nu_g\tau|i\tau\right)}$$
$$\times\left[f_3^4\left(e^{-\pi\tau}\right)\Theta_3\left(i\nu_f\tau|i\tau\right)\Theta_3\left(i\nu_g\tau|i\tau\right) - f_4^4\left(e^{-\pi\tau}\right)\Theta_4\left(i\nu_f\tau|i\tau\right)\Theta_4\left(i\nu_g\tau|i\tau\right)\right.$$
$$\left.- f_2^4\left(e^{-\pi\tau}\right)\Theta_2\left(i\nu_f\tau|i\tau\right)\Theta_2\left(i\nu_g\tau|i\tau\right)\right], \quad (2.11) \text{ and}$$

$$Z_h^0 = -\frac{N}{\left(8\pi^2\alpha'\right)^2}\int d^4x\sqrt{-\det\left(\eta+\hat{F}\right)}\times\int_0^\infty\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}\left[\frac{4\sin\pi\nu_f\sin\pi\nu_g}{\Theta_2^2(0|i\tau)\Theta_1\left(i\nu_f\tau|i\tau\right)\Theta_1\left(i\nu_g\tau|i\tau\right)}\right]$$
$$\times\left[\Theta_4^2(0|i\tau)\Theta_3\left(i\nu_f\tau|i\tau\right)\Theta_3\left(i\nu_g\tau|i\tau\right) - \Theta_3^2(0|i\tau)\Theta_4\left(i\nu_f\tau|i\tau\right)\Theta_4\left(i\nu_g\tau|i\tau\right)\right]$$

$$-\frac{iN}{32\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\alpha\beta}\int_0^\infty \frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}\ , (2.12) \quad \text{where } \tilde{F}_{\alpha\beta}=\frac{1}{2}\varepsilon_{\alpha\beta\delta\gamma}F^{\delta\gamma}.$$

The three terms in Eq.(2.11) come respectively from the NS, NS $(-1)^F$ and R sectors, while the contribution from the R $(-1)^F$ sector vanishes. In Eq.(2.12) the three terms come respectively from the NS, NS $(-1)^F$ and R $(-1)^F$ sectors, while the R contribution vanishes because the projector h annihilates the Ramond vacuum.

The above computation can also be performed in the closed string channel where $Z_e^c$ and $Z_h^c$ are now given by the tree level closed string amplitude between two untwisted and two twisted boundary states respectively:

$$Z_e^c=\frac{\alpha'\pi N}{2}\int_0^\infty dt\ {}^U\left\langle D3;F\right|e^{-\pi t(L_0+\bar{L}_0)}\left|D3\right\rangle^U \ (2.13) \text{ and } Z_h^c=\frac{\alpha'\pi N}{2}\int_0^\infty dt\ {}^T\left\langle D3;F\right|e^{-\pi t(L_0+\bar{L}_0)}\left|D3\right\rangle^T \ (2.14)$$

where $|D3;F>$ is the boundary state dressed with the gauge field F. Hence, we have:

$$Z_e^c=\frac{N}{(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^\infty\frac{dt}{t^3}e^{-\frac{y^2}{2\pi\alpha' t}}\frac{\sin\pi\nu_f\sin\pi\nu_g}{\Theta_1(\nu_f|it)\Theta_1(\nu_g|it)f_1^4(e^{-\pi t})}$$
$$\times\left\{f_3^4(e^{-\pi t})\Theta_3(\nu_f|it)\Theta_3(\nu_g|it)-f_2^4(e^{-\pi t})\Theta_2(\nu_f|it)\Theta_2(\nu_g|it)\right.$$
$$\left.-f_4^4(e^{-\pi t})\Theta_4(\nu_f|it)\Theta_4(\nu_g|it)\right\} \quad (2.15) \text{ and}$$

$$Z_h^c=\frac{N}{(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^\infty\frac{dt}{t}e^{-\frac{y^2}{2\pi\alpha' t}}\frac{4\sin\pi\nu_f\sin\pi\nu_g}{\Theta_2^4(0|it)\Theta_1(\nu_f|it)\Theta_1(\nu_g|it)}$$
$$\times\left\{\Theta_2^2(0|it)\Theta_3(\nu_f|it)\Theta_3(\nu_g|it)-\Theta_3^2(0|it)\Theta_2(\nu_f|it)\Theta_2(\nu_g|it)\right\}$$
$$-\frac{iN}{32\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\alpha\beta}\int\frac{dt}{t}e^{-\frac{y^2}{2\pi\alpha' t}}\ . \quad (2.16)$$

The three terms in Eq.(2.15) respectively come from the NS-NS, R-R and NS-NS $(-1)^F$ sectors, while those in Eq.(2.16) from the NS-NS, R-R and R-R $(-1)^F$ sectors. In particular, the twisted odd R-R $(-1)^F$ spin structure gets a nonvanishing contribution only from the zero modes.

It is useful to write Eq.(2.12) in a more convenient way. Using the notation for the Θ-functions

$$\Theta\begin{bmatrix}a\\b\end{bmatrix}(\nu|t)=\sum_{n=-\infty}^{\infty}e^{2\pi i\left[\frac{1}{2}\left(n+\frac{a}{2}\right)^2 t+\left(n+\frac{a}{2}\right)\left(\nu+\frac{b}{2}\right)\right]}\ , (2.17) \text{ and the identity}$$

$$\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+ab}\prod_{i=1}^{4}\Theta\begin{bmatrix}a & +h_i\\ b & +g_i\end{bmatrix}(\nu_i)=-\prod_{i=1}^{4}\Theta\begin{bmatrix}1 & -h_i\\ 1 & -g_i\end{bmatrix}(\nu'_i)\ , (2.18) \text{ with}$$

$$h_i=g_1=g_2=0;\ g_3=-g_4=1;\ \nu_1=i\nu_f\tau;\ \nu_2=i\nu_g\tau;\ \nu_3=\nu_4=0$$

$$\nu'_1=-\nu'_2=\frac{i}{2}(\nu_g-\nu_f)\tau;\ \nu'_3=\nu'_4=\frac{i}{2}(\nu_g+\nu_f)\tau,$$ we can rewrite Eq.(2.12) as follows:

$$Z_h^o=-\frac{2N}{(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^\infty\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}\left[\frac{4\sin\pi\nu_f\sin\pi\nu_g}{\Theta_2^2(0|i\tau)\Theta_1(i\nu_f\tau|i\tau)\Theta_1(i\nu_g\tau|i\tau)}\right]$$
$$\times\Theta_1\left(i\frac{\nu_g-\nu_f}{2}\tau|i\tau\right)\Theta_1\left(i\frac{\nu_f-\nu_g}{2}\tau|i\tau\right)\Theta_2^2\left(i\frac{\nu_f+\nu_g}{2}\tau|i\tau\right).\ (2.19)$$

By expanding the previous equation up to the second order in F and using the following relations

$$\Theta_{2,3,4}(0|it)=f_1(e^{-\pi t})f^2_{2,3,4}(e^{-\pi t});\ \lim_{\nu\to 0}\frac{\Theta_1(\nu|it)}{2\sin\pi\nu}=-f_1^3(e^{-\pi t})\ (2.20)$$ together with $\nu_f\cong -i\frac{f}{\pi}$ and

$\nu_g \cong -\frac{g}{\pi}$, we get:

$$Z_h^o = \frac{N}{32\pi^2}\int d^4x\left(F^a_{\alpha\beta}F^{a\alpha\beta} - iF^a_{\alpha\beta}\tilde{F}^{a\alpha\beta}\right)\int_0^\infty \frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}$$ , (2.21) which reduces to

$$Z_h^o(F) \to \left[-\frac{1}{4}\int d^4xF^a_{\alpha\beta}F^{a\alpha\beta}\right]\left\{\frac{1}{g^2_{YM}(\Lambda)} - \frac{N}{8\pi^2}\int_{\frac{1}{\alpha'\Lambda^2}}^\infty \frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}\right\}$$

$$-iN\left[\frac{1}{32\pi^2}\int d^4xF^a_{\alpha\beta}\tilde{F}^{a\alpha\beta}\right]\int_{\frac{1}{\alpha'\Lambda^2}}^\infty \frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}} \; . \quad (2.22)$$

In the closed string channel we get instead:

$$Z_h^c(F) \to \left[-\frac{1}{4}\int d^4xF^a_{\alpha\beta}F^{a\alpha\beta}\right]\left\{\frac{1}{g^2_{YM}(\Lambda)} - \frac{N}{8\pi^2}\int_0^{\alpha'\Lambda^2} \frac{dt}{t}e^{-\frac{y^2}{2\pi\alpha' t}}\right\}$$

$$-iN\left[\frac{1}{32\pi^2}\int d^4xF^a_{\alpha\beta}\tilde{F}^{a\alpha\beta}\right]\int_0^{\alpha'\Lambda^2} \frac{dt}{t}e^{-\frac{y^2}{2\pi\alpha' t}} \; . \quad (2.23)$$

Now we study the one-loop vacuum amplitude of an open string stretching between a stack of $N_I\,(I = 1,...,4)$ branes of type I and a D3 fractional brane, with a background SU(N) gauge field turned-on on its world-volume. Due to the structure of the orbifold $C^3/(Z_2 \times Z_2)$, this amplitude is the sum of four terms: $Z = Z_e + \sum_{i=1}^3 Z_{h_i}$, where $Z_e$ and $Z_{h_i}$ are obtained in the open [closed] channel by multiplying Eq.s (2.11) and (2.12) [Eq.s (2.15) and (2.16)] by an extra 1/2 factor due to the orbifold projection. In the open string channel, $Z^o_{h_i}$ is:

$$Z^o_{h_i} = \frac{f_i(N)}{2(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^\infty \frac{d\tau}{\tau}e^{-\frac{y_i^2\tau}{2\pi\alpha'}}\frac{2\sin\pi\nu_f\, 2\sin\pi\nu_g}{\Theta_2^2(0|i\tau)\Theta_1(i\nu_f\tau|i\tau)\Theta_1(i\nu_g\tau|i\tau)}$$
$$\times\left\{\Theta_3^2(0|i\tau)\Theta_4(i\nu_f\tau|i\tau)\Theta_4(i\nu_g\tau|i\tau) - \Theta_4^2(0|i\tau)\Theta_3(i\nu_f\tau|i\tau)\Theta_3(i\nu_g\tau|i\tau)\right\}$$
$$-\frac{if_i(N)}{64\pi^2}\int d^4xF^a_{\alpha\beta}\tilde{F}^{a\alpha\beta}\int\frac{d\tau}{\tau}e^{-\frac{y_i^2\tau}{2\pi\alpha'}} \; . \quad (2.24)$$

The functions $f_i(N)$ introduced in Eq. (2.24) depend on the number of the different kinds of fractional branes $N_I$ and their explicit expressions are:

$$f_1(N_I) = N_1 + N_2 - N_3 - N_4, \quad f_2(N_I) = N_1 - N_2 + N_3 - N_4, \quad f_3(N_I) = N_1 - N_2 - N_3 + N_4 \quad (2.25)$$

Let us now extract in both channels the quadratic terms in the gauge field F. In the open sector, we get:

$$Z_h^o(F) \to \left[-\frac{1}{4}\int d^4xF^a_{\alpha\beta}F^{a\alpha\beta}\right]\left\{\frac{1}{g^2_{YM}(\Lambda)} - \sum_{i=1}^3\frac{f_i(N)}{16\pi^2}\left[\int_{1/(\alpha'\Lambda^2)}^\infty \frac{d\tau}{\tau}e^{-\frac{y_i^2\tau}{2\pi\alpha'}}\right]\right\}$$

$$-i\left[\frac{1}{32\pi^2}\int d^4xF^a_{\alpha\beta}\tilde{F}^{a\alpha\beta}\right]\sum_{i=1}^3\frac{f_i(N)}{2}\int_{1/(\alpha'\Lambda^2)}^\infty \frac{d\tau}{\tau}e^{-\frac{y_i^2\tau}{2\pi\alpha'}} \; , \quad (2.26)$$

while in the closed string channel we obtain:

$$Z_h^c(F) \to \left[-\frac{1}{4}\int d^4xF^a_{\alpha\beta}F^{a\alpha\beta}\right]\left\{\frac{1}{g^2_{YM}(\Lambda)} - \sum_{i=1}^3\frac{f_i(N)}{16\pi^2}\left[\int_0^{\alpha'\Lambda^2} \frac{dt}{t}e^{-\frac{y_i^2}{2\pi\alpha' t}}\right]\right\}$$

$$-i\left[\frac{1}{32\pi^2}\int d^4x F^a_{\alpha\beta}\tilde{F}^{a\alpha\beta}\right]\sum_{i=1}^{3}\frac{f_i(N)}{2}\int_0^{\alpha'\Lambda^2}\frac{dt}{t}e^{-\frac{y_i^2}{2\pi\alpha' t}}, \quad (2.27)$$

where the divergent contribution is due to the massless states in both channels.

Now we consider the validity of the gauge/gravity correspondence in the 26-dimensional bosonic string and we consider it in the orbifold $C^{\delta/2}/Z_2$ with $\delta < 22$. We consider the one-loop vacuum amplitude of an open string stretching between a D3 brane dressed with a background gauge field and a system on N undressed D3 branes. It is given by:

$$Z = N\int_0^{\infty}\frac{d\tau}{\tau}Tr\left[\left(\frac{e+h}{2}\right)(-1)^{G_{bc}}e^{-2\pi\tau L_0}\right] \equiv Z_e^o + Z_h^o, \quad (2.28)$$

where $L_0$ includes the ghost and the matter contribution. By performing the explicit calculation of the one-loop vacuum amplitude one gets:

$$Z_e^o = -\frac{N}{(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^{\infty}\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}$$

$$\times\frac{2e^{\pi\tau(\nu_f^2+\nu_g^2)}\sin\pi\nu_f\sin\pi\nu_g}{f_1^{18}(e^{-\pi\tau})\Theta_1(i\nu_f\tau|i\tau)\Theta_1(i\nu_g\tau|i\tau)} \quad (2.29)$$

and

$$Z_h^o = -\frac{N}{(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^{\infty}\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}\left[\frac{2e^{\pi\tau(\nu_f^2+\nu_g^2)}\sin\pi\nu_f\sin\pi\nu_g}{\Theta_1(i\tau\nu_f|i\tau)\Theta_1(i\tau\nu_g|i\tau)}\right]$$

$$\times 2^{\frac{\delta}{2}}[f_1(k)]^{-(18-\delta)}[f_2(k)]^{-\delta}, \quad (2.30)$$

where the power 18 is obtained from d-8 for the value of the critical dimension d=26. The previous expressions can also be rewritten in the closed string channel and one gets:

$$Z_e^c = \frac{N}{(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^{\infty}\frac{dt}{t^{11}}e^{-\frac{y^2}{2\pi\alpha' t}}\frac{2\sin\pi\nu_f\sin\pi\nu_g}{f_1^{18}(e^{-\pi t})\Theta_1(\nu_f|it)\Theta_1(\nu_g|it)} \quad (2.31)$$

for the untwisted sector and

$$Z_h^c = \frac{N}{(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^{\infty}\frac{dt}{t^{11-\delta/2}}e^{-\frac{y^2}{2\pi\alpha' t}}\left[\frac{2\sin\pi\nu_f\sin\pi\nu_g}{\Theta_1(\nu_f|it)\Theta_1(\nu_g|it)}\right]$$

$$\times 2^{\delta/2}[f_1(q)]^{-(18-\delta)}[f_4(q)]^{-\delta} \quad (2.32)$$

for the twisted sector.

Also these equations can be related with the Palumbo's model. For example, we take the equation of Scherck-Schwarz theory, the equation of heterotic string action and the equation of the one-loop vacuum amplitude of an open string stretching between a D3 brane dressed with a background gauge field and a system of N undressed D3 branes, in bosonic string theory (2.29-2.30), we have:

$$\int_0^{\infty}F_i dF_i = F \rightarrow \int_0^{\infty}\frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right] =$$

$$= \int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] \rightarrow$$

$$\rightarrow -\frac{N}{(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^{\infty}\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}\times\frac{2e^{\pi\tau(\nu_f^2+\nu_g^2)}\sin\pi\nu_f\sin\pi\nu_g}{f_1^{18}(e^{-\pi\tau})\Theta_1(i\nu_f\tau|i\tau)\Theta_1(i\nu_g\tau|i\tau)} +$$

$$-\frac{N}{(8\pi^2\alpha')^2}\int d^4x\sqrt{-\det(\eta+\hat{F})}\int_0^{\infty}\frac{d\tau}{\tau}e^{-\frac{y^2\tau}{2\pi\alpha'}}\left[\frac{2e^{\pi\tau(\nu_f^2+\nu_g^2)}\sin\pi\nu_f\sin\pi\nu_g}{\Theta_1(i\tau\nu_f|i\tau)\Theta_1(i\tau\nu_g|i\tau)}\right]\times$$

$$\times 2^{\frac{\delta}{2}}[f_1(k)]^{-(18-\delta)}[f_2(k)]^{-\delta}. \quad (2.33)$$

3. Mathematical connections between linear subcanonical spinor theory in third order formalism, Dirac-Born-Infeld action, Duality $Het/T^4 - IIA/K3$ and Palumbo's Model.[4]

Linear subcanonical spinor theory in third order formalism.

We concentrate our attention on the investigation of the simplest possible nonlinear spinor theory, namely a theory for a self-coupled 2-component Weyl spinor field $\psi(x)$ which obeys the nonlinear field equation

$$i\sigma \cdot \partial\psi(x) + g'\sigma^{\mu} : \psi(\psi^{*}\sigma_{\mu}\psi) : (x) = 0 \quad (3.1).$$

This is essentially the Heisenberg nonlinear spinor equation in the form as given by Durr. An invariance of this spinor equation under dilatations requires to assume the spinor field to have the subcanonical dimension

$$\dim\psi = 1/2 \quad (3.2)$$

The linear theory corresponding to this subcanonical spinor theory is the third order Weyl equation

$$-i(\sigma \cdot \partial)\partial^2\psi(x) = 0 \quad (3.3)$$

or the set of first order equations

$$i\sigma \cdot \partial\psi = \hat{\psi}$$
$$i\bar{\sigma} \cdot \partial\hat{\psi} = \hat{\hat{\psi}}$$
$$i\sigma \cdot \partial\hat{\hat{\psi}} = 0 \quad (3.4)$$

This linear theory could be shown to be invariant under the full 15-parameter conformal group. The transition back to the nonlinear theory will be essentially performed by the requirement of phase-gauge invariance of the theory, which demands the replacement

$$\partial_{\mu} \to \nabla_{\mu} = \partial_{\mu} + igR_{\mu} \quad (3.5)$$

in the Lagrangian, where $R_{\mu}$ is identified with the bilinear form

$$R_{\mu}(x) = - : \psi^{*}\sigma_{\mu}\psi : (x) \quad (3.6)$$

Now we shortly review the linear subcanonical spinor theory in the third order derivative formalism and explicitly consider its solutions. These solutions span a quantum mechanical state space with indefinite metric.

We consider the free massless third order derivative theory for a 2-component Weyl spinor field with the field equation

$$-i(\sigma \cdot \partial)\partial^2\psi(x) = 0 \quad (3.7)$$

which can be formally derived from the Lagrangian density

$$L = \frac{i^3}{2}\left[\psi^{*}(\sigma \cdot \partial)\partial^2\psi - ((\sigma \cdot \partial)\partial^2\psi)^{*}\psi\right] \quad (3.8)$$

This theory is invariant under the full 15-parameter conformal group if we require the Weyl spinor field to transform according to an irreducible representation with mass dimension

$$\dim\psi = \frac{1}{2} \quad (3.9)$$

Quantization of the spinor field is achieved by the requirement that the anticommutator of pseudo-hermitian conjugate fields is connected with an invariant solution of (3.7) which vanishes for space-like distances, and a normalization which is fixed by the normalization of the Lagrangian density (3.8). One obtains

$$\left\{\psi\left(\frac{x}{2}\right),\psi^*\left(-\frac{x}{2}\right)\right\} = -\frac{1}{2}(\bar{\sigma}\cdot x)\frac{1}{2\pi}\varepsilon(x^0)\delta(x^2) = \frac{i}{(2\pi)^4}\oint d^4p\frac{\bar{\sigma}\cdot p}{(p^2)^2}e^{-ip\cdot x}$$
$$= -\frac{1}{(2\pi)^3}\int d^4p\bar{\sigma}\cdot p\varepsilon(p^0)\delta'(p^2)e^{-ip\cdot x} \quad (3.10)$$

where $\frac{1}{2\pi}\varepsilon(x^0)\delta(x^2)$ is the invariant function of a massless field. The integrand in the momentum integral (3.10) has the form

$$\frac{\bar{\sigma}\cdot p}{(p^2)^2} = \frac{1}{(\sigma\cdot p)(\bar{\sigma}\cdot p)(\sigma\cdot p)} = \frac{1}{(p^0-\vec{\sigma}\cdot\vec{p})^2(p^0+\vec{\sigma}\cdot\vec{p})} \quad (3.11)$$

which indicates that there exists a double pole for positive chirality states (positive-energy positive-helicity or negative-energy negative-helicity states)

$$p^0 = \vec{\sigma}\cdot\vec{p} = |\vec{p}|h \quad (3.12)$$

($h = \vec{\sigma}\cdot\vec{p}/|\vec{p}| = helicity$), and a single pole for negative chirality states (positive-energy negative-helicity or negative-energy positive-helicity states)

$$p^0 = -\vec{\sigma}\cdot\vec{p} = -|\vec{p}|h \quad (3.13)$$

both with zero mass. The field operator $\psi$(x) will contain annihilation operators for a massless right-handed good and bad ghost, $a_g$ and $a_b$, and an annihilation operator $a_n$ for an ordinary massless left-handed state similar to the neutrino, and also the creation operators $b_g, b_b, b_n$ for the corresponding “antiparticles”. It is convenient to use the pseudo-hermitian operators

$$b^* = \eta b^x\eta^{-1} \quad (3.14)$$

constructed with the metric tensor $\eta$ in the quantum mechanical state space, because in a theory with indefinite metric the pseudo-hermitian conjugation takes over the role of the hermitian conjugation in a theory with positive definite metric. In the 1-particle sector of the quantum mechanical state space the metric tensor $\eta$ has the form

$$\eta = \eta^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.15)$$

where the diagonal element refers to the ordinary state, the n-state. Relation (3.14) then states

$$b_b^* = b_g^x \;;\; b_g^* = b_b^x \;;\; b_n^* = b_n^x \quad (3.16)$$

For the creation and annihilation operators we have the anticommutation rules

$$\left\{a_g^{(i)}(\vec{p}), a_b^{(j)*}(\vec{p}\,')\right\} = \delta(\vec{p}-\vec{p}\,')\delta_{ij}\,; \qquad \left\{a_b^{(i)}(\vec{p}), a_g^{(j)*}(\vec{p}\,')\right\} = \delta(\vec{p}-\vec{p}\,')\delta_{ij}\,;$$
$$\left\{a_n^{(i)}(\vec{p}), a_n^{(j)*}(\vec{p}\,')\right\} = \delta(\vec{p}-\vec{p}\,')\delta_{ij} \qquad (3.17)$$

and similar anticommutation rules for the b-operators. All other anticommutators are zero. The superscript (i) refers to the spin degree of freedom. The Weyl spinor field $\psi$(x) can be expanded in terms of these operators

$$\psi_\alpha(x) = \frac{1}{(2\pi)^{3/2}} \sum_{i=1,2} \int d^3p (2|\vec{p}|)^{-1} \Bigg\{ \Bigg[ \left( a_b^{(i)}(\vec{p}) - \left(\frac{1}{2} + 2i|\vec{p}|t\right) a_g^{(i)}(\vec{p}) \right] (h_+(\vec{p}))_{\alpha i} +$$
$$+ a_n^{(i)}(\vec{p})(h_-(\vec{p}))_{\alpha i} \Bigg] e^{-i(|\vec{p}|t - \vec{p}\cdot\vec{x})} -$$
$$- \left[ \left[ b_b^{(i)*}(\vec{p}) - \left(\frac{1}{2} - 2i|\vec{p}|t\right) b_g^{(i)*}(\vec{p}) \right] (h_+(\vec{p}))_{\alpha i} - b_n^{(i)*}(\vec{p})(h_-(\vec{p}))_{\alpha i} \right] e^{+i(|\vec{p}|t - \vec{p}\cdot\vec{x})} \qquad (3.18)$$

with the helicity projection operators

$$h_\pm(\vec{p}) = \frac{1}{2|\vec{p}|}\left(|\vec{p}| \pm \vec{\sigma}\cdot\vec{p}\right) \quad (3.19)$$

The expansion for $\psi_\alpha^*(x)$ is given by the pseudo-hermitian expression of (3.18). With (3.18) we deduce for the anticommutator (3.10) on the basis of the anticommutator rules (3.17)

$$\left\{\psi\left(\frac{x}{2}\right), \psi^*\left(-\frac{x}{2}\right)\right\} = \frac{1}{(2\pi)^3} \int d^3p \frac{1}{(2|\vec{p}|)^2} \Big\{ \Big[ h_- - (1 + 2i|\vec{p}|t)h_+ \Big] e^{-i(|\vec{p}|t - \vec{p}\cdot\vec{x})} +$$
$$+ \left[ h_- - (1 - 2i|\vec{p}|t)h_+ \right] e^{+i(|\vec{p}|t - \vec{p}\cdot\vec{x})}$$
$$= \frac{1}{(2\pi)^3} \int d^4p \Big\{ h_+\delta(p_0 - |\vec{p}|)\frac{d}{dp_0}(p_0 + |\vec{p}|)^{-1} + h_-\delta(p_0 + |\vec{p}|)\frac{d}{dp_0}(p_0 - |\vec{p}|)^{-1} +$$
$$+ (2|\vec{p}|)^{-2}\left[h_-\delta(p_0 - |\vec{p}|) + h_+\delta(p_0 + |\vec{p}|)\right] \Big\} e^{-ip\cdot x}$$
$$= \frac{i}{(2\pi)^4} \oint d^4p \left\{ \frac{h_+}{(p_0 - |\vec{p}|)^2 (p_0 + |\vec{p}|)} + \frac{h_-}{(p_0 + |\vec{p}|)^2 (p_0 - |\vec{p}|)} \right\} e^{-ip\cdot x} \quad (3.20)$$

i.e. the correct expression (3.10).

The situation in the state space is less pathological if we generalize the third order spinor theory (3.7) to include a mass, i.e.

$$-i(\sigma\cdot\partial)(\partial^2 + m^2)\psi(x) = 0 \quad (3.21)$$

In this case, of course, the symmetry under dilatation and special conformal transformation will be broken. The anticommutator then has the form

$$\left\{\psi\left(\frac{x}{2}\right), \psi^*\left(-\frac{x}{2}\right)\right\} = \overline{\sigma}\cdot\partial \frac{1}{m^2}\left[\Delta(x; m^2) - \Delta(x; 0)\right] = \frac{i}{(2\pi)^4} \oint d^4p \frac{\overline{\sigma}\cdot p}{p^2(p^2 - m^2)} e^{-ip\cdot x} =$$

$$= \frac{1}{(2\pi)^3 m^2} \int d^4 p \bar{\sigma} \cdot p \varepsilon(p^0) \left[\delta(p^2 - m^2) - \delta(p^2)\right] e^{-ip\cdot x} . \quad (3.22)$$

From this we deduce that $\psi(x)$ now annihilates positive norm states of mass m, containing positive and negative chirality components, and negative norm zero states with zero mass and positive chirality. The Weyl spinor field has the expansion

$$\psi_\alpha(x) = \frac{1}{(2\pi)^{3/2} m} \sum_{i=1,2} \int d^3 p \left[4E_p (E_p + m)\right]^{1/2} \left[E_p + \vec{\sigma} \cdot \vec{p} + m\right]_{\alpha i} \times$$
$$\times \left[a_m^{(i)}(\vec{p}) \exp\left[-i(E_p t - \vec{p}\cdot\vec{x})\right] - b_m^{(i)*}(\vec{p}) \exp\left[i(E_p t - \vec{p}\cdot\vec{x})\right]\right] +$$
$$+ (h_+)_{\alpha i} \left[a_-^{(i)}(\vec{p}) \exp\left[-i(|\vec{p}|t - \vec{p}\cdot\vec{x})\right] - b_-^{(i)*}(\vec{p}) \exp\left[i(|\vec{p}|t - \vec{p}\cdot\vec{x})\right]\right] \quad (3.23)$$

with $\quad E_p = (\vec{p}^2 + m^2)^{1/2} \quad (3.24)$

The annihilation and creation operators obey the anticommutation rules

$$\left\{a_m^{(i)}(\vec{p}), a_m^{(j)*}(\vec{p}\,')\right\} = \delta(\vec{p} - \vec{p}\,')\delta_{ij} \qquad \left\{a_-^{(i)}(\vec{p}), a_-^{(j)*}(\vec{p}\,')\right\} = -\delta(\vec{p} - \vec{p}\,')\delta_{ij} \quad (3.25)$$

and similar anticommutators for the $b^{(i)}$ . All other anticommutators vanish. The negative sign in the second anticommutator of (3.25) indicates that $a_-^*$ creates a negative norm state. It is possible verify easily that the expansion (3.23) leads back to (3.22):

$$\left\{\psi\left(\frac{x}{2}\right), \psi^*\left(-\frac{x}{2}\right)\right\} = \frac{1}{(2\pi)^3 m^2} \int d^3 p \left\{\frac{E_p + \vec{\sigma}\cdot\vec{p}}{2E_p} \left[\exp\left(-i(E_p t - \vec{p}\cdot\vec{x})\right)\right] + \exp\left[i(E_p t - \vec{p}\cdot\vec{x})\right] - \right.$$
$$\left. - h_+ \left[\exp\left(-i(|\vec{p}|t - \vec{p}\cdot\vec{x})\right)\right] + \exp\left[i(|\vec{p}|t - \vec{p}\cdot\vec{x})\right] \right\} =$$
$$= \frac{1}{(2\pi)^3 m^2} \int d^4 p \bar{\sigma} \cdot p \varepsilon(p^0) \left[\delta(p^2 - m^2) - \delta(p^2)\right] e^{-ip\cdot x} . \quad (3.26)$$

<u>3.1 Born-Infeld action and D-brane actions</u>.[5]

Born and Infeld realized the final version of their non-linear electrodynamics through a manifestly covariant action. In modern language this can be expressed by saying that the world-volume theory of the brane is described by the action

$$S_{(p)} = -\frac{1}{(2\pi)^p g_s} \int d^{p+1}\sigma \sqrt{-\det(G_{\mu\nu} + F_{\mu\nu})} \quad (3.27)$$

where F is the world-volume electromagnetic field strength, measured in units in which $2\pi\alpha' = 1$ . G is the induced metric on the brane

$$G_{\mu\nu} = \eta_{mn} \partial_\mu X^m \partial_\nu X^n \quad (3.28)$$

Thence, we have from (3.27):

$$S_{(p)} = -\frac{1}{(2\pi)^p g_s} \int d^{p+1}\sigma \sqrt{-\det(\eta_{mn} \partial_\mu X^m \partial_\nu X^n + F_{\mu\nu})} \quad (3.29).$$

The action is invariant under arbitrary diffeomorphysms of the world-volume. One way of fixing this freedom is to adopt the so-called “static gauge” for which the world-volume coordinates are equated with the first p+1 space-time coordinates:

$$X^{\mu} \equiv \sigma^{\mu}, \mu = 0,1,..., p. \quad (3.30)$$

This “static gauge” description is most convenient if the brane is indeed positioned along those directions. The rest of the coordinates become world-volume fields

$$X^{m} \equiv \phi^{m}, m = p+1,...,9. \quad (3.31)$$

The Born-Infeld action becomes

$$S'_{(p)} = -\frac{1}{(2\pi)^{p} g_{s}} \int d^{p+1}\sigma \sqrt{-\det\left(\eta_{\mu\nu} + \partial_{\mu}\phi^{i}\partial_{\nu}\phi^{i} + F_{\mu\nu}\right)}. \quad (3.32)$$

Note that this is in some sense a modification of pure Born-Infeld: it has extra scalar fields $\phi$ and that the action (3.27) can be also write as:

$$S = -\frac{1}{g_{p}} \int d^{4}x \sqrt{-Det\left(G_{\mu\nu} + F_{\mu\nu}\right)} \text{ with } g_{p} = (2\pi)^{3} g_{s}\text{, hence:}$$

$$S = -\frac{1}{(2\pi)^{3} g_{s}} \int d^{4}x \sqrt{-Det\left(G_{\mu\nu} + F_{\mu\nu}\right)}. \quad (3.33)$$

The action for a Dp-brane comes in two parts, the Dirac-Born-Infeld part, and the Wess-Zumino part. These are

$$S_{DBI} = -\mu_{p} \int d^{p+1}\zeta e^{-\phi} \sqrt{-\det\left(g_{\alpha\beta} + f_{\alpha\beta}\right)}, \quad (3.34)$$

where $f = 2\pi\alpha' F - B$ is a U(1) field strength (the world volume gauge field therefore transforms as $\delta A = \lambda_{B} / 2\pi\alpha'$ under a SUGRA gauge transformation $\delta B_{2} = d\lambda_{B}$), and

$$S_{WZ} = \mu_{p} \int e^{f} \wedge \oplus_{q} C_{q}, \quad (3.35)$$

where the integral projects onto p+1 forms. The D-brane charge is $\mu_{p} = 1/(2\pi)^{p} \alpha'^{(p+1)/2}$. The coordinates $\zeta^{\alpha}$ are the embedding coordinates of the D-brane. Note that the spacetime fields are pulled back to the world volume. Hence, we have

$$S = -\frac{1}{(2\pi)^{p} \alpha'^{(p+1)/2}} \int d^{p+1}\zeta e^{-\phi} \sqrt{-\det(g_{\alpha\beta} + f_{\alpha\beta})} + \frac{1}{(2\pi)^{p} \alpha'^{(p+1)/2}} \int e^{f} \wedge \oplus_{q} C_{q}. \quad (3.36)$$

With regard to string corrections, the most important corrections are those to the D7-brane action because they give an induced D3-brane charge and tension. There are also corrections to the DBI action that are responsible for modifying the tension of wrapped D7-branes. Considering the bosonic part only, the DBI action becomes

$$S_{DBI} = -\frac{1}{(2\pi)^p \alpha'^{(p+1)/2}} \int d^{p+1}\zeta e^{-\phi} \sqrt{-\det(g+f)} \left[ 1 - \frac{(2\pi\alpha')^2}{192} \left( (R_T)_{\alpha\beta\gamma\delta}(R_T)^{\alpha\beta\gamma\delta} \right. \right.$$
$$\left. \left. -2(R_T)_{\alpha\beta}(R_T)^{\alpha\beta} - (R_N)_{\alpha\beta\hat{a}\hat{b}}(R_N)^{\alpha\beta\hat{a}\hat{b}} + 2\overline{R}_{\hat{a}\hat{b}}\overline{R}^{\hat{a}\hat{b}} \right) \right] \quad (3.37)$$

up to $O(\alpha')^2$. There is an additional contribution at this order with an undetermined coefficient, but it vanishes on-shell, so it does not affect S-matrix elements or dispersion relations. Here, $\hat{a}, \hat{b}$ are normal bundle indices in an orthonormal basis with vielbein $\xi^{\hat{a}}$.

3.2 Duality type I-SO(32).[6]

In these theories, the action is fixed from the supersymmetry. The heterotic action contain the fields $G_{\mu\nu}, B_{\mu\nu}, \phi$ and $A^a_\mu$; the type I $G_{\mu\nu}$ and $\phi$ from the closed sector $(NS)^2$, $B_{\mu\nu}$ from the closed sector $(R)^2$ and $A^a_\mu$ from the open sector. In the Einstein frame for the two actions, we have

$$S^H = \frac{1}{(2\pi)^7} \int d^{10}x\sqrt{-g} \left[ R - \frac{1}{8} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4} g^{\mu\nu} g^{\rho\sigma} e^{-\frac{\phi}{4}} trF_{\mu\nu}F_{\rho\sigma} - \frac{1}{12} g^{\mu\nu} g^{\rho\sigma} g^{\varepsilon\zeta} e^{-\frac{\phi}{2}} H_{\mu\nu\rho}H_{\sigma\varepsilon\zeta} \right],$$
(3.38)

$$S^I = \frac{1}{(2\pi)^7} \int d^{10}x\sqrt{-g} \left[ R - \frac{1}{8} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4} g^{\mu\nu} g^{\rho\sigma} e^{\frac{\phi}{4}} trF_{\mu\nu}F_{\rho\sigma} - \frac{1}{12} g^{\mu\nu} g^{\rho\sigma} g^{\varepsilon\zeta} e^{\frac{\phi}{2}} H_{\mu\nu\rho}H_{\sigma\varepsilon\zeta} \right],$$
(3.39)
where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \sqrt{2}[A_\mu, A_\nu], \qquad H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \frac{1}{2} Tr\left( A_\mu F_{\nu\rho} - \frac{\sqrt{2}}{3} A_\mu [A_\nu, A_\rho] \right) + cicl.$$

These two actions are obtained each other identifying among them the fields corresponding of the two different theories and putting $\phi^H = -\phi^I$; the change of sign in dilaton connected the perturbative aspect of Type I with that non-perturbative of heterotic and vice versa.

3.3 Duality $Het/T^4 - IIA/K3$ [6].

With regard to duality $Het/T^4 - IIA/K3$, the heterotic relation contain, metric, antisymmetric tensor, dilaton, 10+6+64=80 scalars and 8+16=24 vectors; with $M \in O(4,20)$, $M = M^t$ we can write

$$S = \frac{1}{(2\pi)^3} \int d^6x\sqrt{-g} \left[ R - \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{g^{\mu\nu}}{8} Tr(\partial_\mu ML\partial_\nu ML) \right.$$
$$\left. -\frac{1}{4} e^{-\frac{\phi}{2}} g^{\mu\nu} g^{\rho\sigma} F^a_{\mu\rho}(LML)_{ab} F^b_{\nu\sigma} - \frac{1}{12} e^{-\phi} g^{\mu\nu} g^{\rho\sigma} g^{\varepsilon\zeta} H_{\mu\rho\varepsilon} H_{\nu\sigma\zeta} \right], \quad (3.40)$$

where

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \frac{1}{2} A^a_\mu L_{ab} F^b_{\nu\rho} + cicl\,.$$

The duality group is, in this case, $O(4,20;Z)$.
When we compactific the IIA on K3, we have 58 scalars describing the fluctuations in the complex and kahlerian structure of manifold; 22 scalars that we obtain decomposing $B_{mn}$, with respect to the 22 harmonics 2-forms $\omega^p_{mn}$ of K3: $B_{mn}(x,y) \approx \sum_{p=1}^{22} \phi_p(x)\omega^p_{mn}(y)$ with x coordinates on $R^6$ and y coordinates on K3. Altogether we have 80 scalars that parametrize a coset $O(4,20)/O(4)\times O(20)$ as for the heterotic. Decomposing $C_{mnp}$ in the base of 2-forms, $C_{mnp}(x,y) \approx \sum_{p=1}^{22} A^p_\mu(x)\omega^p_{mn}(y)$, we obtain 22 gauge fields; another arise from $A_\mu$ and another on obtain dualizing $C_{\mu\nu\rho}$; hence we have 24 gauge fields. The effective action is

$$S = \frac{1}{(2\pi)^3}\int d^6x\sqrt{-g}\left[R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{g^{\mu\nu}}{8}Tr(\partial_\mu ML\partial_\nu ML) - \frac{1}{4}e^{\frac{\phi}{2}}g^{\mu\nu}g^{\rho\sigma}F^a_{\mu\rho}(LML)_{ab}F^b_{\nu\sigma} - \right.$$
$$\left. - \frac{1}{12}e^{\phi}g^{\mu\nu}g^{\rho\sigma}g^{\varepsilon\zeta}H_{\mu\rho\varepsilon}H_{\nu\sigma\zeta} - \frac{1}{16}\frac{\varepsilon^{\mu\nu\rho\sigma\varepsilon\zeta}}{\sqrt{-g}}B_{\mu\nu}F^a_{\rho\sigma}L_{ab}F^b_{\varepsilon\zeta}\ \right], \quad (3.41)$$

where $\qquad H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + cicl$.

We note that the eqs. (3.10)-(3.22) and (3.26) are connected with eqs. (3.29)-(3.32) and (3.37) with regard to the DBI action, and with (3.38)-(3.39)-(3.40) and (3.41) with regard to the duality type I – SO(32) and duality $Het/T^4 - IIA/K3$, respectively. Furthermore, we have obtained also the connection with Palumbo's model. We find that, for example,

$$\frac{1}{(2\pi)^3}\int d^4p\overline{\sigma}\cdot p\varepsilon(p^0)\left[\delta(p^2-m^2)-\delta(p^2)\right]e^{-ip\cdot x} \Rightarrow$$
$$\Rightarrow -\frac{1}{(2\pi)^p\alpha'^{(p+1)/2}}\int d^{p+1}\zeta e^{-\phi}\sqrt{-\det(g+f)}\left[1 - \frac{(2\pi\alpha')^2}{192}\left((R_T)_{\alpha\beta\gamma\delta}(R_T)^{\alpha\beta\gamma\delta}\right.\right.$$
$$\left.\left. - 2(R_T)_{\alpha\beta}(R_T)^{\alpha\beta} - (R_N)_{\alpha\beta\hat{a}\hat{b}}(R_N)^{\alpha\beta\hat{a}\hat{b}} + 2\overline{R}_{\hat{a}\hat{b}}\overline{R}^{\hat{a}\hat{b}}\ \right)\ \right] \Rightarrow$$
$$\Rightarrow \frac{1}{(2\pi)^7}\int d^{10}x\sqrt{-g}\left[R - \frac{1}{8}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}g^{\mu\nu}g^{\rho\sigma}e^{-\frac{\phi}{4}}trF_{\mu\nu}F_{\rho\sigma} - \frac{1}{12}g^{\mu\nu}g^{\rho\sigma}g^{\varepsilon\zeta}e^{-\frac{\phi}{2}}H_{\mu\nu\rho}H_{\sigma\varepsilon\zeta}\right] \Rightarrow$$
$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$
$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_V\left(\left|F_2\right|^2\right)\right] \quad (3.42)$$

4. <u>On some correlations obtained between some solutions in string theory, Riemann zeta function and Palumbo's model.</u>

In the paper: "Brane Inflation, Solitons and Cosmological Solutions:I", that dealt various cosmological solutions for a D3/D7 system directly from M-theory with fluxes and M2-branes, and in the paper: "General brane geometries from scalar potentials: gauged supergravities and accelerating universes", that dealt time-dependent configurations describing accelerating universes, we have obtained interesting connection between some equations concerning cosmological solutions, some equations concerning the Riemann zeta function and the relationship of Palumbo's model.

4.1 Cosmological solutions from the D3/D7 system.[7]

The full action in M-theory will consist of three pieces: a bulk term, $S_{bulk}$, a quantum correction term, $S_{quantum}$, and a membrane source term, $S_{M2}$. The action is then given as the sum of these three pieces:

$$S = S_{bulk} + S_{quantum} + S_{M2}. \quad (4.1)$$

The individual pieces are:

$$S_{bulk} = \frac{1}{2\kappa^2}\int d^{11}x\sqrt{-g}\left[R - \frac{1}{48}G^2\right] - \frac{1}{12\kappa^2}\int C \wedge G \wedge G, \quad (4.2)$$

where we have defined G = dC, with C being the usual three form of M-theory, and $\kappa^2 \equiv 8\pi G_N^{(11)}$. This is the bosonic part of the classical eleven-dimensional supergravity action. The leading quantum correction to the action can be written as:

$$S_{quantum} = b_1 T_2 \int d^{11}x\sqrt{-g}\left[J_0 - \frac{1}{2}E_8\right] - T_2\int C \wedge X_8. \quad (4.3)$$

The coefficient $T_2$ is the membrane tension. For our case, $T_2 = \left(\frac{2\pi^2}{\kappa^2}\right)^{1/3}$, and $b_1$ is a constant number given explicitly as $b_1 = (2\pi)^{-4}3^{-2}2^{-13}$. The M2 brane action is given by:

$$S_{M2} = -\frac{T_2}{2}\int d^3\sigma\sqrt{-\gamma}\left[\gamma^{\mu\nu}\partial_\mu X^M \partial_\nu X^N g_{MN} - 1 + \frac{1}{3}\varepsilon^{\mu\nu\rho}\partial_\mu X^M \partial_\nu X^N \partial_\rho X^P C_{MNP}\right], \quad (4.4)$$

where $X^M$ are the embedding coordinates of the membrane. The world-volume metric $\gamma_{\mu\nu}, \mu,\nu = 0,1,2$ is simply the pull-back of $g_{MN}$, the space-time metric. The motion of this M2 brane is obviously influenced by the background G-fluxes.

4.2 Classification and stability of cosmological solutions.

The metric that we get in type IIB is of the following generic form:

$$ds^2 = \frac{f_1}{t^\alpha}\left(-dt^2 + dx_1^2 + dx_2^2\right) + \frac{f_2}{t^\beta}dx_3^2 + \frac{f_3}{t^\gamma}g_{mn}dy^m dy^n \quad (4.5)$$

where $f_i = f_i(y)$ are some functions of the fourfold coordinates and $\alpha, \beta$ and $\gamma$ could be positive or negative number. For arbitrary $f_i(y)$ and arbitrary powers of t , the type IIB metric can in general come from an M-theory metric of the form

$$ds^2 = e^{2A}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2B}g_{mn}dy^m dy^n + e^{2C}|dz|^2, \quad (4.6)$$

with three different warp factors A, B and C , given by:

$$A=\frac{1}{2}\log\frac{f_1 f_2^{\frac{1}{3}}}{t^{\alpha+\frac{\beta}{3}}}+\frac{1}{3}\log\frac{\tau_2}{|\tau|^2},\quad B=\frac{1}{2}\log\frac{f_3 f_2^{\frac{1}{3}}}{t^{\gamma+\frac{\beta}{3}}}+\frac{1}{3}\log\frac{\tau_2}{|\tau|^2},\quad C=-\frac{1}{3}\left[\log\frac{f_2}{t^{\beta}}+\log\frac{\tau_2^2}{|\tau|}\right].\ (4.7)$$

To see what the possible choices are for such a background, we need to find the difference B – C . This is given by:

$$B-C=\frac{1}{2}\log\frac{f_2 f_3}{t^{\gamma+\beta}}+\log\frac{\tau_2}{|\tau|}.\ (4.8)$$

Since the space and time dependent parts of (4.8) can be isolated, (4.8) can only vanish if

$$f_2=f_3^{-1}\cdot\frac{|\tau|}{\tau_2},\quad \gamma+\beta=0,\ (4.9)$$

with $\alpha$ and $f_1(y)$ remaining completely arbitrary.

We now study the following interesting case, where $\alpha=\beta=2$, $\gamma=0$ $f_1=f_2$. The internal six manifold is time independent. This example would correspond to an exact de-Sitter background, and therefore this would be an accelerating universe with the three warp factors given by:

$$A=\frac{2}{3}\log\frac{f_1}{t^2},\quad B=\frac{1}{2}\left[\log f_3+\frac{1}{3}\log\frac{f_1}{t^2}\right],\quad C=-\frac{1}{3}\log\frac{f_1}{t^2}.\ (4.10)$$

We see that the internal fourfold has time dependent warp factors although the type IIB six dimensional space is completely time independent. Such a background has the advantage that the four dimensional dynamics that would depend on the internal space will now become time independent.

This case, assumes that the time-dependence has a peculiar form, namely the 6D internal manifold of the IIB theory is assumed constant, and the non-compact directions correspond to a 4D de-Sitter space. Using (4.10), the corresponding 11D metric in the M-theory picture, can then, in principle, be inserted in the equations of motion that follow from (4.1). Hence, for the Palumbo's model, we have the following connection:

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$
$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right]\Rightarrow$$
$$\Rightarrow\frac{1}{2\kappa^2}\int d^{11}x\sqrt{-g}\left[R-\frac{1}{48}G^2\right]-\frac{1}{12\kappa^2}\int C\wedge G\wedge C\quad(4.11),$$

where the third term is the bosonic part of the classical eleven-dimensional super-gravity action.

4.3 Solution applied to ten dimensional IIB supergravity (uplifted 10-dimensional solution).[8]

This solution can be oxidized on a three sphere $S^3$ to give a solution to ten dimensional IIB supergravity. This 10D theory contains a graviton, a scalar field, and the NSNS 3-form among other fields, and has a ten dimensional action given by

$$S_{10}=\int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R-\frac{1}{2}(\partial\phi)^2-\frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right].\ (4.12)$$

We have a ten dimensional configuration given by

$$ds_{10}^2 = \left(\frac{2}{r}\right)^{3/4}\left[-h(r)dt^2 + r^2 dx_{0,5}^2 + \frac{r^2}{h(r)}dr^2\right] + \left(\frac{r}{2}\right)^{5/4}\left[d\theta^2 + d\psi^2 + d\varphi^2 + \left(d\psi + \cos\theta d\varphi - \frac{Q}{5r^5}dt\right)^2\right]$$

$$\phi = -\frac{5}{4}\log\frac{r}{2},$$

$$H_3 = -\frac{Q}{r^6}dr \wedge dt \wedge (d\psi + \cos\theta d\varphi) - \frac{g}{\sqrt{2}}\sin\theta d\theta \wedge d\varphi \wedge d\psi. \quad (4.13)$$

This uplifted 10-dimensional solution describes NS-5 branes intersecting with fundamental strings in the time direction.

Now we make the manipulation of the angular variables of the three sphere simpler by introducing the following left-invariant 1-forms of SU(2):

$$\sigma_1 = \cos\psi d\theta + \sin\psi \sin\theta d\varphi, \quad \sigma_2 = \sin\psi d\theta - \cos\psi \sin\theta d\varphi, \quad \sigma_3 = d\psi + \cos\theta d\varphi, \quad (4.14)$$

and

$$h_3 = \sigma_3 - \frac{Q}{5}\frac{1}{r^5}dt. \quad (4.15)$$

Next, we perform the following change of variables

$$\frac{r}{2} = \rho^{\frac{4}{5}}, \quad t = \frac{5}{32}\tilde{t}, \quad dx_4 = \frac{1}{2\sqrt{2}}d\tilde{x}_4, \quad dx_5 = \frac{1}{2}dZ, \quad g = \sqrt{2}\tilde{g}, \quad Q = \sqrt{2}2^7\tilde{Q}, \quad \sigma_i = \frac{1}{\tilde{g}}\tilde{\sigma}_i. \quad (4.16)$$

It is straightforward to check that the 10-dimensional solution (4.13) becomes, after these changes

$$d\tilde{s}_{10}^2 = \frac{1}{2}\rho^{-1}\left[d\tilde{s}_6^2\right] + \frac{\rho}{\tilde{g}^2}\left[\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 + \left(\tilde{\sigma}_3 - \frac{\tilde{g}\tilde{Q}}{4\sqrt{2}}\frac{1}{\rho^4}d\tilde{t}\right)^2\right] + \rho dZ^2,$$

$$\phi = -\ln\rho,$$

$$H_3 = -\frac{1}{\tilde{g}^2}\tilde{\sigma}_1 \wedge \tilde{\sigma}_2 \wedge \tilde{h}_3 + \frac{\tilde{Q}}{\sqrt{2}\tilde{g}\rho^5}d\tilde{t} \wedge d\rho \wedge \tilde{h}_3, \quad (4.17)$$

where we define

$$d\tilde{s}_6^2 = -\tilde{h}(\rho)d\tilde{t}^2 + \frac{\rho^2}{\tilde{h}(\rho)}d\rho^2 + \rho^2 d\tilde{x}_{0,4}^2 \quad (4.18)$$

and, after re-scaling M,

$$\tilde{h} = -\frac{2\tilde{M}}{\rho^2} + \frac{\tilde{g}^2}{32}\rho^2 + \frac{\tilde{Q}^2}{8}\frac{1}{\rho^6}. \quad (4.19)$$

We now transform the solution from the Einstein to the string frame. This leads to

$$d\overline{s}_{10}^2 = \frac{1}{2}\rho^{-2}\left[d\widetilde{s}_6^2\right] + \frac{1}{\widetilde{g}^2}\left[\widetilde{\sigma}_1^2 + \widetilde{\sigma}_2^2 + \left(\widetilde{\sigma}_3 - \frac{\widetilde{g}\widetilde{Q}}{4\sqrt{2}}\frac{1}{\rho^4}d\widetilde{t}\right)^2\right] + dZ^2,$$

$$\overline{\phi} = -2\ln\rho,$$

$$\overline{H}_3 = H_3. \qquad (4.20)$$

We have a solution to 10-dimensional IIB supergravity with a nontrivial NSNS field. If we perform an S-duality transformation to this solution we again obtain a solution to type-IIB theory but with a nontrivial RR 3-form, $F_3$. The S-duality transformation acts only on the metric and on the dilaton, leaving invariant the three form. In this way we are led to the following configuration, which is S-dual to the one derived above

$$d\overline{s}_{10}^2 = \frac{1}{2}\left[d\widetilde{s}_6^2\right] + \frac{\rho^2}{\widetilde{g}^2}\left[\widetilde{\sigma}_1^2 + \widetilde{\sigma}_2^2 + \left(\widetilde{\sigma}_3 - \frac{\widetilde{g}\widetilde{Q}}{4\sqrt{2}}\frac{1}{\rho^4}d\widetilde{t}\right)^2\right] + \rho^2 dZ^2,$$

$$\overline{\phi} = 2\ln\rho,$$

$$F_3 = H_3. \qquad (4.21)$$

With regard the T-duality, in the string frame we have

$$d\overline{s}_{10}^2 = \frac{1}{2}\left[ds_6^2\right] + \frac{r^2}{g^2}\left[\sigma_1^2 + \sigma_2^2 + \left(\sigma_3 - \frac{gQ}{4\sqrt{2}}\frac{1}{r^4}dt\right)^2\right] + r^{-2}dZ^2. \quad (4.22)$$

This gives a solution to IIA supergravity with excited RR 4-form, $C_4$. We proceed by performing a T-duality transformation, leading to a solution of IIB theory with nontrivial RR 3-form, $C_3$. The complete solution then becomes

$$d\overline{s}_{10}^2 = \frac{1}{2}\left[ds_6^2\right] + \frac{r^2}{g^2}\left[\sigma_1^2 + \sigma_2^2 + \left(\sigma_3 - \frac{gQ}{4\sqrt{2}}\frac{1}{r^4}dt\right)^2\right] + r^2 dZ^2,$$

$$\overline{\phi} = 2\ln r$$

$$C_3 = -\frac{1}{g^2}\sigma_1 \wedge \sigma_2 \wedge h_3 - \frac{Q}{\sqrt{2}g}\frac{1}{r^5}dt \wedge dr \wedge h_3. \quad (4.23)$$

We are led in this way to precisely the same 10D solution as we found earlier [see formula (4.21)].
With regard the Palumbo's model, we have the following connection:

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$
$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\widetilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right] \rightarrow$$

$$\to \int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R-\frac{1}{2}(\partial\phi)^2-\frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right]. \quad (4.24)$$

4.4 Connections with some equations concerning the Riemann zeta function.[9]

We have obtained interesting connections between some cosmological solutions of a D3/D7 system, some solutions concerning ten dimensional IIB supergravity and some equations concerning the Riemann zeta function, specifying the Goldston-Montgomery theorem.
In the chapter "Goldbach's numbers in short intervals" of Languasco's paper "The Goldbach's conjecture", is described the Goldston-Montgomery theorem.
Assume the Riemann hypothesis. We have the following implications: if $0 < B_1 \le B_2 \le 1$ and $F(X,T) \approx \frac{1}{2\pi}T\log T$ uniformly for $\frac{X^{B_1}}{\log^3 X} \le T \le X^{B_2}\log^3 X$, then $\int_1^X (\psi((1+\delta)x)-\psi(x)-\delta(x))^2 dx \approx \frac{1}{2}\delta X^2 \log\frac{1}{\delta}$, (4.25) uniformly for $\frac{1}{X^{B_2}} \le \delta \le \frac{1}{X^{B_1}}$. We take the Lemma 3 of this theorem:

Lemma 3.

Let $f(t) \ge 0$ a continuous function defined on $[0,+\infty)$ so that $f(t) << \log^2(t+2)$. If

$$I(k) = \int_0^\infty \left(\frac{\sin ku}{u}\right)^2 f(u)du = \left(\frac{\pi}{2}+\varepsilon'(k)\right)k\log\frac{1}{k}, \quad (4.26) \text{ then}$$

$$J(T) = \int_0^T f(t)dt = (1+\varepsilon')T\log T \ , \quad (4.27)$$

with $|\varepsilon'|$ small if $|\varepsilon(k)| \le \varepsilon$ uniformly for $\frac{1}{T\log T} \le k \le \frac{1}{T}\log^2 T$.

Now, we take the equation (4.10) and precisely $A = \frac{2}{3}\log\frac{f_1}{t^2}$. We note that from equation (4.27) for $\varepsilon' = -\frac{2}{3}$ and T = 2, we have $J(T) = \int_0^T f(t)dt = (1+\varepsilon')T\log T = \frac{2}{3}\log 2$. This result is related to $A = \frac{2}{3}\log\frac{f_1}{t^2}$ putting $\frac{f_1}{t^2} = 2$, hence with the Lemma 3 of Goldston-Montgomery Theorem. Then, we have the following interesting relation

$$A = \frac{2}{3}\log\frac{f_1}{t^2} \Rightarrow \int_0^T f(t)dt = (1+\varepsilon')T\log T, \quad (4.28)$$

hence the connection between the cosmological solution and the equation related to Riemann zeta function.
Now, we take the equations (4.13) and (4.21) and precisely $\phi = -\frac{5}{4}\log\frac{r}{2}$ and $\bar{\phi} = 2\ln\rho$. We note that from equation (4.27) for $\varepsilon' = \frac{3}{2}$ and T = 1/2 , we have $J(T) = \int_0^T f(t)dt = (1+\varepsilon')T\log T = \frac{5}{4}\log\frac{1}{2}$.

Furthermore, for $\varepsilon' = 3$ and T = 1/2 , we have $J(T) = \int_0^T f(t)dt = (1+\varepsilon')T\log T = 2\log\frac{1}{2}$.

These results are related to $\phi = -\frac{5}{4}\log\frac{r}{2}$ putting r = 1 and to $\bar{\phi} = 2\ln\rho$ putting $\rho = 1/2$, hence with the Lemma 3 of Goldston-Montgomery Theorem. Then, we have the following interesting relations

$$\phi = -\frac{5}{4}\log\frac{r}{2} \Rightarrow -\int_0^T f(t)dt = -[(1+\varepsilon')T\log T], \qquad \bar{\phi} = 2\ln\rho \Rightarrow \int_0^T f(t)dt = (1+\varepsilon')T\log T, \quad (4.29)$$

hence the connection between the 10-dimensional solutions and the equation related to Riemann zeta function.

4.5 Further connections between some equations of string theory and lemma 3 of Goldston-Montgomery theorem.[10]

We now show that, in a large class of string constructions with NS-NS tadpoles, including brane-antibrane pairs and brane supersymmetry breaking models, the one-loop threshold corrections are UV finite, despite the presence of tadpoles.
In order to obtain a field-theory interpretation, one can turn windings into momenta via a pair of T-dualities that also convert D9 and D5 branes into D7 and D3. The one-loop threshold corrections for the D3 gauge couplings are found to be

$$\Delta = -\frac{4}{v_3}\left(TrQ^2\right)\int_0^\infty dl\left(P^{(2)} - P_e^{(2)}\right) - \left(TrQ^2\right)\frac{N}{2^8\pi^2 v_1 v_2 v_3}\int_0^\infty dl\frac{\vartheta_2^4}{\eta^{12}}\left(\frac{2\pi^2}{3} + \frac{\vartheta_2''}{\vartheta_2} - \frac{\vartheta_1''}{6\pi\eta^3}\right)(-1)^m P^{(2)}P^{(4)},$$
(4.30)

where Q is a gauge generator for the D3 gauge group, $v_1, v_2, v_3$ are the volumes of the three internal tori, $P^{(2)}$ and $P^{(4)}$ are Kaluza-Klein momentum sums along the torus where the T-duality was performed and along the other two tori, respectively, $P_e^{(2)}$ is a corresponding even momentum sum, $\eta$ and $\vartheta_i$ are Jacobi functions. The non-supersymmetric contribution in the second line of (4.30) is IR and UV finite, where IR and UV refer to the open (loop) channel. The UV finiteness can be explained from the supergravity point of view, while the IR finiteness is guaranteed by the separation between the D3 (branes) and the $\overline{D3}$ (antibranes) in the internal space. In the field theory (large volume) limit the non-supersymmetric contribution is negligible, while the explicit evaluation of the first term in (4.30) gives

$$\Delta = -\frac{1}{4}b^{(N=2)}\ln\left(\sqrt{G}\mu^2|\eta(U)|^4\,\mathrm{Im}U\right), \quad (4.31)$$

where for a rectangular torus of radii $R_1, R_2$, $\sqrt{G} = R_1R_2$ and $\mathrm{Im}U = R_1/R_2$. In (4.31), $b^{(N=2)}$ denote beta function coefficients for Kaluza-Klein excitations in the compact torus where the T-dualities were performed, that fill N = 2 multiplets. The first, BPS-like contribution in (4.30), is similar to the standard N = 2 one in orientifold models, and is finite. The non-supersymmetric one originates from the cylinder and reflects the $D3 - \overline{D3}$ interactions between branes and antibranes located at different orbifold fixed points. This explains, in particular, the origin of the alternating factor $(-1)^m$. The remarkable property of (4.30) is that the threshold corrections are UV finite, despite the presence of the NS-NS tadpole. This can be understood noting that in the $l \to \infty$ limit the string amplitudes acquire a field-theory interpretation in terms of dilaton and graviton exchanges between Dp-branes and Op-planes. For parallel localized sources, the relevant terms in the effective Lagrangian are

$$S=\frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}\left\{R-\frac{1}{2}(\partial\varphi)^2-\frac{1}{2(p+2)!}e^{(5-p-2)\varphi/2}F_{p+2}^2\right\}$$
$$-\int_{y=y_i}d^{p+1}\xi\left\{\sqrt{-\gamma}\left[T_p e^{(p-3)\varphi/4}+e^{(p-7)\varphi/4}trF_{\mu\nu}^2\right]+qC^{(p+1)}\right\}, \quad (4.32)$$

where $\xi$ are brane world-volume coordinates, $q=\pm 1$ distinguishes between branes or O-planes and antibranes or $\overline{O}$ -planes, G is the 10-dimensional metric, $\gamma$ is the induced metric and $C^{(p+1)}$ denotes a R-R form that couples to the branes.
We note that the eq. (4.32) is related to the Palumbo's model. Indeed, we have the following connection:

$$\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$
$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right]\Rightarrow$$
$$\Rightarrow\frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}\left\{R-\frac{1}{2}(\partial\varphi)^2-\frac{1}{2(p+2)!}e^{(5-p-2)\varphi/2}F_{p+2}^2\right\}$$
$$-\int_{y=y_i}d^{p+1}\xi\left\{\sqrt{-\gamma}\left[T_p e^{(p-3)\varphi/4}+e^{(p-7)\varphi/4}trF_{\mu\nu}^2\right]+qC^{(p+1)}\right\}. \quad (4.33)$$

From (4.31), we have $\Delta=-\frac{1}{4}b^{(N=2)}\ln\left(R_1R_2\mu^2\left|\eta(U)\right|^4R_1/R_2\right)$, where putting $b^{(N=2)}=A$ and $\left(R_1R_2\mu^2\left|\eta(U)\right|^4R_1/R_2\right)=B$, we obtain $\Delta=-\frac{1}{4}A\ln B$. Also this equation can be related to the Riemann zeta function and precisely to the lemma 3 of Goldston-Montgomery theorem, with the change of sign. Then:

$-\left[\int_0^T f(t)dt\right]=-\left[(1+\varepsilon')T\log T\right]$ that for $T=2$ and $\varepsilon'=-\frac{3}{4}$ lead to $-\frac{1}{2}\log 2$. For $\Delta=-\frac{1}{4}A\ln B$ and A, B = 2 , we have $-\frac{1}{2}\ln 2$. Thence, we obtain the following relation:

$$\Delta=-\frac{1}{4}b^{(n=2)}\ln\left(R_1R_2\mu^2\left|\eta(U)\right|^4R_1/R_2\right)\Rightarrow-\left[\int_0^T f(t)dt\right]=-\left[(1+\varepsilon')T\log T\right]. \quad (4.34)$$

5. <u>On the solutions of some differential equations describing configurations with naked singularities and mathematical connections between naked singularities and some differential elliptic equations concerning open sets.</u>

In this chapter, we have related some differential equations describing configurations with naked singularities, with some theorems applied to differential equations concerning open sets of Stampacchia's papers.

5.1 <u>On some equations whose cosmological solutions leads to the naked singularities</u>.[8]

Now we consider the following action in (q+n+2) dimensions, containing the metric, $g_{\mu\nu}$, a dilaton field, $\phi$, with a general scalar potential, $V(\phi)$, and a (q+2)-form field strength, $F_{q+2} = dA_{q+1}$, conformally coupled to the dilaton:

$$S = \int_{M_{q+n+2}} d^{q+n+2}x\sqrt{|g|}\left[\alpha R - \beta(\partial\phi)^2 - \frac{\eta}{(q+2)!}e^{-\sigma\phi}F_{q+2}^2 - V(\phi)\right]. \quad (5.1)$$

Here R is the Ricci scalar built from the metric. The Ricci scalar is given by the simple expression

$$R = g(r)\frac{(MS'(\ln r))^2}{2r^2} + \frac{(n+q+2)V(\phi)}{n+q} + \frac{(q+2-n)\eta Q^2 h(r)}{(n+q)g(r)}e^{\sigma MS(\ln r)}r^{2(1-n-N)}. \quad (5.2)$$

The field equations obtained for the action of eq. (5.1) are given by:

$$\alpha G_{\mu\nu} = \beta T_{\mu\nu}[\phi] + \frac{\eta}{(q+2)!}e^{-\sigma\phi}T_{\mu\nu}[F_{q+2}] - \frac{1}{2}V(\phi)g_{\mu\nu}$$
$$2\beta\nabla^2\phi = -\sigma\frac{\eta}{(q+2)!}e^{-\sigma\phi}F_{q+2}^2 + \frac{d}{d\phi}V(\phi)$$
$$\nabla_\mu\left(e^{-\sigma\phi}F^{\mu\ldots}\right) = 0, \quad (5.2b)$$

where $T_{\mu\nu}[\phi] = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2$ and $T_{\mu\nu}[F_{q+2}] = (q+2)F_\mu^{\cdots}F_{\nu\ldots} - \frac{1}{2}g_{\mu\nu}F_{q+2}^2$.

We look for solutions having the symmetries of the well-known black q-branes. To this end we consider the following metric ansatz:

$$ds^2 = -h(\tilde{r})dt^2 + h(\tilde{r})^{-1}d\tilde{r}^2 + \tilde{f}^2(\tilde{r})dx_{k,n}^2 + \tilde{g}^2(\tilde{r})dy_q^2, \quad (5.2c)$$

where $dx_{k,n}^2$ describes the metric of an n-dimensional maximally-symmetric space with constant curvature k = -1 , 0 , 1 and $dy_q^2$ describes the flat spatial q-brane directions. Let us assume the metric component g can be written in the form $\tilde{g} = r^c$ for constant c, and with the new variable r defined by the redefinition $r = \tilde{f}(\tilde{r})$. It is also convenient to think of the dilaton as being a logarithmic function of r, with $\phi(r) = MS(\ln r)$, where M is a constant. Subject to these ansatze the solutions to the previous system of equations are given by

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{g(r)} + r^2dx_{k,n}^2 + r^{2c}d_q^y, \quad (5.2d) \qquad F^{try_1\cdots y_q} = Qe^{\sigma MS(\ln r) - L(\ln r)}r^{-M^2-(N-1)}\varepsilon^{try_1\cdots y_q},$$

with $g(r) = h(r)r^{-2(N-1)}e^{-2L(\ln r)}$, (5.2e) and the function L(lnr) is given in terms of S(lnr) by $\frac{dL}{dx}(x) = \frac{\beta}{\alpha}\left(\frac{dS}{dx}(x)\right)^2$. The constants M and N are related to the parameters n, q and c by $M^2 = n + cq$, $N = \frac{n + c^2q}{M^2}$.

To proceed further, we must choose a particular form for $V(\phi)$. We take the following Liouville potential

$$V(\phi) = \Lambda e^{-\lambda\phi}. \quad (5.3)$$

Now, we present three classes of solutions for the Liouville potential (5.3), with $\Lambda \neq 0$.
Let us start by rewriting the general form of the solutions in this case, substituting in (5.2d) and (5.2e) the form of S given by formula $S(\ln r) = \rho \ln r$. We find in this way:

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{g(r)} + r^2 dx_{k,n}^2 + r^{2c} dy_q^2, \qquad \phi(r) = \rho M \ln r, \qquad F^{try_1 \cdots y_q} = Q r^{\sigma\rho M - N - M^2 - \beta\rho^2/\alpha + 1} \varepsilon^{try_1 \cdots y_q},$$
(5.3a)

with $g(r) = h(r) r^{-2(N + \beta\rho^2/\alpha - 1)}$. With these expressions the (tt) and (rr) components of Einstein's equations imply the following condition for h:

$$M^2 h(r) = \left[\frac{n(n-1)k}{(M^2 - 2 + N + \beta\rho^2/\alpha)}\right] r^{2(\beta\rho^2/\alpha + N - 1)} - 2\mathrm{M} M^2 r^{N + \beta\rho^2/\alpha - M^2} -$$
$$-\left[\frac{\eta Q^2}{\alpha(\sigma\rho M - 2n + M^2 + N + \beta\rho^2/\alpha)}\right]\frac{r^{\sigma\rho M + 2\beta\rho^2/\alpha}}{r^{2(n-N)}} - \left[\frac{\Lambda}{\alpha(M^2 + N + \beta\rho^2/\alpha - \lambda\rho M)}\right]\frac{r^{2(N + \beta\rho^2/\alpha)}}{r^{\lambda\rho M}},$$
(5.3b)

where $\mathrm{M}$ is an integration constant. On the other hand the dilaton equation implies h(r) must also satisfy:

$$M h(r) = -2\mathrm{M} M r^{N + \beta\rho^2/\alpha - M^2} + \left[\frac{\eta\sigma Q^2}{2\beta\rho(\sigma\rho M - 2n + M^2 + N + \beta\rho^2/\alpha)}\right]\frac{r^{\sigma\rho M + 2\beta\rho^2/\alpha}}{r^{2(n-N)}} -$$
$$-\left[\frac{\lambda\Lambda}{2\beta\rho(M^2 + N + \beta\rho^2/\alpha - \lambda\rho M)}\right]\frac{r^{2(N + \beta\rho^2/\alpha)}}{r^{\lambda\rho M}}. \quad (5.3c)$$

The $(y_q y_r)$ components of the Einstein's equations impose the further conditions

$$q \cdot (c-1)\left[\frac{n(n-1)k}{M^2} - \frac{\eta Q^2}{\alpha M^2} r^{\sigma\rho M - 2(n-1)} - \frac{\Lambda}{\alpha M^2} r^{-\lambda\rho M + 2}\right] = q \cdot \left[(n-1)k - \frac{\eta Q^2}{\alpha} r^{\sigma\rho M - 2(n-1)}\right]. \quad (5.3d)$$

In order to obtain solutions we must require that eqs. (5.3b) and (5.3c) imply consistent conditions for h(r), and we must also impose eq. (5.3d). We find these conditions can be satisfied by making appropriate choices for the parameters in the solutions. We identify three classes of possibilities which now enumerate, giving interesting solutions for extended objects.

**Class I.** This class of solutions are defined for zero spatial curvature k = 0. The form of the metric in this case is given by

$$h(r) = -2\mathrm{M}\frac{r^{\beta\rho^2/\alpha}}{r^{M^2-1}} - \frac{\Lambda r^2}{\alpha M^2[M^2 - \beta\rho^2/\alpha + 1]} + \frac{\eta Q^2}{\alpha M^2[2n - M^2 - 1 + \beta\rho^2/\alpha]r^{2(n-1)}}, \quad (5.4)$$

and $g(r) = h(r) r^{-2\beta\rho^2/\alpha}$. The dilaton and gauge fields are given by

$$\phi(r) = M S(\ln r), \ (5.5) \quad \text{and} \quad F^{try_1 \cdots y_q} = Q e^{\sigma M S(\ln r) - L(\ln r)} r^{-M^2 - (N-1)} \varepsilon^{try_1 \cdots y_q}, \ (5.6)$$

with the relevant values of the parameters. Let $\beta\rho^2 > \alpha(M^2+1)$ and $\mathrm{M}<0$. For $\Lambda > 0$, the solution is static everywhere and there are no horizons at all. There is a naked singularity at the origin and the asymptotic infinity is null-like.

**Class II.** These solutions are defined for non zero spatial curvature k = -1 , 1. The form of the metric is given by

$$h(r) = -2\mathrm{M}\frac{r^{\beta\rho^2/\alpha}}{r^{M^2-1}} - \frac{\lambda\Lambda r^{2\beta\rho^2/\alpha}}{2\beta\rho M\left[M^2-1+\beta\rho^2/\alpha\right]} + \frac{\eta Q^2}{\alpha M^2\left[2n-M^2+\beta\rho^2/\alpha-1\right]r^{2(n-1)}}, \quad (5.7)$$

and $g(r) = h(r)r^{-2\beta\rho^2/\alpha}$. Let $\beta\rho^2 \geq \alpha$ and $\mathrm{M}<0$. For $\Lambda<0$, and k = -1 the solutions are static everywhere with a naked time-like singularity at the origin.

**Class III.** These solutions are defined only for positive spatial curvature k = 1. The metric is given by

$$h(r) = -2\mathrm{M}\frac{r^{\beta\rho^2/\alpha}}{r^{M^2-1}} - \frac{\Lambda r^2}{\alpha M^2\left[M^2+1-\beta\rho^2/\alpha\right]} + \frac{\sigma\eta Q^2 r^{2\beta\rho^2/\alpha}}{2\beta\rho M\left[M^2-1+\beta\rho^2/\alpha\right]}, \quad (5.8)$$

and $g(r) = h(r)r^{-2\beta\rho^2/\alpha}$. Let $\beta\rho^2 > \alpha(M^2+1)$ and $\mathrm{M}<0$. For $\Lambda>0$, the solution is static everywhere with a naked singularity at the origin.

5.2 On further equations having naked singularities solutions.[2]

We start from the differential equation

$$\partial_{z\bar{z}} \ln\psi = c\frac{(1-\psi)}{\psi^2}. \quad (5.9)$$

Configurations with naked singularities can be solutions of this equation. An exact solution of this equation can be obtained by asking that $\psi$ depends on some real combination of $(z,\bar{z})$, for example by $x \equiv z+\bar{z}$. In this case, it is simple to show that (5.9) can be reduced to a first order differential equation

$$\left(\frac{d}{dx}\ln\psi\right)^2 = c\left(\frac{2\psi-1}{\psi^2}\right)+\alpha^2, \quad (5.10)$$

where $\alpha^2$ is a positive real constant. Eq. (5.10) can be reassembled in the following way

$$\alpha^2\left(\psi+\frac{c}{\alpha^2}\right)^2 - \left(\frac{d}{dx}\psi\right)^2 = \left(c+\frac{c^2}{\alpha^2}\right). \quad (5.11)$$

At this point, it is easy to show that the general solution for the equation (5.11) is given by

$$\psi = \frac{1}{e^{\alpha x}}\left[M+Ne^{\alpha x}+Pe^{2\alpha x}\right], \quad (5.12)$$

where the real numbers M, N, P are integration constants that satisfy the condition

$$N = -\frac{c}{\alpha^2} = \frac{1}{2}\left(1 - \sqrt{1+16MP}\right). \quad (5.13)$$

Since $\psi$ is real and positive, this implies that $M, P \geq 0$.
The general supersymmetric solution above, eq. (5.12), can be seen to constitute the most general axially symmetry solution that preserves supersymmetry, and maximal space-time symmetry in 4D. The general solution depending on the variable x with the coordinates

$$e^{2z} = re^{i\theta}, \qquad e^{2\bar{z}} = re^{-i\theta}, \quad (5.14)$$

depends only on the radial coordinate r, and, consequently, it is axially symmetric. In terms of these coordinates, the solution is:

$$ds_6^2 = \eta_{\mu\nu} dx^\mu dx^\nu + e^{2B(r)}\left(dr^2 + r^2 d\theta^2\right), \quad \phi = \psi^{\frac{1}{2}} e^{ig'T/2}, \quad \varphi = \varphi_0,$$
$$F_{r\theta} = -\frac{g' e^{-\varphi_0} \tilde{c}^2}{8} \frac{(1-\psi)}{r\psi^2}, \quad A_\theta = -\frac{r}{g'} \frac{\psi'}{\psi} + \partial_\theta T. \quad (5.15)$$

with the definitions and constraints:

$$e^{2B} = \frac{\tilde{c}^2}{4} \frac{(1-\psi)^2}{r^2 \psi^2}, \quad \psi = \frac{1}{r^\alpha}\left(M - \frac{c}{\alpha^2} r^\alpha + \Pr^{2\alpha}\right), \quad c = \frac{g'^2 e^{-\varphi_0} \tilde{c}^2}{8} = \frac{\alpha^2}{2}\left(\sqrt{1+16MP} - 1\right). \quad (5.16)$$

The limit $\psi \to 0$, is obtained by properly sending M, c and P to zero. The function $e^{2B}$ can be rewritten as

$$e^{2B} = \frac{2ce^{\varphi_0}}{MPg'^2} \frac{1}{r^2} \frac{\left(r^\alpha - M + \frac{c}{\alpha^2} r^\alpha - \Pr^{2\alpha}\right)^2}{\left[\left(\frac{M}{P}\right)^{\frac{1}{2}} - \frac{c}{\sqrt{MP}\alpha^2} r^\alpha + \left(\frac{P}{M}\right)^{\frac{1}{2}} r^{2\alpha}\right]^2}. \quad (5.17)$$

The singularity structure can be read from the metric function $e^{2B}$ given in formula (5.17). When the hyperscalars are turned on, the solution has unavoidable, timelike singularities at the points at which this function vanishes, or diverges. This occurs at the positive zeros of the function $1-\psi = 0$, where the conformal factor $e^{2B}$ vanishes. These are located at

$$r_\pm^\alpha = \frac{1}{2P}\sqrt{\frac{1+\sqrt{1+16MP}}{2}}\left(\sqrt{\frac{1+\sqrt{1+16MP}}{2}} \pm 1\right) = \frac{1}{2P}\sqrt{1+\frac{c}{\alpha^2}}\left(\sqrt{1+\frac{c}{\alpha^2}} \pm 1\right). \quad (5.18)$$

We have the presence of these singularities because the 6D potential and target-space metric, blow up at these positions. The physical space-time lies in the coordinate range $r_- \leq r \leq r_+$. We now consider the limit $r \to r_-$. The relevant part of the metric is

$$ds_2^2 = e^{2B(r)}\left(dr^2 + r^2 d\theta^2\right), \quad (5.19)$$

with $e^{2B}$ given in eq. (5.17). Performing the coordinate transformation

$$r^{\alpha} = \sqrt{\rho}\sqrt{\frac{4}{\tilde{c}}\frac{\alpha r_{-}^{\alpha}}{\left(r_{+}^{\alpha} - r_{-}^{\alpha}\right)}} + r_{-}^{\alpha}, \quad (5.20)$$

brings the metric (5.19), for $\rho \to 0$ (that is, $r \to r_{-}$), to the form

$$ds_2^2 \approx d\rho^2 + \gamma\rho d\theta^2, \quad (5.21)$$

with $\gamma = 4\tilde{c}\,\alpha r_{-}^{\alpha}\left(r_{-}^{\alpha} - r_{+}^{\alpha}\right)$. This implies that near $r_{-}$ the metric does not have a conical singularity, but a more serious one: a naked time-like singularity.

5.3 On some mathematical theorems concerning open sets applied to the naked singularities.[11]-[12]

If an open set is a set formed only from the internal points, without the points belonging to the boundary, hence without consider the boundary, and a naked singularity is a singularity formed only from the internal parts, without events horizon and no bounded from a black hole, hence without the boundary, then open sets and naked singularities can be related and the mathematical theorems concerning the open sets (differential equations and boundary conditions) can be applied to the naked singularities, obtaining new interesting mathematical considerations.

Let $R^m$ an euclidean space of m dimensions (m > 2) of generic point $x \equiv (x_1, x_2, ..., x_m)$ , $y \equiv (y_1, y_2, ..., y_m), ...$ . We denote with $I(y, \rho)$ the sphere of $R^m$ with centre in y and radius $\rho$ and with $\Gamma(y, \rho)$ the spherical hyper-surface boundary of $I(y, \rho)$. Furthermore, we denote with $\Sigma(x)$ an measurable set of $\Gamma(x,1)$ and with $|\Sigma(x)|$ the measure m-1 dimensional of it. In relation to $\Sigma(x)$ we denote with $S(x, \rho)$ the set of points of $I(x, \rho)$ that are projected from x in $\Sigma(x)$. If we have a bounded and open set $\Omega$ of $R^m$, we'll tell that $\Omega$ is of type (S) if there are two positive numbers: $\omega$ and $\rho$ $(\omega \le \omega_{m-1})$ so that for each $x \in \overline{\Omega}$ can be determined a set $\Sigma(x)$ with $|\Sigma(x)| \ge \omega$ hence $S(x, \rho) \subset \overline{\Omega}$.

Let $C^1(\Omega)$ the real functions space u(x) continuous with the partial first derivative in $\Omega$ so that $u \in L^a(\Omega), D_i u \in L^a(\Omega), (i = 1,2,...,m)$ and we introduce the norm

$$\|u\|_{1,a} = \|u\|_{L^a(\Omega)} + \sum_{i}^{1...m} \|D_i u\|_{L^a(\Omega)} . \quad (5.22)$$

We denote with $H^{1,a}(\Omega)$ the completion of $C^{1,a}(\Omega)$ as regards the norm $\|u\|_{1,a}$; $H^{1,2}(\Omega) \equiv H^1(\Omega)$ is a Hilbert's space. Then, we denote with $H_0^{1,a}(\Omega)$ the sub-space of $H^{1,a}(\Omega)$ formed from the close, in $H^{1,a}(\Omega)$, of the functions of $C^1(\Omega)$ having contained support in $\Omega$; in $H^{1,2}(\Omega) \equiv H_0^1(\Omega)$ the two norms $\|u\|_1$ and $\|u\|_1 = \sum_{i}^{1...m} \|D_i u\|_{L^a(\Omega)}$ are equivalents.

Let V a closed manifold of $H^1(\Omega)$ so that $H_0^1(\Omega) \subset V \subset H^1(\Omega)$. Furthermore, let $a_{ij}(x), (i, j = 1,2,...,m)$ real functions bounded and measurable in $\Omega$ that satisfy the following condition:

$$\mu \sum_{i}^{1...m} \lambda_i^2 \le \sum_{ij}^{1...m} a_{ij}(x) \lambda_i \lambda_j \le M \sum_{i}^{1...m} \lambda_i^2 \quad (5.23)$$

$\{\lambda_i \in R^1, (i = 1,2,...,m); x \in \overline{\Omega}, \mu > 0\}$, $f_0(x), ..., f_m(x)$ m+1 functions $\in L^p(\Omega)$ with $p \ge 2$, while

$c(x), b_1(x), ..., b_n(x)$ are measurable and limited in $\Omega$ with $c(x) \geq 0$, $|b_1(x)| \leq M$. Now, we put for $u, v \in H^1(\Omega)$:

$$a(u,v) = \int_\Omega \left\{ \sum_{ij}^{1...m} a_{ij}(x) D_i u D_j v + \sum_i^{1...m} b_i D_i u v + c(x) u v \right\} dx \quad (5.24), \text{ and}$$

$$\langle f, v \rangle = \int_\Omega \left\{ \sum_i^{1...m} f_i D_i v + f_0 v \right\} dx . \quad (5.25)$$

One function $u(x) \in V$ that, for each $v \in V$, satisfy the relation $a(u,v) = \langle f, v \rangle$, is denoted shortly with $u(x) \equiv \mathrm{E}(\Omega, V)$. Hence, we have the following relation:

$$\int_\Omega \left\{ \sum_{ij}^{1...m} a_{ij}(x) D_i u D_j v + \sum_i^{1...m} b_i D_i u v + c(x) u v \right\} dx = \int_\Omega \left\{ \sum_i^{1...m} f_i D_i v + f_0 v \right\} dx . \quad (5.26)$$

We denote with $H^+, [H^-]$ the values of $k \in R^1$ hence $t_k^+(u(x)) \in V, [t_k^-(u(x)) \in V]$ for each $u \in V$.
If $u(x) \in V$, we denote with $A^+(k), [A^-(k)]$ the set of points $x \in \overline{\Omega}$ where $u(x) \geq k, [u(x) \leq k]$. We now denote with A(k) the sets $A^+(k)$ and $A^-(k)$, and with H the sets $H^+ \cap (0, +\infty)$ and $H^- \cap (-\infty, 0)$.
LEMMA 1.
If $u(x) \equiv \mathrm{E}(\Omega, V)$, it is possible to determine two constants $\gamma, \Lambda : \gamma = \gamma(\mu, M), \Lambda \equiv \Lambda(\mu, M, \Omega)$ so that, for each $k \in H$, we have:

$$\int_{A(k)} \sum_i^{1...m} (D_i u)^2 dx \leq \gamma \int_{A(k)} (u-k)^2 dx + \Lambda \int_{A(k)} \sum_i^{0...m} f_i^2 dx . \quad (5.27)$$

LEMMA 2.
We suppose that $\Omega$ is the type (S), fixed q with $1 \leq q \leq 2$, it is possible to determine two positive constants, deriving from $\Omega$ and $q : \eta$, and $\beta$ so that for each function $u(x) \in H^1(\Omega)$, and for each $k \in R^1$ hence $misA(k) < \eta$, (with mis we denote the Lebesgue's measure m-dimensional) we have:

$$\int_{A(k)} |u(x) - k|^q dx \leq \beta \int_{A(k)} \sum_i^{1...m} |D_i u|^q dx \{misA(k)\}^{q/m} . \quad (5.28)$$

PROPOSITION 1.

Let $\Omega$ an open set of type (S), $a_{ij}(x), [a_{ij} = a_{ji}], b_i(x), c(x)$ are measurable and limited functions in $\Omega$ and (5.23) let satisfied, furthermore let $4\mu c(x) - \sum b_i^2(x) \geq v > 0$; $f_i \in L^p(\Omega)$, (i = 1,…,m) with p > m.
If the function $u(x) \in H_0^1(\Omega)$ satisfy the relation

$$\int_\Omega \left\{ \sum_{ij}^{1...m} a_{ij}(x) D_i u D_j v + \sum_i^{1...m} b_i D_i u v + c(x) u v \right\} dx = \sum_i^{1...m} \int_\Omega f_i D_i v dx , \quad (5.29)$$

for any $v \in H_0^1(\Omega)$, then we have the following increase: $\sup_{x\in\overline{\Omega}}|u(x)| \le C\sum_i^{1...m}\|f_i\|_{L^p(\Omega)}$, with C deriving only from $\Omega$ and from the constants $\mu$ and M of the (5.23).

PROPOSITION 2.

Dirichlet's Problem (with boundary conditions not homogeneous). In the similar hypotheses of proposition 1 and if $\psi$ is the trace of a function $\overline{u}$ having first derivatives in $\Omega$ (in $L^p(\Omega)$) with p > m, we argue, for each function $u(x) \in H^1(\Omega)$ having trace $\psi$ on $\partial\Omega$ and that satisfy the relation (5.29), the following increase:
$\sup_{x\in\overline{\Omega}}|u(x)| \le C\sum_i^{1...m}\|f_i\|_{L^p(\Omega)} + \sum_i^{1...m}\|D_i\overline{u}\|_{L^p(\Omega)} + \max_{\overline{\Omega}}|\overline{u}|$.
Putting $W = u - \overline{u}$, we have $W \in H_0^1(\Omega)$ and from (5.29):

$$\int_\Omega \left\{ \sum_{ij}^{1...m} a_{ij}(x) D_i W D_j v + \sum_i^{1...m} b_i D_i W v + c(x) W v \right\} dx =$$
$$= \sum_i^{1...m} \int_\Omega \left[ f_i - \sum_j^{1...m} a_{ij}(x) D_j \overline{u} \right] D_i v dx - \int_\Omega \left\{ \sum_i^{1...m} b_i \frac{\partial \overline{u}}{\partial x_i} + c(x)\overline{u} \right\} v dx, \quad (5.30)$$

PROPOSITION 3.

Neumann's Problem (with boundary conditions homogeneous). In the similar hypotheses on $\Omega, a_{ij}(x), [a_{ij} = a_{ji}], c(x)$ formulated in the proposition 1, let $g \in L^p$ with p > m and c(x) > v > 0. If $u(x) \in H^1(\Omega)$ satisfy the relation:

$$\int_\Omega \left\{ \sum_{ij}^{1...m} a_{ij}(x) D_i u D_j v + c(x) u v \right\} dx = \int_\Omega g v dx, \quad (5.30)$$

for any $v \in H^1(\Omega)$, we have the increase: $\sup_{x\in\overline{\Omega}}|u(x)| \le A\|g\|_{L^p(\Omega)}$.

PROPOSITION 4.

Dirichlet – Neumann's mixed Problem. Let $\partial\Omega = \partial_1\Omega \cup \partial_2\Omega$, and $u \in H_1(\Omega)$ satisfy the relation:

$$\int_\Omega \left\{ \sum_{ij}^{1...m} a_{ij} D_i u D_j v + c(x) u v \right\} dx = \int_\Omega g v dx \quad (5.31)$$

for each $v \in V$, where V is the sub-space of the v of $H^1(\Omega)$ and the trace on $\partial_1\Omega : \gamma v$ is vanish.
Because $H^+ \equiv (0,+\infty), H^- \equiv (-\infty,0)$, we have, as for proposition 1, the limitation $\sup_{\overline{\Omega}}|u(x)| \le C\|g\|_{L^p(\Omega)}$.

LEMMA 3.

If $u(x) \in H^1(\Omega)$ with $\Omega$ of type (S), it is possible to determine two constants $\delta_1 = \delta_1(\Omega), \delta_2 = \delta_2(\Omega)$ so that we have:

$$|u(x)| \le \delta_1 \int_\Omega \frac{|u(t)|}{|x-t|^{m-1}} dt + \delta_2 \int_\Omega \sum_i^{1...m} |D_i u| \frac{1}{|x-t|^{m-1}} dt . \quad (5.32)$$

for almost all $x \in \Omega$.

LEMMA 4.

If $u(x) \in H^1(\Omega)$ with $\Omega$ of type (S), for each q with $1 \le q \le 2$ there is a constant $\beta_1 = \beta_1(q, \Omega)$ so that we have:

$$\left(mis\{E_{x\in\Omega}[|u| \ge \sigma > 0]\}\right)^{(m-q)/m} \le \frac{\beta_1}{\sigma^q}\left\{\int_\Omega |u(x)|^q dx + \int_\Omega \sum_i^{1...m} |D_i u|^q dx\right\}. \quad (5.33)$$

From this lemma, we obtain:

LEMMA 5.

In the similar hypotheses of lemma 4, and preserving the similar notations, there is the following inequality:

$$\int_\Omega |u|^q dx \le \beta_1 \left\{\int_\Omega |u|^q dx + \int_\Omega \sum_i^{1...m} |D_i u|^q dx\right\}[mis\Omega_0]^{q/m} , \quad (5.34)$$

where $\Omega_0$ denote the set of the points of $\Omega$ where $u \ne 0$ $\left(\Omega_0 = \Omega \cap E[|u| > 0]\right)$.

Let $\Omega$ be a bounded connected open set in the n-dimensional real Euclidean space $R^n$, $\overline{\Omega}$ its closure and $\partial\Omega$ its boundary. We shall denote by V the subspace of $H^1(\Omega)$ consisting of all distributions $u \in H^1(\Omega)$ such that u = 0 on $\partial_1\Omega$. The space V provided with the norm induced from that of $H^1(\Omega)$, being a closed subspace, becomes a Hilbert space. We shall assume that $\Omega$ and $\partial_1\Omega$ are such that the following Poincarè type inequality holds for all $u \in V$: There exists a constant $C = C(\Omega, \partial_1\Omega) > 0$ such that $\|u\|_{2,\Omega} \le C\|u_x\|_{2,\Omega}$.

<u>Assumption A</u>. We require that there exist a constant $\mu_0 > 0$ such that $\left[\sum(x)\right] > \mu_0$ for all $x \in \Omega$.

<u>Assumption A'</u>. $\Omega$ and $\partial_1\Omega$ are the images under a bi-Lipschitz mapping of some $\Omega'$ and $\partial_1\Omega'$ which satisfy the assumption A.

Let A be a bounded open set in $R^n$ and $\beta > 0$ be a constant. $F(\beta, A)$ denotes the family of all subsets B of $\overline{A}$ such that the following inequality holds for all $u \in C^1(\overline{A})$ vanishing on B $\|u\|_{q^*,A} \le \beta\|u_x\|_{q,A}$ where $1/q^* = 1/q - 1/n$ for all $1 < q \le n$. We shall require that $\Omega$ satisfies a mild assumption of admissibility described below.

<u>Assumption B</u>. For all $y \in \partial\Omega$ we have $\liminf_{\rho \to 0} \frac{|\Omega(y,\rho)|}{|I(y,\rho)|} > 0$. There exist a constant $\beta > 0$ and, for all $y \in \partial\Omega$, a $\overline{\rho}(y) > 0$ such that (1) for all $y \in \overline{\partial_1\Omega}$ and $0 < \rho < \overline{\rho}(y)$, $\Omega \cap S(y,\rho) \in F(\beta, \Omega(y,\rho))$; (2) for all $y \in \partial_2\Omega$ and $0 < \rho < \overline{\rho}(y)$, every subset E of $\Omega(y,\rho)$ such that $|E| > 1/2|\Omega(y,\rho)|$ belongs to the family $F(\beta, \Omega(y,\rho))$.

We consider on $\overline{\Omega}$ a linear uniformly elliptic second order differential operator of the form

$Au = -\frac{\partial}{\partial x_k}\left(a_{jk}(x)u_{xj}\right)$ (5.35) where the coefficients $a_{jk}$ are bounded measurable functions defined on $\overline{\Omega}$ satisfying $m|\xi|^2 \le a_{jk}(x)\xi_i\xi_k \le M|\xi|^2$, for all $\xi \in R^n$ and a.e. in $\overline{\Omega}$, with some constant of ellipticity m > 0. We shall write $a(u,v) = \int_\Omega a_{jk}(x)u_{xj}(x)v_{xk}(x)dx$ (5.36).

Then it is clear that there exists a constant C > 0 such that $|a(u,v)| \le C\|u\|_V \|v\|_V$, for all $u, v \in V$, and hence A maps V continuously into its dual space V'.

Let us set $K = \{u \in V; u \ge \psi \text{ in } \Omega\} = \{u \in V; u - \psi \ge 0 \text{ in } \Omega\}$. It is clear that K is a closed convex subset of V.
Let $T \in V'$ be given. We shall be concerned with the variational inequality $u \in K; a(u, v-u) \ge \langle T, v-u \rangle$, for all $v \in K$ (5.37), where $\langle , \rangle$ denotes the pairing between V and V'. When $\partial_2\Omega$ is Lipschitz, the functionals of the form

$$\langle T, v\rangle = \int_\Omega (f_0 v + f_j v_{xj})dx + \int_{\partial_2\Omega} gv d\sigma \quad (5.38)\text{, for all } v \in V,$$

belong to V' provided that

$f_0 \in L^r(\Omega), r \ge 2n/(n+2); f_j \in L^p(\Omega), p \ge 2$, for j = 1,…,n;
$g \in L^q(\partial_2\Omega), q \ge 2(n-1)/n$

where $d\sigma$ denotes the (n-1)-dimensional volume element on $\partial_2\Omega$.
Let u be the solution of the variational inequality (5.37) and $p \ge 2$. Let $k_0 = \max(\max_{\overline{\Omega}} \psi, 0)$. For any real number $k \ge k_0$ let $v = \min(u,k)$ which is clearly in the convex set K. If A(k) denotes the set $\{x \in \overline{\Omega}; u(x) > k\}$ then, since v-u vanishes in $\overline{\Omega} - A(k)$, we obtain on substituting this v in the variational inequality (5.37):

$$\int_{A(k)} a_{jl}u_{xj}u_{xl}dx \le \int_{A(k)} \left(f_0(u-k) + f_j u_{xj}\right)dx + \int_{A(k)\cap\partial_2\Omega} g(u-k)d\sigma \quad (5.39).$$

<u>Assumption C</u>. In the sense of distributions, $A\psi$ is a measure on $\Omega$ and $\partial\psi/\partial v$ is a measure on $\partial_2\Omega$ such that:

$$\max(A\psi - f, 0) \in L^p(\Omega), p > n/2; \max((\partial\psi/\partial v) - g, 0) \in L^q(\partial_2\Omega), q > n-1.$$

If the Assumption A (or A'), B and C are satisfied, then u is a solution of the variational inequality

$$u \in K; a(u, v-u) \ge \int_\Omega f(v-u)dx + \int_{\partial_2\Omega} g(v-u)d\sigma \quad \text{for all } v \in K. \quad (5.40)$$

We have that $u \in K$. If $v \in K$, then $v - u_m^{'} \in V$ for each m. Since the quasi-linear form $b_m^{'}(u,v)$, corresponding to the function $\theta_m^{'}$, is monotone and (hemi-) continuous it follows that

$$b_m'(v,v-u_m') \geq b_m'(u_m',v-u_m') = a(u_m',v-u_m') - \int_\Omega P(\psi,f)\theta_m'(u_m'-\psi)(v-u_m')dx$$
$$- \int_{\partial_2\Omega} Q(\psi,g)\theta_m'(u_m'-\psi)(v-u_m')d\sigma = \int_\Omega f(v-u_m')dx + \int_{\partial_2\Omega} g(v-u_m')d\sigma. \quad (5.41)$$

Since $v \in K$ implies that $v-\psi \geq 0$ so that $\theta_m'(v-\psi)=0$, we have $P(\psi,f)\theta_m'(v-\psi)=0$ in $\Omega$, $Q(\psi,g)\theta_m'(v-\psi)=0$ on $\partial_2\Omega$, and hence $a(v,v-u_m') = b_m'(v,v-u_m')$ for $v \in K$.
We thus obtain the inequality

$$a(v,v-u_m') \geq \int_\Omega f(v-u_m')dx + \int_{\partial_2\Omega} g(v-u_m')d\sigma. \quad (5.42)$$

Here since $u_m' \to u$ weakly in V, we can pass to the limits on both sides and we find that

$$a(v,v-u) \geq \int_\Omega f(v-u)dx + \int_{\partial_2\Omega} g(v-u)d\sigma, \text{ for all } v \in K. \quad (5.43)$$

Now we give an interpretation of the boundary conditions formally imposed by the variational inequality (5.43). We have show that the solutions $u_m'$ (a subsequence of $u_m'$) of the non-linear mixed boundary value problems converge in $V \cap C^{0,\lambda}(\overline{\Omega})$ to the solution of the variational inequality (5.43). Thus the variational inequality (5.43) can be formally described as follows:

$$Au - f \in \max(A\psi - f, 0)\tilde{\theta}(u-\psi) \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial_1\Omega,\ \partial u/\partial v - g \in \max((\partial\psi/\partial v) - g, 0)\tilde{\theta}(u-\psi) \text{ on } \partial_2\Omega. \quad (5.44)$$

We observe that if $\omega$ is an open subset of $\overline{\Omega}$ where $u > \psi$, then $\tilde{\theta}(u-\psi)=0$ and so u is a solution of the linear mixed boundary value problem

$$Au = f \text{ in } \omega \cap \Omega \text{ (in the sense of distribution)},$$
$$u = 0 \text{ on } \omega \cap \partial_1\Omega, \partial u/\partial v = g \text{ on } \omega \cap \partial_2\Omega. \quad (5.45)$$

If $\partial_2\Omega$ is of class $C^1$, then it admits a continuously varying tangent space at each of its points and a continuous normal vector field $v_0$ oriented towards the interior of $\Omega$. Then, for any $u \in C^1(\overline{\Omega}) \cap D(A)$, we obtain by applying Green's formula

$$\int_\Omega (Au)v dx = a(u,v) - \int_{\partial_2\Omega} \frac{\partial u}{\partial v} v d\sigma \quad (5.46)$$

where $\dfrac{\partial u}{\partial v} = a_{jk}(x)v_k(x)u_{xj}$. Thus we see that if $u \in C^1(\overline{\Omega}) \cap D(A)$, then

$$a(u,v) = \int_\Omega fv dx + \int_{\partial_2\Omega} (\partial u/\partial v)v d\sigma, \text{ for all } v \in V. \quad (5.47)$$

Let $V(\partial_2\Omega) = V/V_0$ ($V_0$ being the space of all functions v in V having its trace on $\partial_2\Omega$ zero) be provided with the quotient norm. There exists a unique element $G(u) \in [V(\partial_2\Omega)]'$, the dual space of $V(\partial_2\Omega)$, such that $\langle G(u),v\rangle = \langle Au,v\rangle - a(u,v)$. By definition we set $(\partial u/\partial v) = G(u)$ on $\partial_2\Omega$. We know that $V(\partial_2\Omega) \subset L^s(\partial_2\Omega)$ and the inclusion mapping is continuous so that every $g \in L^{s'}(\partial_2\Omega)$ defines a continuous linear functional on $V(\partial_2\Omega)$. Moreover, we can then write

$\langle G(u), v\rangle = \int_{\partial_2\Omega} gv d\sigma$ , (5.48) that is $(\partial u/\partial v) = g$ on $\partial_2\Omega$ in a "generalized sense".

These considerations lead us to the following formal interpretation of the **boundary conditions**.

1) If there exists an open subset $E_1$ of $\partial_2\Omega$ where $u > \psi$, then $\partial u/\partial v = g$ on $E_1$.
2) If $u = \psi$ and $g - \partial\psi/\partial v$ is a positive measure on a subset $E_2$ of $\partial_2\Omega$, then again $\partial u/\partial v = g$ on $E_2$.
3) If $u = \psi$ and $\partial\psi/\partial v - g$ is a positive measure on a subset $E_3$ of $\partial_2\Omega$ then, since $0 \le \tilde{\theta}(t) \le 1$, we have $g \le \partial u/\partial v \le \partial\psi/\partial v$ on $E_3$.

The solution u of the variational inequality (5.43) can also be obtained by another approximation procedure of potential theoretic nature.
Suppose $u \in K$ is the solution of the variational inequality (5.43). Let $K_u$ denote the cone of all w$\in V$ which can be written in the form w = t (u-v) for some $v \in K$ and t > 0, and $\overline{K}_u$ be its closure in V. Then it is clear that

$$a(u,w) \ge \int_\Omega fwdx + \int_{\partial_2\Omega} gwd\sigma, \text{ for all } w \in \overline{K}_u. \quad (5.49)$$

We next observe that the positive cone $\{w \in V; w \ge 0 in \overline{\Omega}\}$ is contained in $\overline{K}_u$ and in particular, (5.49) is satisfied. These considerations lead us to introduce the following definition:
A distribution $w \in H^1(\Omega)$ is said to be a super solution with respect to V, A, f and g if

$$a(w,\phi) \ge \int_\Omega f\phi dx + \int_{\partial_2\Omega} g\phi d\sigma, \text{ for all } \phi \in C^1(\overline{\Omega}) \text{ with } \phi = 0 \text{ on } \partial_1\Omega \text{ and } \phi \ge 0 \text{ in } \overline{\Omega}. \quad (5.50)$$

We have the following Theorem: If $u \in K$ is the solution of the variational inequality (5.43) and W denotes the set of all super-solutions with respect to V, A, f and g such that $w \ge 0$ on $\partial_1\Omega$ and $w \ge \psi$ in $\Omega$ (5.51) then $u = \min\{w; w \in W\}$.
Let $w \in W$ be arbitrary and let v = min(u, w). Then $v \in K$ because of (5.51) and we shall show that v = u. Substituting v in the variational inequality we get

$$a(u, v-u) \ge \int_\Omega f(v-u)dx + \int_{\partial_2\Omega} g(v-u)d\sigma. \quad (5.52)$$

Since w is a super solution and $v - u \in V$ with $v - u \le 0$ in $\overline{\Omega}$, we have

$$a(w, v-u) \le \int_\Omega f(v-u)dx + \int_{\partial_2\Omega} g(v-u)d\sigma. \quad (5.53)$$

We can write the left hand side as

$a(w,v-u) = \left(\int_{(u=w)} + \int_{(u>w)}\right) a_{jk} w_{xk} (v-u)_{xj} dx$ where we have $v - u = 0$ and $v_x = u_x$ on the set $\{x \in \overline{\Omega}; u(x) = w(x)\}$ and v = w and $v_x = w_x$ on the set $\{x \in \overline{\Omega}; u(x) > w(x)\}$. Hence the first integral vanishes and we have

$$a(v,v-u)=a(w,v-u)\le\int_\Omega f(v-u)dx+\int_{\partial_2\Omega}g(v-u)d\sigma. \quad (5.54)$$

Let $u_0$ and $\psi$ be two functions belonging to $H^1(\Omega)$ such that $\psi\le 0$ on $\partial_1\Omega$. Consider the closed convex set $K_0$ in $H^1(\Omega)$ defined by $K_0=\{v\in H^1(\Omega);v-u_0\in Vandv-u_0\ge\psi in\Omega\}$. Then all our results can be extended to the variational inequality

$$u\in K_0;a(u,v-u)\ge\int_\Omega\left[f_0(v-u)+f_j(v-u)_{xj}\right]dx+\int_{\partial_2\Omega}g(v-u)d\sigma, \text{ for all } v\in K_0. \quad (5.55)$$

The variational inequality (5.55) formally corresponds to the mixed **boundary value problem**:

$$Aw=f_0-(f_j)_{xj} \text{ in } \Omega \text{ (in the sense of distributions)}$$
$$w=u_0 \text{ on } \partial_1\Omega,\ \partial w/\partial v=g \text{ on } \partial_2\Omega. \quad (5.56)$$

<u>Examples of equations concerning open sets applied to equations whose solutions describing naked singularities</u>.

Now we take the following equation:

$$S=\int d^{10}x\sqrt{|g|}\left[R-\frac{1}{2}(\partial\phi)^2-\frac{1}{2}m^2e^{5\phi/2}\right], \quad (5.57)$$

which is a special form of eq. (5.1) obtained by choosing $\alpha=1$, $\beta=1/2$, $\Lambda=\frac{1}{2}m^2$ and in the formula $V(\phi)=\Lambda e^{-\lambda\phi}$, $\lambda=-5/2$. Furthermore, we take the eq. (5.4). If $\Omega=h(r)$, where $h(r)$ is equal to eq. (5.4), we know that for $\beta\rho^2>\alpha(M^2+1)$ and $M<0$, $\Lambda>0$, there is a naked singularity at the origin. Then, from the eqs. (5.30) or (5.31), we obtain the following relation:

$$\int_\Omega\left\{\sum_{ij}^{1...m}a_{ij}(x)D_iuD_jv+c(x)uv\right\}dx=\int_\Omega gvdx=\int_\Omega\int d^{10}x\sqrt{|g|}\left[R-\frac{1}{2}(\partial\phi)^2-\frac{1}{2}m^2e^{5\phi/2}\right]. \quad (5.58)$$

From the eqs. (5.46), (5.47) and (5.57) we obtain:

$$\int_\Omega(Au)vdx=a(u,v)-\int_{\partial_2\Omega}\left(\frac{\partial u}{\partial v}\right)vd\sigma,\quad a(u,v)=\int_\Omega fvdx+\int_{\partial_2\Omega}\left(\frac{\partial u}{\partial v}\right)vd\sigma, \text{ hence}$$

$$\int_\Omega(Au)vdx=\int_\Omega fvdx+\int_{\partial_2\Omega}\left(\frac{\partial u}{\partial v}\right)vd\sigma-\int_{\partial_2\Omega}\left(\frac{\partial u}{\partial v}\right)vd\sigma=$$

$$\int_\Omega(Au)vdx=\int_\Omega fvdx=\int_\Omega\int d^{10}x\sqrt{|g|}\left[R-\frac{1}{2}(\partial\phi)^2-\frac{1}{2}m^2e^{5\phi/2}\right]. \quad (5.59)$$

We note that also these equations can be related with the Palumbo's model. Indeed, we have the following connections:

$$\int_{\Omega}\left\{\sum_{ij}^{1...m} a_{ij}(x)D_i u D_j v + c(x)uv\right\}dx = \int_{\Omega} gvdx = \int_{\Omega}\int d^{10}x\sqrt{|g|}\left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2 e^{5\phi/2}\right] \Rightarrow$$

$$\Rightarrow \int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right], \quad (5.60) \text{ and}$$

$$\int_{\Omega}(Au)vdx = \int_{\Omega} fvdx = \int_{\Omega}\int d^{10}x\sqrt{|g|}\left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2 e^{5\phi/2}\right] \Rightarrow$$

$$\Rightarrow \int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right]. \quad (5.61)$$

## **Conclusions**.

Our conviction is that the following theorems, as so for open sets, can be applied also to the naked singularities. Principally the expressions concerning the boundary conditions for these equations describing open sets, must be considered and applied to the equations whose solutions describing naked singularities.

**Acknowledgments**

I would like to thank G. Tasinato of Rudolf Peierls Centre for Theoretical Physics, Oxford University, and M. Zagermann of Max-Planck-Institut fur Physik, Munich, for stimulating and useful discussions.
Principally G. Tasinato for his friendship and availability.
Furthermore, I would like to thank A. Palumbo whose advices has been invaluable for me.

**REFERENCES.**

[1] G.Dvali, R.Kallosh, A.Van Proeyen “D-term strings” hep-th/0312005.
[2] S.L.Parameswaran, G.Tasinato, I.Zavala “The 6D SuperSwirl” hep-th/0509061.
[3] P.Di Vecchia, A.Liccardo, R.Marotta, F.Pezzella “On the Gauge/Gravity Correspondence and the Open/Closed String Duality” hep-th/0503156.

[4] H.P.Durr "On a Gauge and Conformal Invariant Nonlinear Spinor Theory" Max Planck Institut fur Physik und Astrophysik, Munchen (Germany) – October 1974.
[5] K.G.Savvidy "Born-Infeld Action in String Theory" hep-th/9906075.
[6] G.D'Appollonio "Aspetti non perturbativi della Teoria delle Stringhe" hep-th/0111284.
[7] P.Chen, K.Dasgupta, K.Narayan, M.Shmakova, M.Zagermann "Brane Inflation, Solitons and Cosmological Solutions: I" hep-th/0501185.
[8] C.P.Burgess, C.Nunez, F.Quevedo, I.Zavala C., G.Tasinato "General brane geometries from scalar potentials: gauged supergravities and accelerating universes" JHEP08 (2003) 056.
[9] A.Languasco "La congettura di Goldbach" Tesi Dottorato di Ricerca in Matematica VI ciclo (1995).
[10] E.Dudas, M.Nicolosi, G.Pradisi, A.Sagnotti "On tadpoles and vacuum redefinitions in String Theory" hep-th/0410101.
[11] G.Stampacchia "Contributi alla regolarizzazione delle soluzioni dei problemi al contorno per equazioni del secondo ordine ellittiche", Ann. Sc. Norm. Sup. Pisa, (III), XII (1958), pp. 223-245.
[12] G. Stampacchia, V. Murthy "A variational inequality with mixed boundary conditions", Israel J. of Math., 13, (1-2), (1972), pp. 188-334.