On some mathematical connections between Fermat's Last Theorem, Modular Functions, Modular Elliptic Curves and some sector of String Theory II

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Abstract

This paper is fundamentally a review, a thesis, of principal results obtained in some sectors of Number Theory and String Theory of various authoritative theoretical physicists and mathematicians.
Precisely, we have described some mathematical results regarding the Fermat's Last Theorem, the Mellin transform, the Riemann zeta function, the Ramanujan's modular equations, how primes and adeles are related to the Riemann zeta functions and the p-adic and adelic string theory.
Furthermore, we show that also the fundamental relationship concerning the Palumbo-Nardelli model (a general relationship that links bosonic string action and superstring action, i.e. bosonic and fermionic strings in all natural systems), can be related with some equations regarding the p-adic (adelic) string sector.
Thence, in conclusion, we have described some new interesting connections that are been obtained between String Theory and Number Theory, with regard the arguments above mentioned.

Chapter 4.

On p-adic and adelic strings

4.1 Open and closed p-adic strings.

Let us now discuss the question of the construction of a dynamical theory for open and closed p-adic strings. It was proposed (Volovich, 1987) to consider p-adic generalization of the Veneziano string amplitude in two ways, according to two equivalent representations

$$A(a,b) = \int_0^1 |x|^{a-1} |1-x|^{b-1} dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \quad (1)$$

The first way corresponds to an interpretation of the amplitude A(a, b) as a convolution of two characters and the second one to the p-adic interpolation of the gamma function. Using the first approach a complex-valued string amplitude over a finite Galois field has been constructed.

Consideration of string amplitudes as a convolution of characters is a very general concept applicable to characters on number fields, groups and algebras.
Now, we have the string amplitudes of the following form

$$A(\gamma_a,\gamma_b)=\int_K \gamma_a(x)\gamma_b(1-x)dx\,, \quad (2)$$

where K is a field F, i.e., K = F, $\gamma_a(x)$ is a multiplicative character on K, and dx is a measure on K. Note that the range of integration in (2) is over the entire field F, and hence this p-adic generalization is rather one of the Virasoro-Shapiro amplitude

$$A=\int_C |z|^{a-1}|1-z|^{b-1}dz\,, \quad (3)$$

than of the Veneziano amplitude (1), where the integration is over the unit segment on the real axis. The equation (3) is just a particular case of (2) for K = C and $\gamma_a(z)=|z|^{a-1}$. The ordinary Veneziano amplitude can be rewritten in the following way

$$A=\int_R |x|^{a-1}|1-x|^{b-1}\theta_{[0,1]}(x)dx\,, \quad (4)$$

where $\theta(x)$ is the characteristic function of the segment $[0,1]$. In particular, it can be written in terms of the Heaviside function $\theta_{[0,1]}(x)=\theta(x)\theta(1-x)$. Hence, in order to have a generalization of the expression (4) on an arbitrary field F one should have on F an analogue of the Heaviside function or the function sign x.
We have a generalization of the amplitude (4), in the case of an arbitrary locally compact disconnected field F, in the following form

$$A_{F,\tau}^{open}(\gamma_a,\gamma_b)=\int_F |x|^{a-1}|1-x|^{b-1}\theta_{\tau[0,1]}dx \quad (5)$$

where $\theta_{\tau[0,1]}(x)$ is a p-adic generalization of the characteristic function of the segment $[0,1]$ on F related to a quadratic extension $F(\sqrt{\tau})$. In particular one can take the function $\theta_{\tau[0,1]}(x)$ in the form $\theta_\tau(x)\theta_\tau(1-x)$ where $\theta_\tau(x)$ is a p-adic analogue of the Heaviside function.
In the ordinary case there is an important relation between amplitudes of the open and the closed strings. This relation give a connection on the tree level as follows

$$A_{tree}^{closed}(s,t,u)=\sin\left(\frac{\pi t}{8}\right)A_{tree}^{open}\left(\frac{s}{4},\frac{t}{4}\right)A_{tree}^{open}\left(\frac{t}{4},\frac{u}{4}\right), \quad (6)$$

where s, t, u are the Mandelstam variables.
Let F in eq. (5) be a non-discrete totally disconnected and locally compact field and define also the generalized Heaviside function in the form

$$\theta_\tau(\omega)=\frac{1+sign_\tau\omega}{2} \quad (7)$$

which is an analogue of the ordinary one.
Now we will consider the amplitude (5) with the characteristic function in one of the following forms:

$$\theta_{\tau[0,1]}(x) = \theta_\tau(x)\theta_\tau(1-x), \qquad (8.1)$$

$$\theta_{\tau[0,1]}(x) = \frac{1}{2}\left(1 + Sign_\tau x \cdot Sign_\tau(1-x)\right), \qquad (8.2)$$

$$\theta_{\tau[0,1]}(x) = \frac{1}{2}\left(Sign_\tau x + Sign_\tau(1-x)\right), \qquad (8.3)$$

$$\theta_{\tau[0,1]}(x) = \frac{1}{2}\left(Sign_\tau x - Sign_\tau(-1) \cdot Sign_\tau(1-x)\right), \quad \tau = \varepsilon, \qquad (8.4)$$

$$\theta_{\tau[0,1]}(x) = \frac{1}{2}\left(1 - Sign_\tau(-1) \cdot Sign_\tau x \cdot Sign_\tau(1-x)\right). \qquad (8.5)$$

The corresponding amplitudes (5) can be calculated with the help of the general formula

$$B(\pi_1, \pi_2) = \frac{\Gamma(\pi_1)\Gamma(\pi_2)}{\Gamma(\pi_1\pi_2)}, \qquad (9)$$

which connects the beta function

$$B(\pi_1, \pi_2) = \int_F \pi_1(x)|x|^{-1} \pi_2(1-x)|1-x|^{-1} dx, \qquad (10)$$

where $\pi(x)$ is a multiplicative character with the gamma function defined by an additive character χ

$$\Gamma(\pi) = \int_F \chi(x)\pi(x)|x|^{-1} dx. \qquad (11)$$

Consider now the string amplitudes, constructed over the p-adic fields Q_p and their quadratic extension $Q_p\left(\sqrt{\tau}\right)$, from the point of view of the product formulae (6) which relates amplitudes of closed and open strings in a very simple form. With regard the case $\tau = \varepsilon$, the closed string amplitude defined on the quadratically extended field $K = Q_p\left(\sqrt{\varepsilon}\right)$, has the form

$$A^{closed}_{Q_p(\sqrt{\varepsilon})}(a,b,c) = \int_{Q_p(\sqrt{\varepsilon})} |x|^{a-1}|1-x|^{b-1} dx = \frac{1-q^{a-1}}{1-q^{-a}} \cdot \frac{1-q^{b-1}}{1-q^{-b}} \cdot \frac{1-q^{c-1}}{1-q^{-c}}, \qquad (12)$$

where $q = p^2$. There are no such formulae as simple as (7) for the above constructed open string amplitudes. However, there exists a formula in the following form

$$A^{closed}_{Q_p(\sqrt{\varepsilon})}(a,b,c) = A^{open,total}_{Q_p}(a,b,c)\tilde{A}_{Q_p}(a,b,c), \qquad (13)$$

where

$$A^{open,total}_{Q_p}(a,b,c) = \int_{Q_p} |x|^{a-1}|1-x|^{b-1} dx = \frac{1-p^{a-1}}{1-p^{-a}} \cdot \frac{1-p^{b-1}}{1-p^{-b}} \cdot \frac{1-p^{c-1}}{1-p^{-c}} \qquad (14)$$

is a p-adic analogue of the totally crossing symmetric Veneziano amplitude.
Furthermore, the p-adic generalization of the N-point tree amplitude for vector particles in the bosonic case, can be proposed in the following form

$$A(\zeta_1 k_1,\dots,\zeta_n k_n) = g^{n-2} \int_{(Q_p)^{n-3}} \theta_{[0,y_{n-1},\dots,y_{3,1}]}(y) F(\zeta,k,y) \cdot \prod_{3\le i<j\le n-1} |y_i - y_j|_p^{k_i k_j} \prod_{3\le i\le n-1} \left(|y_i|_p^{k_n k_i} |1-y_i|^{k_2 k_i} dy_i \right), \quad (15)$$

where $\theta_{[0,y_{n-1},\dots,y_{3,1}]}(y)$ is a p-adic generalization of the characteristic function of the simplex $0 \le y_{n-1} \le \dots \le y_4 \le y_3 \le 1$ and $F(\zeta,k,y)$ is the part of $\exp\sum_{i=j}\left\{\frac{1}{2}(\zeta_i\zeta_j)/(y_i-y_j)^2 - k_i k_j / y_i - y_j\right\}$ that is multilinear in all the polarization vectors ζ_i.

4.2 On adelic strings.

The set of all adeles A may be given in the form

$$A = \bigcup_S \mathrm{A}(S), \quad \mathrm{A}(S) = R \times \prod_{p\in S} Q_p \times \prod_{p\notin S} Z_p . \quad (16)$$

A has the structure of a topological ring.
We recall that quantum amplitudes defined by means of path integral may be symbolically presented as

$$A(K) = \int A(X)\chi\left(-\frac{1}{h}S[X]\right)DX , \quad (17)$$

where K and X denote classical momenta and configuration space, respectively. $\chi(a)$ is an additive character, $S[X]$ is a classical action and h is the Planck constant.
Now we consider simple p-adic and adelic bosonic string amplitudes based on the functional integral (17). The scattering of two real bosonic strings in 26-dimensional space-time at the tree level can be described in terms of the path integral in 2-dimensional quantum field theory formalism as follows:

$$A_\infty(k_1,\dots,k_4) = g_\infty^2 \int DX \exp\left(\frac{2\pi i}{h} S_0[X]\right) \times \prod_{j=1}^{4} \int d^2\sigma_j \exp\left[\frac{2\pi i}{h} k_\mu^{(j)} X^\mu(\sigma_j,\tau_j)\right], \quad (18)$$

where $DX = DX^0(\sigma,\tau)DX^1(\sigma,\tau)\dots DX^{25}(\sigma,\tau)$, $d^2\sigma_j = d\sigma_j d\tau_j$ and

$$S_0[X] = -\frac{T}{2}\int d^2\sigma \partial_\alpha X^\mu \partial^\alpha X_\mu \quad (19)$$

with $\alpha = 0,1$ and $\mu = 0,1,\dots,25$. Using the usual procedure one can obtain the crossing symmetric Veneziano amplitude

$$A_\infty(k_1,\dots,k_4) = g_\infty^2 \int_R |x|_\infty^{k_1 k_2} |1-x|_\infty^{k_2 k_3} dx \quad (20)$$

and similarly the Virasoro-Shapiro one for closed bosonic strings.

As p-adic Veneziano amplitude, it was postuled p-adic analogue of (20), i.e.

$$A_p(k_1,...,k_4)=g_p^2\int_{Q_p}|x|_p^{k_1k_2}|1-x|_p^{k_2k_3}dx\,, \quad (21)$$

where only the string world sheet (parametrized by x) is p-adic. Expressions (20) and (21) are Gel'fand-Graev beta functions on R and Q_p, respectively.
Now we take p-adic analogue of (18), i.e.

$$A_p(k_1,...,k_4)=g_p^2\int DX\chi_p\left(-\frac{1}{h}S_0[X]\right)\times\prod_{j=1}^{4}\int d^2\sigma_j\chi_p\left(-\frac{1}{h}k_\mu^{(j)}X^\mu(\sigma_j,\tau_j)\right), \quad (22)$$

to be p-adic string amplitude, where $\chi_p(u)=\exp(2\pi i\{u\}_p)$ is p-adic additive character and $\{u\}_p$ is the fractional part of $u\in Q_p$. In (22), all space-time coordinates X_μ, momenta k_i and world sheet (σ,τ) are p-adic.
Evaluation of (22), in analogous way to the real case, leads to

$$A_p(k_1,...,k_4)=g_p^2\prod_{j=1}^{4}\int d^2\sigma_j\times\chi_p\left[\frac{\sqrt{-1}}{2hT}\sum_{i<j}k_ik_j\log\left((\sigma_i-\sigma_j)^2+(\tau_i-\tau_j)^2\right)\right]. \quad (23)$$

Adelic string amplitude is product of real and all p-adic amplitudes, i.e.

$$A_A(k_1,...,k_4)=A_\infty(k_1,...,k_4)\prod_p A_p(k_1,...,k_4). \quad (24)$$

In the case of the Veneziano amplitude and $(\sigma_i,\tau_j)\in \mathrm{A}(S)\times\mathrm{A}(S)$, where $\mathrm{A}(S)$ is defined in (16), we have

$$A_A(k_1,...,k_4)=g_\infty^2\int_R|x|_\infty^{k_1k_2}|1-x|_\infty^{k_2k_3}dx\times\prod_{p\in S}g_p^2\prod_{j=1}^{4}\int d^2\sigma_j\times\prod_{p\notin S}g_p^2. \quad (25)$$

There is the sense to take adelic coupling constant as

$$g_A^2=|g|_\infty^2\prod_p|g|_p^2=1, \quad 0\neq g\in Q. \quad (26)$$

Hence, it follows that p-adic effects in the adelic Veneziano amplitude induce discreteness of string momenta and contribute to an effective coupling constant in the form

$$g_{ef}^2=g_A^2\prod_{p\in S}\prod_{j=1}^{4}\int d^2\sigma_j\geq 1. \quad (26b)$$

4.3 Solitonic q-branes of p-adic string theory.

Now we consider the expressions for various amplitudes in ordinary bosonic open string theory, written as integrals over the boundary of the world sheet which is the real line R. Now replace the integrals over R by integrals over the p-adic field Q_p with appropriate measure, and the norms of

the functions in the integrand by the p-adic norms. Using p-adic analysis, it is possible to compute N tachyon amplitudes at tree-level for all $N \geq 3$.
This leads to an exact action for the open string tachyon in d dimensional p-adic string theory. This action is:

$$S = \int d^d x L = \frac{1}{g^2}\frac{p^2}{p-1}\int d^d x\left[-\frac{1}{2}\phi p^{-\frac{1}{2}\Box}\phi + \frac{1}{p+1}\phi^{p+1}\right], \quad (27)$$

where $\Box$ denotes the d dimensional Laplacian, ϕ is the tachyon field, g is the open string coupling constant, and p is an arbitrary prime number.
The equation of motion derived from this action is,

$$p^{-\frac{1}{2}\Box}\phi = \phi^p . \quad (28)$$

The following configuration

$$\phi(x) = f\left(x^{q+1}\right)f\left(x^{q+2}\right)\ldots f\left(x^{d-1}\right) \equiv F^{(d-q-1)}\left(x^{q+1},\ldots,x^{d-1}\right), \quad (29)$$

with

$$f(\eta) \equiv p^{\frac{1}{2(p-1)}}\exp\left(-\frac{1}{2}\frac{p-1}{p\ln p}\eta^2\right), \quad (30)$$

describes a soliton solution with energy density localised around the hyperplane $x^{q+1} = \ldots = x^{d-1} = 0$.
This follows from the identity:

$$p^{-\frac{1}{2}\partial_\eta^2} f(\eta) = (f(\eta))^p . \quad (31)$$

We shall call (29), with f as in (30), the solitonic q-brane solution. Let us denote by $x_\perp = \left(x^{q+1},\ldots,x^{d-1}\right)$ the coordinates transverse to the brane and by $x_\parallel = \left(x^0,\ldots,x^q\right)$ those tangential to it. The energy density per unit q-volume of this brane, which can be identified as its tension T_q, is given by

$$T_q = -\int d^{d-q-1}x_\perp L\left(\phi = F^{(d-q-1)}(x_\perp)\right) = \frac{1}{2g_q^2}\frac{p^2}{p+1} \quad (32)$$

where

$$g_q = g\left[\frac{p^2-1}{2\pi p^{2p/(2p-1)}\ln p}\right]^{(d-q-1)/4} . \quad (33)$$

Hence, we obtain the following equation

$$T_q = -\int d^{d-q-1}x_\perp L\left(F^{(d-q-1)}(x_\perp)\right) = \frac{1}{2\left\{g\left[\frac{p^2-1}{2\pi p^{2p/(2p-1)}\ln p}\right]^{(d-q-1)/4}\right\}^2}\frac{p^2}{p+1} . \quad (33b)$$

Let us now consider a configuration of the type

$$\phi(x) = F^{(d-q-1)}(x_\perp)\psi(x_\parallel), \quad (34)$$

with $F^{(d-q-1)}(x_\perp)$ as defined in (29), (30). For $\psi = 1$ this describes the solitonic q-brane. Fluctuations of ψ around 1 denote fluctuations of ϕ localised on the soliton; thus $\psi(x_\parallel)$ can be regarded as one of the fields on its world-volume. We shall call this the tachyon field on the solitonic q-brane world-volume. Substituting (34) into (28) and using (31) we get

$$p^{-\frac{1}{2}\Box_\parallel}\psi = \psi^p, \quad (35)$$

where $\Box_\parallel$ denotes the (q+1) dimensional Laplacian involving the world-volume coordinates $x_\parallel$ of the q-brane. The action involving ψ can be obtained by substituting (34) into (27):

$$S_q(\psi) = S\left(\phi = F^{(d-q-1)}(x_\perp)\psi(x_\parallel)\right) = \frac{1}{g_q^2}\frac{p^2}{p-1}\int d^{q+1}x_\parallel\left[-\frac{1}{2}\psi p^{-\frac{1}{2}\Box_\parallel}\psi + \frac{1}{p+1}\psi^{p+1}\right], \quad (36)$$

where g_q has been defined in eq.(33).
In conclusion, we shall now show the world-volume action on the Dirichlet q-brane. Let us consider the situation where we start with the action (27) with g replaced by another coupling constant $\overline{g}$, and compactify (d – q – 1) directions on circles of radii $1/\sqrt{2}$. Let u^i denote the compact coordinates and z^μ the non-compacts ones, and consider an expansion of the field ϕ of the form:

$$\phi(x) = \tilde{\psi}(z) + \sqrt{\frac{C}{p}}\sum_{i=1}^{d-q-1}\tilde{\xi}^i(z)\left(\sqrt{2}\cos\left(\sqrt{2}u^i\right)\right) + \ldots . \quad (37)$$

Substituting this into (27), with g replaced by $\overline{g}$, we get the action:

$$\frac{1}{\overline{g}^2}\frac{p^2}{p-1}\left(\frac{2\pi}{\sqrt{2}}\right)^{d-q-1}\int d^{q+1}z\left[-\frac{1}{2}\tilde{\psi}p^{-\frac{1}{2}\Box_z}\tilde{\psi} + \frac{1}{p+1}\tilde{\psi}^{p+1} - C\left\{\frac{1}{2}\tilde{\xi}^i p^{-\frac{1}{2}\Box_z}\tilde{\xi}^i - \frac{1}{2}\tilde{\psi}^{p-1}\tilde{\xi}^i\tilde{\xi}^i\right\} + O\left(\tilde{\xi}^3\right) + \ldots\right]. \quad (38)$$

4.4 Open and closed scalar zeta strings.

The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$L_p = \frac{1}{g^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2}}\varphi + \frac{1}{p+1}\varphi^{p+1}\right], \quad (39)$$

where p is any prime number, $\Box = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alembertian and we adopt metric with signature (– + … +).

Now we want to show a model which incorporates the p-adic string Lagrangians in a restricted adelic way. The eq. (39) take the form:

$$L = \sum_{n\geq 1} C_n L_n = \sum_{n\geq 1} \frac{n-1}{n^2} L_n = \frac{1}{g^2}\left[-\frac{1}{2}\phi \sum_{n\geq 1} n^{-\frac{\Box}{2}} \phi + \sum_{n\geq 1} \frac{1}{n+1}\phi^{n+1} \right]. \quad (39b)$$

Recall that the Riemann zeta function is defined as

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}, \quad s = \sigma + i\tau, \quad \sigma > 1. \quad (40)$$

Employing usual expansion for the logarithmic function and definition (40) we can rewrite (39b) in the form

$$L = -\frac{1}{g^2}\left[1/2\phi\zeta(\Box/2)\phi + \phi + \ln(1-\phi) \right], \quad (41)$$

where $|\phi| < 1$. $\zeta(\Box/2)$ acts as pseudo-differential operator in the following way:

$$\zeta(\Box/2)\phi(x) = \frac{1}{(2\pi)^D} \int e^{ixk} \zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk, \quad -k^2 = k_0^2 - \vec{k}^2 > 2 + \varepsilon, \quad (42)$$

where $\tilde{\phi}(k) = \int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.

Dynamics of this field ϕ is encoded in the (pseudo)differential form of the Riemann zeta function. When the d'Alembertian is an argument of the Riemann zeta function we shall call such string a *zeta string*. Consequently, the above ϕ is an open scalar zeta string. The equation of motion for the zeta string ϕ is

$$\zeta(\Box/2)\phi = \frac{1}{(2\pi)^D} \int_{k_0^2 - \vec{k}^2 > 2+\varepsilon} e^{ixk} \zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} \quad (43)$$

which has an evident solution $\phi = 0$.

For the case of time dependent spatially homogeneous solutions, we have the following equation of motion

$$\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t) = \frac{1}{(2\pi)} \int_{|k_0| > \sqrt{2}+\varepsilon} e^{-ik_0 t} \zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0 = \frac{\phi(t)}{1-\phi(t)}. \quad (44)$$

Finally, with regard the open and closed scalar zeta strings, the equations of motion are

$$\zeta(\Box/2)\phi = \frac{1}{(2\pi)^D} \int e^{ixk} \zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \sum_{n\geq 1} \theta^{\frac{n(n-1)}{2}} \phi^n, \quad (45)$$

$$\zeta(\Box/4)\theta = \frac{1}{(2\pi)^D} \int e^{ixk} \zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk = \sum_{n\geq 1} \left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)} \theta^{\frac{n(n-1)}{2}-1} \left(\phi^{n+1} - 1\right) \right], \quad (46)$$

and one can easily see trivial solution $\phi = \theta = 0$.

Chapter 5

On some correlations obtained between some solutions in string theory, Riemann zeta function and Palumbo-Nardelli Model.

With regard the paper: "Brane Inflation, Solitons and Cosmological Solutions:I", that dealt various cosmological solutions for a D3/D7 system directly from M-theory with fluxes and M2-branes, and the paper: "General brane geometries from scalar potentials: gauged supergravities and accelerating universes", that dealt time-dependent configurations describing accelerating universes, we have obtained interesting connections between some equations concerning cosmological solutions, some equations concerning the Riemann zeta function and the relationship of Palumbo-Nardelli model.

5.1 Cosmological solutions from the D3/D7 system.

The full action in M-theory will consist of three pieces: a bulk term, S_{bulk}, a quantum correction term, $S_{quantum}$, and a membrane source term, S_{M2}. The action is then given as the sum of these three pieces:

$$S = S_{bulk} + S_{quantum} + S_{M2}. \quad (1)$$

The individual pieces are:

$$S_{bulk} = \frac{1}{2\kappa^2}\int d^{11}x\sqrt{-g}\left[R - \frac{1}{48}G^2\right] - \frac{1}{12\kappa^2}\int C \wedge G \wedge G, \quad (2)$$

where we have defined G = dC, with C being the usual three form of M-theory, and $\kappa^2 \equiv 8\pi G_N^{(11)}$. This is the bosonic part of the classical eleven-dimensional supergravity action. The leading quantum correction to the action can be written as:

$$S_{quantum} = b_1 T_2 \int d^{11}x\sqrt{-g}\left[J_0 - \frac{1}{2}E_8\right] - T_2\int C \wedge X_8. \quad (3)$$

The coefficient T_2 is the membrane tension. For our case, $T_2 = \left(\frac{2\pi^2}{\kappa^2}\right)^{1/3}$, and b_1 is a constant number given explicitly as $b_1 = (2\pi)^{-4}3^{-2}2^{-13}$. The M2 brane action is given by:

$$S_{M2} = -\frac{T_2}{2}\int d^3\sigma\sqrt{-\gamma}\left[\gamma^{\mu\nu}\partial_\mu X^M \partial_\nu X^N g_{MN} - 1 + \frac{1}{3}\varepsilon^{\mu\nu\rho}\partial_\mu X^M \partial_\nu X^N \partial_\rho X^P C_{MNP}\right], \quad (4)$$

where X^M are the embedding coordinates of the membrane. The world-volume metric $\gamma_{\mu\nu}, \mu,\nu = 0,1,2$ is simply the pull-back of g_{MN}, the space-time metric. The motion of this M2 brane is obviously influenced by the background G-fluxes.

5.2 Classification and stability of cosmological solutions.

The metric that we get in type IIB is of the following generic form:

$$ds^2 = \frac{f_1}{t^\alpha}\left(-dt^2 + dx_1^2 + dx_2^2\right) + \frac{f_2}{t^\beta}dx_3^2 + \frac{f_3}{t^\gamma}g_{mn}dy^m dy^n \quad (5)$$

where $f_i = f_i(y)$ are some functions of the fourfold coordinates and α, β and γ could be positive or negative number. For arbitrary $f_i(y)$ and arbitrary powers of t , the type IIB metric can in general come from an M-theory metric of the form

$$ds^2 = e^{2A}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2B}g_{mn}dy^m dy^n + e^{2C}|dz|^2, \quad (6)$$

with three different warp factors A, B and C , given by:

$$A = \frac{1}{2}\log\frac{f_1 f_2^{\frac{1}{3}}}{t^{\alpha+\frac{\beta}{3}}} + \frac{1}{3}\log\frac{\tau_2}{|\tau|^2}, \quad B = \frac{1}{2}\log\frac{f_3 f_2^{\frac{1}{3}}}{t^{\gamma+\frac{\beta}{3}}} + \frac{1}{3}\log\frac{\tau_2}{|\tau|^2}, \quad C = -\frac{1}{3}\left[\log\frac{f_2}{t^\beta} + \log\frac{\tau_2^2}{|\tau|}\right]. \quad (7)$$

To see what the possible choices are for such a background, we need to find the difference B – C . This is given by:

$$B - C = \frac{1}{2}\log\frac{f_2 f_3}{t^{\gamma+\beta}} + \log\frac{\tau_2}{|\tau|}. \quad (8)$$

Since the space and time dependent parts of (8) can be isolated, (8) can only vanish if

$$f_2 = f_3^{-1}\cdot\frac{|\tau|}{\tau_2}, \qquad \gamma + \beta = 0, \quad (9)$$

with α and $f_1(y)$ remaining completely arbitrary.
We now study the following interesting case, where $\alpha = \beta = 2$, $\gamma = 0$ $f_1 = f_2$. The internal six manifold is time independent. This example would correspond to an exact de-Sitter background, and therefore this would be an accelerating universe with the three warp factors given by:

$$A = \frac{2}{3}\log\frac{f_1}{t^2}, \quad B = \frac{1}{2}\left[\log f_3 + \frac{1}{3}\log\frac{f_1}{t^2}\right], \quad C = -\frac{1}{3}\log\frac{f_1}{t^2}. \quad (10)$$

We see that the internal fourfold has time dependent warp factors although the type IIB six dimensional space is completely time independent. Such a background has the advantage that the four dimensional dynamics that would depend on the internal space will now become time independent.
This case, assumes that the time-dependence has a peculiar form, namely the 6D internal manifold of the IIB theory is assumed constant, and the non-compact directions correspond to a 4D de-Sitter space. Using (10), the corresponding 11D metric in the M-theory picture, can then, in principle, be inserted in the equations of motion that follow from (1). Hence, for the Palumbo-Nardelli model, we have the following connection:

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi\right]=$$

$$=\int_{0}^{\infty}\frac{1}{2\kappa_{10}^{2}}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_{\mu}\Phi\partial^{\mu}\Phi-\frac{1}{2}\left|\tilde{H}_{3}\right|^{2}-\frac{\kappa_{10}^{2}}{g_{10}^{2}}Tr_{v}\left(\left|F_{2}\right|^{2}\right)\right]\Rightarrow$$

$$\Rightarrow\frac{1}{2\kappa^{2}}\int d^{11}x\sqrt{-g}\left[R-\frac{1}{48}G^{2}\right]-\frac{1}{12\kappa^{2}}\int C\wedge G\wedge C \quad (11),$$

where the third term is the bosonic part of the classical eleven-dimensional super-gravity action.

5.3 Solution applied to ten dimensional IIB supergravity (uplifted 10-dimensional solution).

This solution can be oxidized on a three sphere S^3 to give a solution to ten dimensional IIB supergravity. This 10D theory contains a graviton, a scalar field, and the NSNS 3-form among other fields, and has a ten dimensional action given by

$$S_{10}=\int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R-\frac{1}{2}(\partial\phi)^{2}-\frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right]. \quad (12)$$

We have a ten dimensional configuration given by

$$ds_{10}^{2}=\left(\frac{2}{r}\right)^{3/4}\left[-h(r)dt^{2}+r^{2}dx_{0,5}^{2}+\frac{r^{2}}{h(r)}dr^{2}\right]+\left(\frac{r}{2}\right)^{5/4}\left[d\theta^{2}+d\psi^{2}+d\varphi^{2}+\left(d\psi+\cos\theta d\varphi-\frac{Q}{5r^{5}}dt\right)^{2}\right]$$

$$\phi=-\frac{5}{4}\log\frac{r}{2},$$

$$H_{3}=-\frac{Q}{r^{6}}dr\wedge dt\wedge\left(d\psi+\cos\theta d\varphi\right)-\frac{g}{\sqrt{2}}\sin\theta d\theta\wedge d\varphi\wedge d\psi. \quad (13)$$

This uplifted 10-dimensional solution describes NS-5 branes intersecting with fundamental strings in the time direction.
Now we make the manipulation of the angular variables of the three sphere simpler by introducing the following left-invariant 1-forms of SU(2):

$$\sigma_{1}=\cos\psi d\theta+\sin\psi\sin\theta d\varphi,\quad \sigma_{2}=\sin\psi d\theta-\cos\psi\sin\theta d\varphi,\quad \sigma_{3}=d\psi+\cos\theta d\varphi,\quad (14)$$

and

$$h_{3}=\sigma_{3}-\frac{Q}{5}\frac{1}{r^{5}}dt. \quad (15)$$

Next, we perform the following change of variables

$$\frac{r}{2}=\rho^{\frac{4}{5}},\quad t=\frac{5}{32}\tilde{t},\quad dx_{4}=\frac{1}{2\sqrt{2}}d\tilde{x}_{4},\quad dx_{5}=\frac{1}{2}dZ,\quad g=\sqrt{2}\tilde{g},\quad Q=\sqrt{2}2^{7}\tilde{Q},\quad \sigma_{i}=\frac{1}{\tilde{g}}\tilde{\sigma}_{i}. \quad (16)$$

It is straightforward to check that the 10-dimensional solution (13) becomes, after these changes

$$d\tilde{s}_{10}^2 = \frac{1}{2}\rho^{-1}\left[d\tilde{s}_6^2\right] + \frac{\rho}{\tilde{g}^2}\left[\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 + \left(\tilde{\sigma}_3 - \frac{\tilde{g}\tilde{Q}}{4\sqrt{2}}\frac{1}{\rho^4}d\tilde{t}\right)^2\right] + \rho dZ^2,$$

$$\phi = -\ln\rho,$$

$$H_3 = -\frac{1}{\tilde{g}^2}\tilde{\sigma}_1 \wedge \tilde{\sigma}_2 \wedge \tilde{h}_3 + \frac{\tilde{Q}}{\sqrt{2}\tilde{g}\rho^5}d\tilde{t} \wedge d\rho \wedge \tilde{h}_3, \quad (17)$$

where we define

$$d\tilde{s}_6^2 = -\tilde{h}(\rho)d\tilde{t}^2 + \frac{\rho^2}{\tilde{h}(\rho)}d\rho^2 + \rho^2 d\tilde{x}_{0,4}^2 \quad (18)$$

and, after re-scaling M,

$$\tilde{h} = -\frac{2\tilde{M}}{\rho^2} + \frac{\tilde{g}^2}{32}\rho^2 + \frac{\tilde{Q}^2}{8}\frac{1}{\rho^6}. \quad (19)$$

We now transform the solution from the Einstein to the string frame. This leads to

$$d\bar{s}_{10}^2 = \frac{1}{2}\rho^{-2}\left[d\tilde{s}_6^2\right] + \frac{1}{\tilde{g}^2}\left[\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 + \left(\tilde{\sigma}_3 - \frac{\tilde{g}\tilde{Q}}{4\sqrt{2}}\frac{1}{\rho^4}d\tilde{t}\right)^2\right] + dZ^2,$$

$$\bar{\phi} = -2\ln\rho,$$

$$\bar{H}_3 = H_3. \qquad (20)$$

We have a solution to 10-dimensional IIB supergravity with a nontrivial NSNS field. If we perform an S-duality transformation to this solution we again obtain a solution to type-IIB theory but with a nontrivial RR 3-form, F_3. The S-duality transformation acts only on the metric and on the dilaton, leaving invariant the three form. In this way we are led to the following configuration, which is S-dual to the one derived above

$$d\bar{s}_{10}^2 = \frac{1}{2}\left[d\tilde{s}_6^2\right] + \frac{\rho^2}{\tilde{g}^2}\left[\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 + \left(\tilde{\sigma}_3 - \frac{\tilde{g}\tilde{Q}}{4\sqrt{2}}\frac{1}{\rho^4}d\tilde{t}\right)^2\right] + \rho^2 dZ^2,$$

$$\bar{\phi} = 2\ln\rho,$$

$$F_3 = H_3. \qquad (21)$$

With regard the T-duality, in the string frame we have

$$d\bar{s}_{10}^2 = \frac{1}{2}\left[ds_6^2\right] + \frac{r^2}{g^2}\left[\sigma_1^2 + \sigma_2^2 + \left(\sigma_3 - \frac{gQ}{4\sqrt{2}}\frac{1}{r^4}dt\right)^2\right] + r^{-2}dZ^2. \quad (22)$$

This gives a solution to IIA supergravity with excited RR 4-form, C_4. We proceed by performing a T-duality transformation, leading to a solution of IIB theory with nontrivial RR 3-form, C_3. The complete solution then becomes

$$d\bar{s}_{10}^2 = \frac{1}{2}\left[ds_6^2\right] + \frac{r^2}{g^2}\left[\sigma_1^2 + \sigma_2^2 + \left(\sigma_3 - \frac{gQ}{4\sqrt{2}}\frac{1}{r^4}dt\right)^2\right] + r^2 dZ^2 \ ,$$

$$\bar{\phi} = 2\ln r$$

$$C_3 = -\frac{1}{g^2}\sigma_1 \wedge \sigma_2 \wedge h_3 - \frac{Q}{\sqrt{2}g}\frac{1}{r^5}dt \wedge dr \wedge h_3 . \ (23)$$

We are led in this way to precisely the same 10D solution as we found earlier [see formula (21)]. With regard the Palumbo-Nardelli model, we have the following connection:

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$
$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right] \rightarrow$$
$$\rightarrow \int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right]. \ (24)$$

<u>5.4 Connections with some equations concerning the Riemann zeta function.</u>

We have obtained interesting connections between some cosmological solutions of a D3/D7 system, some solutions concerning ten dimensional IIB supergravity and some equations concerning the Riemann zeta function, specifying the Goldston-Montgomery theorem.
In the chapter “Goldbach’s numbers in short intervals” of Languasco’s paper “The Goldbach’s conjecture”, is described the Goldston-Montgomery theorem.

THEOREM 1

Assume the Riemann hypothesis. We have the following implications: (1) If $0 < B_1 \le B_2 \le 1$ and $F(X,T) \approx \frac{1}{2\pi}T\log T$ uniformly for $\frac{X^{B_1}}{\log^3 X} \le T \le X^{B_2}\log^3 X$, then

$$\int_1^X \left(\psi((1+\delta)x) - \psi(x) - \delta(x)^2\right) dx \approx \frac{1}{2}\delta X^2 \log\frac{1}{\delta}, \ (25)$$

uniformly for $\frac{1}{X^{B_2}} \le \delta \le \frac{1}{X^{B_1}}$.

(2) If $1 < A_1 \le A_2 < \infty$ and $\int_1^X (\psi((1+\delta)x) - \psi(x) - \delta x)^2 dx \approx \frac{1}{2}\delta X^2 \log\frac{1}{\delta}$ uniformly for $\frac{1}{X^{1/A_1}\log^3 X} \le T \le \frac{1}{X^{1/A_2}}\log^3 X$, then $F(X,T) \approx \frac{1}{2\pi} T\log T$ uniformly for

$$T^{A_1} \le X \le T^{A_2}.$$

Now, for show this theorem, we must to obtain some preliminary results .

Preliminaries Lemma. (Goldston-Montgomery)

Lemma 1.

We have $f(y) \ge 0 \quad \forall y \in R$ and let $I(Y) = \int_{-\infty}^{+\infty} e^{-2|y|} f(Y+y)dy = 1 + \varepsilon(Y)$. If R(y) is a Riemann-integrable function, we have:

$$\int_a^b R(y) f(Y+y)dy = \left(\int_a^b R(y)dy\right)(1+\varepsilon'(y)).$$

Furthermore, fixed R, $|\varepsilon'(Y)|$ is little if $|\varepsilon(y)|$ is uniformly small for $Y + a - 1 \le y \le Y + b + 1$.

Lemma 2.

Let $f(t) \ge 0$ a continuous function defined on $[0,+\infty)$ such that $f(t) << \log^2(t+2)$.
If

$$J(T) = \int_0^T f(t)dt = (1+\varepsilon(T))T\log T,$$

then

$$\int_0^\infty \left(\frac{\sin ku}{u}\right)^2 f(u)du = \left(\frac{\pi}{2} + \varepsilon'(k)\right) k\log\frac{1}{k},$$

with $|\varepsilon'(k)|$ small for $k \to 0^+$ if $|\varepsilon(T)|$ is uniformly small for

$$\frac{1}{k\log^2 k} \le T \le \frac{1}{k}\log^2 k.$$

Lemma 3.

Let $f(t) \ge 0$ a continuous function defined on $[0,+\infty)$ such that $f(t) << \log^2(t+2)$. If

$$I(k) = \int_0^\infty \left(\frac{\sin ku}{u}\right)^2 f(u)du = \left(\frac{\pi}{2} + \varepsilon'(k)\right) k\log\frac{1}{k}, \quad (26) \qquad \text{then}$$

$$J(T)=\int_0^T f(t)dt=(1+\varepsilon')T\log T \quad , \quad (27)$$

with $|\varepsilon'|$ small if $|\varepsilon(k)|\le\varepsilon$ uniformly for $\frac{1}{T\log T}\le k\le\frac{1}{T}\log^2 T$.

Lemma 4.

Let $F(X,T):=\sum_{0<\gamma,\gamma'<T}\frac{4X^{i}(\gamma-\gamma')}{4+(\gamma-\gamma')^2}$. Then (i) $F(X,T)\ge 0$; (ii) $F(X,T)=F(1/X,T)$; (iii) If

The Riemann hypothesis is preserved, then we have

$$F(X,T)=T\left(\frac{1}{X^2}\log^2 T+\log X\right)\left(\frac{1}{2\pi}+O\left(\sqrt{\frac{\log\log T}{\log T}}\right)\right)$$

uniformly for $1\le X\le T$.

Lemma 5.

Let $\delta\in(0,1]$ and $a(s)=\frac{(1+\delta)^s-1}{s}$. If $c(\gamma)\le 1 \quad \forall y$ we have that

$$\int_{-\infty}^{+\infty}|a(it)|^2\left|\sum_\gamma\frac{c(\gamma)}{1+(t-\gamma)^2}\right|^2 dt=\int_{-\infty}^{+\infty}\left|\sum_{|\gamma|\le Z}a(1/2+i\gamma)\frac{c(\gamma)}{1+(t-\gamma)^2}\right|^2 dt+O\left(\delta^2\log^3\frac{2}{\delta}\right)+O\left(\frac{1}{Z}\log^3 Z\right)$$

for $Z>\frac{1}{\delta}$.

For to show the Theorem 1, there are two parts. We go to prove (1).
We define

$$J(X,T)=4\int_0^T\left|\sum_\gamma\frac{X^{i\gamma}}{1+(t-\gamma)^2}\right|^2 dt.$$

Montgomery has proved that $J(X,T)=2\pi F(X,T)+O(\log^3 T)$ and thence the hypothesis $F(X,T)\approx\frac{1}{2\pi}T\log T$ is equal to $J(X,T)=(1+o(1))T\log T$. Putting $k=\frac{1}{2}\log(1+\delta)$, we have

$$|a(it)|^2=4\left(\frac{\sin kt}{t}\right)^2.$$

For the Lemma 2, we obtain that

$$\int_0^\infty|a(it)|^2\left|\sum_\gamma\frac{X^{i\gamma}}{1+(t-\gamma)^2}\right|^2 dt=\left(\frac{\pi}{2}+o(1)\right)k\log\frac{1}{k}=\left(\frac{\pi}{4}+o(1)\right)\delta\log\frac{1}{\delta}$$

for $$\frac{1}{\delta\log^2\frac{1}{\delta}} \le T \le \frac{3}{\delta}\log^2\frac{1}{\delta}.$$

For the Lemma 5 and the parity of the integrand, we have that

$$\int_{-\infty}^{+\infty}\left|\sum_{|\gamma|\le Z} a(\rho)\frac{X^{i\gamma}}{1+(t-\gamma)^2}\right|^2 dt = \left(\frac{\pi}{2}+o(1)\right)\delta\log\frac{1}{\delta} \qquad \text{(a)}$$

if $Z \ge \frac{1}{\delta}\log^3\frac{1}{\delta}$.

From the $S(t)=\sum_{|\gamma|\le Z} a(\rho)\frac{X^{i\gamma}}{1+(t-\gamma)^2}$ we note that the Fourier's transformed verify that

$$\hat{S}(u) = \pi\sum_{|\gamma|\le Z} a(\rho)X^{i\gamma}e(-\gamma u)e^{-2\pi|u|}.$$

From the Plancherel identity, we have that

$$\int_{-\infty}^{+\infty}\left|\sum_{|\gamma|\le Z} a(\rho)X^{i\gamma}e(-\gamma u)\right|^2 e^{-4\pi|u|}du = \left(\frac{2}{\pi}+o(1)\right)\delta\log\frac{1}{\delta}.$$

For the substitution $Y=\log X$, $-2\pi u = y$ we obtain

$$\int_{-\infty}^{+\infty}\left|\sum_{|\gamma|\le Z} a(\rho)e^{i\gamma(Y+y)}\right|^2 e^{-2|y|}dy = (1+o(1))\delta\log\frac{1}{\delta}. \qquad \text{(b)}$$

Using the Lemma 1 with $R(y)=e^{2y}$ if $0\le y\le \log 2$ and $R(y)=0$ otherwise, and putting $x=e^{Y+y}$ we have that

$$\int_{X}^{2X}\left|\sum_{|\gamma|\le Z} a(\rho)x^{\rho}\right|^2 dx = \left(\frac{3}{2}+o(1)\right)\delta X^2\log\frac{1}{\delta}.$$

Substituting X with $X2^{-j}$, summarizing on j, $1\le j\le K$, and using the explicit formula for $\psi(x)$ with $Z = X\log^3 X$ we obtain

$$\int_{X2^{-K}}^{X}(\psi((1+\delta)x)-\psi(x)-\delta x)^2 dx = \frac{1}{2}\left(1-2^{-2K}+o(1)\right)\delta X^2\log\frac{1}{\delta}.$$

Furthermore, we put $K=[\log\log X]$ and we utilize, for the interval $1\le x\le X2^{-K}$, the estimate of Lemma 4 (placing $X2^{-K}$ for X). Thus, we obtain (1).
Now, we prove (2).

We fix an real number X_1. Making an integration for parts between X_1 and $X_2 = X_1 \log^{2/3} X_1$ we obtain, remembering that for hypothesis we have

$$\int_1^X (\psi((1+\delta)x) - \psi(x) - \delta x)^2 dx \approx \frac{1}{2}\delta X^2 \log\frac{1}{\delta},$$

that $$\int_{X_1}^{X_2} (\psi((1+\delta)x) - \psi(x) - \delta x)^2 x^{-4} dx = \left(\frac{1}{2} + o(1)\right)\delta X_1^{-2} \log\frac{1}{\delta}. \quad \text{(c)}$$

Utilizing the estimate, valid under the Riemann hypothesis

$$\int_1^X (\psi((1+\delta)x) - \psi(x) - \delta x)^2 dx << \delta X^2 \log^2\frac{2}{\delta},$$

we obtain analogously as before that

$$\int_{X_2}^{\infty} (\psi((1+\delta)x) - \psi(x) - \delta x)^2 x^{-4} dx << \delta X_2^{-2} \log^2\frac{1}{\delta} = o\left(\delta X_1^{-2} \log\frac{1}{\delta}\right). \quad \text{(d)}$$

Now, summarizing (c) and (d) and multiplying the sum for X_1^2 we obtain

$$\int_1^{\infty} \min\left(\frac{x^2}{X_1^2}, \frac{X_1^2}{x^2}\right)(\psi((1+\delta)x) - \psi(x) - \delta x)^2 x^{-2} dx = (1 + o(1))\delta \log\frac{1}{\delta}.$$

Putting $X_1 = X$, $Y = \log X$, $x = e^{Y+y}$ and using the explicit formula for $\psi(x)$ with $Z = X \log^3 X$, we obtain the equation (b).

Now, we take the equation (10) and precisely $A = \frac{2}{3}\log\frac{f_1}{t^2}$. We note that from the equation (27) for $\varepsilon' = -\frac{2}{3}$ and T = 2, we have $J(T) = \int_0^T f(t)dt = (1+\varepsilon')T\log T = \frac{2}{3}\log 2$. This result is related to $A = \frac{2}{3}\log\frac{f_1}{t^2}$ putting $\frac{f_1}{t^2} = 2$, hence with the Lemma 3 of Goldston-Montgomery theorem. Then, we have the following interesting relation

$$A = \frac{2}{3}\log\frac{f_1}{t^2} \Rightarrow \int_0^T f(t)dt = (1+\varepsilon')T\log T, \quad (28)$$

hence the connection between the cosmological solution and the equation related to the Riemann zeta function.

Now, we take the equations (13) e (21) and precisely $\phi = -\frac{5}{4}\log\frac{r}{2}$ and $\bar{\phi} = 2\ln\rho$. We note that from the equation (27) for $\varepsilon' = \frac{3}{2}$ and T = 1/2 , we have $J(T) = \int_0^T f(t)dt = (1+\varepsilon')T\log T = \frac{5}{4}\log\frac{1}{2}$.

Furthermore, for $\varepsilon' = 3$ and T = 1/2 , we have $J(T) = \int_0^T f(t)dt = (1+\varepsilon')T\log T = 2\log\frac{1}{2}$.

These results are related to $\phi = -\frac{5}{4}\log\frac{r}{2}$ putting r = 1 and to $\bar{\phi} = 2\ln\rho$ putting $\rho = 1/2$, hence with the Lemma 3 of Goldston-Montgomery theorem. Then, we have the following interesting relations:

$$\phi = -\frac{5}{4}\log\frac{r}{2} \Rightarrow -\int_0^T f(t)dt = -[(1+\varepsilon')T\log T], \quad (29a) \qquad \bar{\phi} = 2\ln\rho \Rightarrow \int_0^T f(t)dt = (1+\varepsilon')T\log T, \Rightarrow$$

$$\Rightarrow \int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right] \quad (29b)$$

hence the connection between the 10-dimensional solutions and some equations related to the Riemann zeta function.
From this the possible connection between cosmological solutions concerning string theory and some mathematical sectors concerning the zeta function, whose the Goldston-Montgomery Theorem and the related Goldbach's Conjecture.

5.5 The P-N Model (Palumbo-Nardelli model) and the Ramanujan identities.

Palumbo (2001) ha proposed a simple model of the birth and of the evolution of the Universe. Palumbo and Nardelli (2005) have compared this model with the theory of the strings, and translated it in terms of the latter obtaining:

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$
$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(|F_2|^2\right)\right], \quad (30)$$

A general relationship that links bosonic and fermionic strings acting in all natural systems.
It is well-known that the series of Fibonacci's numbers exhibits a fractal character, where the forms repeat their similarity starting from the reduction factor $1/\phi$ = 0,618033 = $\frac{\sqrt{5}-1}{2}$ (Peitgen et al. 1986). Such a factor appears also in the famous fractal Ramanujan identity (Hardy 1927):

$$0{,}618033 = 1/\phi = \frac{\sqrt{5}-1}{2} = R(q) + \frac{\sqrt{5}}{1 + \frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}, \qquad (31)$$

and $$\pi = 2\Phi - \frac{3}{20}\left[R(q) + \frac{\sqrt{5}}{1 + \frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right], \qquad (32)$$

where $$\Phi = \frac{\sqrt{5}+1}{2}.$$

Furthermore, we remember that π arises also from the following identity:

$$\pi = \frac{12}{\sqrt{130}}\log\left[\frac{(2+\sqrt{5})(3+\sqrt{13})}{\sqrt{2}}\right], \ (32a) \ \text{and} \ \pi = \frac{24}{\sqrt{142}}\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]. \ (32b)$$

The introduction of (31) and (32) in (30) provides:

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16G}\cdot\frac{1}{2\Phi - \frac{3}{20}\left(R(q) + \frac{\sqrt{5}}{1 + \frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f\left(-t^{1/5}\right)}\frac{dt}{t^{4/5}}\right)}\right)} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) + \right.$$

$$\left. -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \ \right] = \int_0^\infty \frac{R}{\kappa_{11}^2}\cdot 2\Phi - \frac{3}{20}\left[R(q) + \frac{\sqrt{5}}{1 + \frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right].$$

$$\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{11}^2}{\left(2\Phi - \frac{3}{20}\left(R(q) + \frac{\sqrt{5}}{1 + \frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right)\right)2Rg_{10}^2} Tr_\nu\right.$$

$$\left.\left(\left|F_2\right|^2\right)\ \right], \qquad (33)$$

which is the translation of (30) in the terms of the Theory of the Numbers, specifically the possible connection between the Ramanujan identity and the relationship concerning the Palumbo-Nardelli model.

5.6 Interactions between intersecting D-branes.

Let us consider two Dp-branes in type II string theory, intersecting at n angles inside the ten-dimensional space.
The interaction between the branes can be computed from the exchange of massless closed string modes. This can be computed from the one-loop vacuum amplitude for the open strings stretched between the two Dp-branes, that is given by

$$A = 2\int \frac{dt}{2t} Tr e^{-tH} , \quad (34)$$

where H is the open string Hamiltonian. For two Dp-branes making n angles in ten dimensions this amplitudes can be computed to give

$$A = V_p \int_0^\infty \frac{dt}{t} \exp^{-\frac{tY^2}{2\pi^2\alpha'}} \left(8\pi^2\alpha' t\right)^{-\frac{p-3}{2}} \left(-iL\eta(it)^{-3}\left(8\pi\alpha' t\right)^{-1/2}\right)^{4-n} \left(Z_{NS} - Z_R\right), \quad (35)$$

with

$$Z_{NS} = \left(\Theta_3(0|it)\right)^{4-n} \prod_{j=1}^{n} \frac{\Theta_3(i\Delta\theta_j t|it)}{\Theta_1(i\Delta\theta_j t|it)} - \left(\Theta_4(0|it)\right)^{4-n} \prod_{j=1}^{n} \frac{\Theta_4(i\Delta\theta_j t|it)}{\Theta_1(i\Delta\theta_j t|it)},$$

$$Z_R = \left(\Theta_2(0|it)\right)^{4-n} \prod_{j=1}^{n} \frac{\Theta_2(i\Delta\theta_j t|it)}{\Theta_1(i\Delta\theta_j t|it)}, \quad (36)$$

being the contributions coming from the NS and R sectors. Thence, the eq. (35) can be rewritten also

$$A = V_p \int_0^\infty \frac{dt}{t} \exp^{-\frac{tY^2}{2\pi^2\alpha'}} \left(8\pi^2\alpha' t\right)^{-\frac{p-3}{2}} \left(-iL\eta(it)^{-3}\left(8\pi\alpha' t\right)^{-1/2}\right)^{4-n}$$

$$\left[\ \left(\Theta_3(0|it)\right)^{4-n} \prod_{j=1}^{n} \frac{\Theta_3(i\Delta\theta_j t|it)}{\Theta_1(i\Delta\theta_j t|it)} - \left(\Theta_4(0|it)\right)^{4-n} \prod_{j=1}^{n} \frac{\Theta_4(i\Delta\theta_j t|it)}{\Theta_1(i\Delta\theta_j t|it)}\ \right] - \left[\ \left(\Theta_2(0|it)\right)^{4-n} \prod_{j=1}^{n} \frac{\Theta_2(i\Delta\theta_j t|it)}{\Theta_1(i\Delta\theta_j t|it)}\ \right]. \quad (36b)$$

Also in (36) Θ_i are the usual Jacobi functions and η is the Dedekin function. Furthermore, in (35) by Y we mean the distance between both branes, $Y = \sqrt{\Sigma_k Y_k^2}$ where k labels the dimensions in which the branes are separated and Y_k the distance between both branes along the k direction.
Now we take the small t limit of (35), that is, the large distance limit $(Y >> l_s)$. This is the right limit that takes into account the contributions coming from the massless closed strings exchanged between the branes.
Using the well known modular properties of the Θ and η functions we obtain, in the $t \to 0$ limit, that the amplitude is just given by

$$A(Y, \Delta\theta_j) = \frac{V_p L^{4-n} F(\Delta\theta_j)}{2^{p-2}\left(2\pi^2\alpha'\right)^{(p+1-n)/2}} \int t^{-\frac{p+n-5}{2}} \exp^{-\frac{tY^2}{2\pi^2\alpha'}} dt , \quad (37)$$

where the function F contains the dependence on the relative angles between the branes, and is extracted from the small t limit of (36). The exact form of this function is given by

$$F(\Delta\theta_j) = \frac{(4-n) + \Sigma_{j+1}^{n} \cos 2\Delta\theta_j - 4\Pi_{j=1}^{n} \cos\Delta\theta_j}{2\Pi_{j=1}^{n} \sin\Delta\theta_j} . \quad (38)$$

Hence, the eq. (37) can be rewritten also

$$\mathrm{A}(Y,\Delta\theta_j) = \frac{V_p L^{4-n}}{2^{p-2}(2\pi^2\alpha')^{(p+1-n)/2}} \frac{(4-n) + \Sigma_{j+1}^{n} \cos 2\Delta\theta_j - 4\Pi_{j=1}^{n} \cos\Delta\theta_j}{2\Pi_{j=1}^{n} \sin\Delta\theta_j} \int t^{-\frac{p+n-5}{2}} \exp^{-\frac{tY^2}{2\pi^2\alpha'}} dt . \quad (38b)$$

The interaction potential between the branes can then be calculated by performing the integral (37). This integral is just given in terms of the Euler Γ-function, so the potential has the following form

$$V(Y,\Delta\theta_j) = -\frac{V_p L^{4-n} F(\Delta\theta_j)}{2^{p-2}(2\pi^2\alpha')^{p-3}} \Gamma\left(\frac{7-p-n}{2}\right) Y^{(p+n-7)} . \quad (39)$$

Note that for $p+n=7$ this expression is not valid as $\Gamma(0)$ is not a well defined function. In fact in that case the integral (37) is divergent, so we need to introduce a lower cutoff to perform it. If we denote by Λ_c the cutoff, the integral becomes

$$V(Y,\Delta\theta_j) = \frac{V_p L^{p-3} F(\Delta\theta_j)}{(4\pi^2\alpha')^{p-3}} \ln\frac{Y}{\Lambda_c} . \quad (40)$$

When dealing with compact spaces the expression (37) is modified in the following way

$$\mathrm{A}(Y,\Delta\theta_j) = \frac{V_p L^{4-n} F(\Delta\theta_j)}{2^{p-2}(2\pi^2\alpha')^{(p+1-n)/2}} \sum_{\omega_k \in Z} \int_0^\infty t^{-\frac{p+n-5}{2}} \exp^{-\frac{t\Sigma_k (Y_k + 2\pi\omega_k R)^2}{2\pi^2\alpha'}} dt , \quad (41)$$

where ω_k represents the winding modes of the strings on the directions transverse to the branes. That means that the summation over k in (41) has only one term in the D6-brane case and it will be $Y_9 = \left|x_9^{(1)} - x_9^{(2)}\right|$. In the D5-brane case we will have two terms: $Y_8 = \left|x_8^{(1)} - x_8^{(2)}\right|$ and $Y_9 = \left|x_9^{(1)} - x_9^{(2)}\right|$. Also in both cases we will denote $Y = \sqrt{\Sigma_k Y_k^2}$. Nevertheless, if the distance between the branes is small compared with the compactification radii $(Y << (2\pi R))$, the winding modes would be too massive and then will not contribute to the low energy regime. That is, it will cost a lot of energy to the strings to wind around the compact space. If we translate this assumption to (41), the dominant mode will be the zero mode, and the potential can be written as in (39), (40), taking into account that we focus on the case where the number of angles is $n = 2$. In this case the potential, when normalised over the non-compact directions, for branes of different dimensions is just given by

$$V_{Dp}(Y,\Delta\theta_j) = -\frac{(2\pi R)^{(p-5)} F(\Delta\theta_j)}{2^{p-2}(2\pi^2\alpha')^{p-3}} \Gamma\left(\frac{5-p}{2}\right) Y^{(p-5)}, \quad (42) \qquad V_{D5}(Y,\Delta\theta_j) = \frac{F(\Delta\theta_j)}{(4\pi^2\alpha')^2} \ln\frac{Y}{\Lambda_c}, \quad (43)$$

where the $(2\pi R)^{p-5}$ factor arises from the dimensions in which the branes become parallel on the compact dimensions. Furthermore, remember that R denotes the radius of the torus.
Now we note that the eq. (37) can be rewritten substituting to π the corresponding Ramanujan's identity (32). Hence, we obtain

$$A(Y,\Delta\theta_j)=\frac{V_p L^{4-n}F(\Delta\theta_j)}{2^{p-2}(2\alpha')^{(p+1-n)/2}\left\{2\Phi-\frac{3}{20}\left[R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q\frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\right\}}\int t^{-\frac{p+n-5}{2}}\exp^{-\frac{tY^2}{2\pi^2\alpha'}}dt$$

(43a)

With regard the eq. (40), we note that can be related with the expression (29b) concerning the lemma 3 of Goldston-Montgomery Theorem and with the Palumbo-Nardelli Model. Hence, we can write the following interesting connections:

$$\frac{V_p L^{p-3}F(\Delta\theta_j)}{(4\pi^2\alpha')^{p-3}}\ln\frac{Y}{\Lambda_c}=2\ln\rho\Rightarrow\int_0^T f(t)dt=(1+\varepsilon')T\log T\,,\Rightarrow$$
$$\Rightarrow\int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R-\frac{1}{2}(\partial\phi)^2-\frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right]\Rightarrow$$

$$\Rightarrow\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right]=$$
$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right].\qquad(43b)$$

5.7 General action and equations of motion for a probe D3-brane moving through a type IIB supergravity background.

Now we will show the general action and equations of motion for a probe D3-brane moving through a type IIB supergravity background describing a configuration of branes and fluxes.
We start by specifying the ansatz for the background fields that we consider, and the form of the brane action. We are interested in compactifications of type IIB theory, in which the metric takes the following general form (in the Einstein frame)

$$ds^2=h^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu+h^{1/2}g_{mn}dy^m dy^n\,.\qquad(44)$$

We now embed a probe D3-brane in this background, with its four infinite dimensions parallel to the four large dimensions of the background solution. The motion of such a brane is described by the sum of the Dirac-Born-Infeld (DBI) action and the Wess-Zumino (WZ) action. The DBI action is given, in the string frame, by

$$S_{DBI} = -T_3 g_s^{-1} \int d^4\xi e^{-\phi} \sqrt{-\det(\gamma_{ab} + F_{ab})}, \quad (45)$$

where $F_{ab} = B_{ab} + 2\pi\alpha' f_{ab}$, with B_2 the pullback of the 2-form field to the brane and f_2 the world-volume gauge field. $\gamma_{ab} = g_{MN}\partial_a x^M \partial_b x^N$, is the pullback of the ten-dimensional metric g_{MN} in the string frame. Finally $\alpha' = \ell_s^2$ is the string scale and ξ^a are the brane world-volume coordinates.
The WZ part is given by

$$S_{WZ} = qT_3 \int_W C_4, \quad (46)$$

where W is the world-volume of the brane and $q = 1$ for a probe D3-brane and $q = -1$ for a probe anti-brane. We are interested in exploring the effect of angular momentum on the motion of the brane, and therefore assume that there are no gauge fields living in the world-volume of the probe brane, $f_{ab} = 0$. For convenience we take the static gauge, that is, we use the non-compact coordinates as our brane coordinates: $\xi^a = x^{\mu=a}$. Since, in addition, we are interested in cosmological solutions for branes, we consider the case where the perpendicular positions of the brane, y^m, depend only on time. Thus

$$\gamma_{00} = g_{00} + g_{mn}\dot{y}^m \dot{y}^n h^{1/2} = -h^{-1/2}\left(1 - hv^2\right) \quad (47)$$

and $B_{ab} = 0$. Hence

$$S_{DBI} = -T_3 g_s^{-1} \int d^4x e^{-3\phi} \sqrt{1 - hv^2}, \quad (48)$$

in the Einstein frame. Thence, summing the DBI and WZ actions, we have the total action for the probe brane

$$S = -T_3 g_s^{-1} \int d^4x h^{-1} \left[e^{-3\phi}\sqrt{1 - hv^2} - q\right]. \quad (49)$$

This action is valid for arbitrarily high velocities. Furthermore, this equation correspond to the Born-Infeld action for the D-brane embedded in the 10-dimensional space of type IIB theory.
The functions appearing in the following equations

$$h(\eta) = \frac{27\pi\alpha'^2}{4\eta^4}\left[g_s N + \frac{3(g_s M)^2}{2\pi}\left(\ln\frac{\eta}{\tilde{\eta}} + \frac{1}{4}\right)\right] = \frac{c}{\eta^4}(1 + b\ln\eta), \quad (50)$$

$$N_{eff} = N + \frac{3(g_s M)^2}{2\pi}\ln\frac{\eta}{\eta_0}, \quad (51)$$

are the solutions of the equations of motion for the IIB theory in 10-dimensions, defining the background. Thence, putting eqs. (50) and (51) in (49), we can determine the trajectory of the brane in ten dimensions.
Here, $\eta = \tilde{\eta}$ determines the UV scale at which the KT throat joins to the Calabi-Yau space. This solution has a naked singularity at the point where $h(\eta_0) = 0$, located at $\eta_0 = \tilde{\eta}e^{-1/b}$. In this configuration, the supergravity approximation is valid when $g_s M, g_s N >> 1$: in this limit the curvatures are small, and we keep $g_s < 1$.
We note that also the eqs. (50) and (51), can be related with the expression (29b) and with the relationship concerning the Palumbo-Nardelli Model. Hence, we obtain the following connections:

$$h(\eta) - \frac{c}{\eta^4} = \frac{c}{\eta^4} b \ln \eta \Rightarrow 2 \ln \rho \Rightarrow \int_0^T f(t) dt = (1+\varepsilon')T \log T \,, \Rightarrow$$

$$\Rightarrow \int d^{10} x \sqrt{|g|} \left[\frac{1}{4} R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{12} e^{-2\phi} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right] \Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{1}{2\kappa_{10}^2} \int d^{10} x (-G)^{1/2} e^{-2\Phi} \left[R + 4 \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} \left| \tilde{H}_3 \right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_\nu \left(|F_2|^2 \right) \right] =$$

$$- \int d^{26} x \sqrt{g} \left[- \frac{R}{16\pi G} - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} Tr(G_{\mu\nu} G_{\rho\sigma}) f(\phi) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right], \quad (51a)$$

$$\frac{2\pi N + 3(g_s M)^2}{2\pi} \ln \frac{\eta}{\eta_0} \Rightarrow 2 \ln \rho \Rightarrow \int_0^T f(t) dt = (1+\varepsilon')T \log T \,, \Rightarrow$$

$$\Rightarrow \int d^{10} x \sqrt{|g|} \left[\frac{1}{4} R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{12} e^{-2\phi} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right] \Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{1}{2\kappa_{10}^2} \int d^{10} x (-G)^{1/2} e^{-2\Phi} \left[R + 4 \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} \left| \tilde{H}_3 \right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_\nu \left(|F_2|^2 \right) \right] =$$

$$- \int d^{26} x \sqrt{g} \left[- \frac{R}{16\pi G} - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} Tr(G_{\mu\nu} G_{\rho\sigma}) f(\phi) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right]. \quad (51b)$$

Furthermore, the eq. (49) is also related with the relationship concerning the Palumbo-Nardelli Model applied to the D-branes. Hence, we have:

$$- \mu_{25} \int d^{26} \xi Tr \left\{ e^{-\Phi} \left[- \det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab}) \right]^{1/2} \right\} = \int_0^\infty - \frac{1}{(2\pi\alpha')^2 g_{YM}^2} \int d^{10} x Tr \left\{ \left[- \det(\eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}) \right]^{1/2} \right\}$$

$$\Rightarrow - \int d^{26} x \sqrt{g} \left[- \frac{R}{16\pi G} - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} Tr(G_{\mu\nu} G_{\rho\sigma}) f(\phi) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] = \int_0^\infty \frac{1}{2\kappa_{10}^2} \int d^{10} x (-G)^{1/2} e^{-2\Phi}$$

$$\left[R + 4 \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} \left| \tilde{H}_3 \right|^2 - \frac{k_{10}^2}{g_{10}^2} Tr_\nu \left(|F_2|^2 \right) \right] \Rightarrow - T_3 g_s^{-1} \int d^4 x h^{-1} \left[e^{-3\phi} \sqrt{1 - h v^2} - q \right]. \quad (52)$$

Chapter 6

Connections.

Now we take the eq. (20) of chapter 1. We note that can be related with the Godston-Montgomery equation, hence we have the following connection:

$$\delta_k(u) := \delta_{k,\mathrm{B}}(u) = \left(\frac{1}{\lambda'_{\hat{E}_{\mathrm{B}}}(\mathrm{T})} \frac{d}{d\mathrm{T}} \right)^k \log f_{u,\mathrm{B}}(\mathrm{T})\Big|_{\mathrm{T}=0} \Rightarrow 2\ln\rho \Rightarrow \int_0^T f(t)dt = (1+\varepsilon')T\log T \ ,\Rightarrow$$

$$\Rightarrow \int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right] \Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right] =$$

$$- \int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]. \quad (1)$$

Now we take the eq. (29) of chapter 1. We note that can be related with the equation regarding the Palumbo-Nardelli model and with the Ramanujan's identity concerning π. Hence, we have the following connections:

$$\left\langle f_\varphi, f_\varphi \right\rangle = \frac{1}{16\pi^3}N^2\left\{\prod_{\substack{q|N \\ q\notin S_\varphi}}\left(1-\frac{1}{q}\right)\right\}L_N\left(2,\varphi^2\bar{\hat{\chi}}\right)L_N(1,\psi) \Rightarrow$$

$$\Rightarrow \int_0^\infty \pi^2\left\langle f_\varphi, f_\varphi \right\rangle \cdot \frac{1}{N^2\left\{\prod_{q|N}\left(1-\frac{1}{q}\right)\right\}L_N\left(2,\varphi^2\bar{\hat{\chi}}\right)L_N(1,\psi)G_N}\int d^{10}x(-G)^{1/2}e^{-2\Phi} \cdot$$

$$\cdot \left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(\left|F_2\right|^2\right)\right] =$$

$$= -\int d^{26}x\sqrt{g}\left[\left(-R\cdot\pi^2\left\langle f_\varphi, f_\varphi \right\rangle \cdot \frac{1}{N^2\left\{\prod_{q|N}\left(1-\frac{1}{q}\right)\right\}L_N\left(2,\varphi^2\bar{\hat{\chi}}\right)L_N(1,\psi)G}\right) - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) + \right.$$

$$\left. - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right], \quad (2)$$

$$\left\langle f_\varphi, f_\varphi \right\rangle = \frac{1}{16\pi^3} N^2 \left\{ \prod_{\substack{q|N \\ q \notin S_\varphi}} \left(1 - \frac{1}{q}\right) \right\} L_N\left(2, \varphi^2 \overline{\hat{\chi}}\right) L_N(1, \psi) \Rightarrow$$

$$\Rightarrow \frac{1}{16\left\{2\Phi - \frac{3}{20}\left[R(q) + \frac{\sqrt{5}}{1 + \frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f\left(-t^{1/5}\right)}\frac{dt}{t^{4/5}}\right)}\right]\right\}^3} N^2 \left\{ \prod_{\substack{q|N \\ q \notin S_\varphi}} \left(1 - \frac{1}{q}\right) \right\} L_N\left(2, \varphi^2 \overline{\hat{\chi}}\right) L_N(1, \psi). \tag{3}$$

Now we take the eqs. (8) and (9) and (11) of the chapter 2. We note that can be related with the Ramanujan's modular equation (32b) and the Ramanujan's identity concerning π (32). Thence, we have the following connection:

$$\Delta = \sum_{n \ge 1} \tau(n) q^n = q \prod_{n \ge 1} \left(1 - q^n\right)^{24} \Rightarrow \frac{\pi\sqrt{142}}{\ln\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]} \Rightarrow$$

$$\Rightarrow 2\Phi - \frac{3}{20}\left[R(q) + \frac{\sqrt{5}}{1 + \frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right] \cdot \frac{\sqrt{142}}{\ln\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \tag{4}$$

Also for the eqs. (11) and (37), we obtain of the similar connections:

$$L_\Delta(s) := \sum_{n \ge 1} \tau(n) n^{-s} = \prod_p \left(1 - \tau(p) p^{-s} + p^{11} p^{-2s}\right)^{-1} \Rightarrow$$

$$\Rightarrow 2\Phi - \frac{3}{20}\left[R(q) + \frac{\sqrt{5}}{1 + \frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right] \cdot \frac{\sqrt{142}}{\ln\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}, \tag{5}$$

$$\varphi(s)=\prod_p \frac{1}{1-\tau(p)p^{-s}+p^{11-2s}} \Rightarrow$$

$$\Rightarrow 2\Phi-\frac{3}{20}\left[R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\cdot\frac{\sqrt{142}}{\ln\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (6)$$

Also with regard the eqs. (101) and (106) of **Chapter 2**, we note that can be related with the Ramanujan's identity concerning π. Thence, we have the following connections:

$$\sum_{k\in N(n/r-a)}\left(1-(k+a)\frac{r}{n}\right)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{r^s}\zeta(s,a)\frac{n^s}{s(s+1)}ds \Rightarrow$$

$$\Rightarrow \frac{1}{2\left\{2\Phi-\frac{3}{20}\left[R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\right\}i}\int_{c-i\infty}^{c+i\infty}\frac{1}{r^s}\zeta(s,a)\frac{n^s}{s(s+1)}ds, \quad (7)$$

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\zeta(s,a)\frac{n^s}{4^s s(s+1)}ds=\varepsilon(n,a)-\frac{1}{8}n-\frac{1}{2}+a \Rightarrow$$

$$\Rightarrow \frac{1}{2\left\{2\Phi-\frac{3}{20}\left[R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\right\}i}\int_{c-i\infty}^{c+i\infty}\frac{1}{r^s}\zeta(s,a)\frac{n^s}{4^s s(s+1)}ds=$$

$$=\varepsilon(n,a)-\frac{1}{8}n-\frac{1}{2}+a. \quad (8)$$

Now we take the eqs. (79), (82), (83), (98) and (105) of **Chapter 3**. We note that can be related with the Goldston-Montgomery equation (29b) and with the Palumbo-Nardelli relationship (30) of chapter 5. Hence, we obtain the following connections:

$$A=-\int_{R^*}\alpha\left(\pi^{-1}y\right)(\log q)dy=-\log q\left(\int_R \alpha\left(\pi^{-1}y\right)dy-\int_P dy\right)=\frac{1}{q}\log q \Rightarrow$$

$$\Rightarrow 2\ln\rho \Rightarrow \int_0^T f(t)dt=(1+\varepsilon')T\log T \Rightarrow$$

$$\Rightarrow \int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R-\frac{1}{2}(\partial\phi)^2-\frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right]\Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right]=$$

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right], \quad (9)$$

$$Pfw\int_{R^*}f_0^3(|u|)\frac{|u|^{1/2}}{|1-u|}d^*u=\log\pi+\gamma\Rightarrow 2\ln\rho\Rightarrow\int_0^T f(t)dt=(1+\varepsilon')T\log T\ ,\Rightarrow$$

$$\Rightarrow \int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R-\frac{1}{2}(\partial\phi)^2-\frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right]\Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right]=$$

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right], \quad (10)$$

$$PF_0\int_{R_+^*}f_0^4\times\left(1-f_0^4\right)^{-1}d^*u=\left[\log(2\pi)+\lim_{t\to\infty}\left(\int_{R_+^*}\left(1-f_0^{2t}\right)f_0^4\left(1-f_0^4\right)^{-1}d^*u-\log t\right)\right]=\log 2\pi+\gamma-\log 2$$

$$\Rightarrow 2\ln\rho\Rightarrow\int_0^T f(t)dt=(1+\varepsilon')T\log T\ ,\Rightarrow$$

$$\Rightarrow \int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R-\frac{1}{2}(\partial\phi)^2-\frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right]\Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right]=$$

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right], \quad (11)$$

$$Pfw\int f_2\left(|u|_C\right)\frac{1}{|1-u|_C}d^*u = PF_0\int f_0 f_1^{-1} d^*\nu = 2(\log 2\pi + \gamma) \Rightarrow$$

$$\Rightarrow 2\ln\rho \Rightarrow \int_0^T f(t)dt = (1+\varepsilon')T\log T \;, \Rightarrow$$

$$\Rightarrow \int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right] \Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(|F_2|^2\right)\right] =$$

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right], \quad (12)$$

$$4\int_{-\infty}^{\log 2} Arc\sin\left(e^x/2\right)dx = -4\int_0^{\pi/2}\log(\sin u)du = 2\pi\log 2 \Rightarrow$$

$$\Rightarrow 2\ln\rho \Rightarrow \int_0^T f(t)dt = (1+\varepsilon')T\log T \;, \Rightarrow$$

$$\Rightarrow \int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right] \Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(|F_2|^2\right)\right] =$$

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]. \quad (13)$$

Now, we take the eqs. (15), (22), (25) and (27) of **Chapter 4**. We note that can be related with the Palumbo-Nardelli relationship. Thence, we have the following connections:

$$A(\zeta_1 k_1,\ldots,\zeta_n k_n) = g^{n-2}\int_{(Q_p)^{n-3}}\theta_{[0,y_{n-1},\ldots,y_{3,1}]}(y)F(\zeta,k,y) \cdot \prod_{3\le i<j\le n-1}|y_i - y_j|_p^{k_i k_j}\prod_{3\le i\le n-1}\left(|y_i|_p^{k_n k_i}|1-y_i|^{k_2 k_i}dy_i\right) \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(|F_2|^2\right)\right], \quad (14)$$

$$A_p(k_1,\ldots,k_4) = g_p^2\int DX\chi_p\left(-\frac{1}{h}S_0[X]\right)\times\prod_{j=1}^4\int d^2\sigma_j\chi_p\left(-\frac{1}{h}k_\mu^{(j)}X^\mu\left(\sigma_j,\tau_j\right)\right) \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(|F_2|^2\right)\right], \quad (15)$$

$$A_A(k_1,...,k_4)=g_\infty^2\int_R |x|_\infty^{k_1k_2}|1-x|_\infty^{k_2k_3}dx\times\prod_{p\in S}g_p^2\prod_{j=1}^4\int d^2\sigma_j\times\prod_{p\notin S}g_p^2\Rightarrow$$

$$\Rightarrow-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$

$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right],\quad(16)$$

$$S=\int d^dxL=\frac{1}{g^2}\frac{p^2}{p-1}\int d^dx\left[-\frac{1}{2}\phi p^{-\frac{1}{2}\square}\phi+\frac{1}{p+1}\phi^{p+1}\right]\Rightarrow$$

$$\Rightarrow\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right]=$$

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right].\quad(17)$$

While, if we take the eqs. (18), (33b), (38), (43) and (46) of **Chapter 4**, we note that can be related with the Ramanujan's identity concerning π and with Palumbo-Nardelli model. Then, we obtain the following connections:

$$A_\infty(k_1,...,k_4)=g_\infty^2\int DX\exp\left(\frac{2\pi i}{h}S_0[X]\right)\times\prod_{j=1}^4\int d^2\sigma_j\exp\left[\frac{2\pi i}{h}k_\mu^{(j)}X^\mu(\sigma_j,\tau_j)\right]\Rightarrow$$

$$\Rightarrow-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$

$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right]\Rightarrow$$

$$\Rightarrow g_\infty^2\int DX\exp\left\{2\left\{2\Phi-\frac{3}{20}\left[R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q\frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\right\}i\frac{1}{h}S_0X\right\}\times$$

$$\times\prod_{j=1}^4\int d^2\sigma_j\exp\left\{2\left\{2\Phi-\frac{3}{20}\left[R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q\frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\right\}\frac{i}{h}k_\mu^{(j)}X^\mu(\sigma_j,\tau_j)\right\},$$

(18)

$$T_q=-\int d^{d-q-1}x_\perp L\left(F^{(d-q-1)}(x_\perp)\right)=\frac{1}{2\left\{g\left[\frac{p^2-1}{2\pi p^{2p/(2p-1)}\ln p}\right]^{(d-q-1)/4}\right\}^2}\frac{p^2}{p+1}\Rightarrow$$

$$\Rightarrow-\int d^{d-q-1}x_\perp L\left(F^{(d-q-1)}(x_\perp)\right)=$$

$$= \frac{1}{2\left\{ g\left[\frac{p^2-1}{2\left(2\Phi - \frac{3}{20}\left(R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}})}\right)\right)p^{2p/(2p-1)}\ln p} \right]^{(d-q-1)/4} \right\}^2} \frac{p^2}{p+1}$$

$\Rightarrow$

$$-\mu_{25}\int d^{26}\xi Tr\left\{e^{-\Phi}\left[-\det\left(G_{ab}+B_{ab}+2\pi\alpha' F_{ab}\right)\right]^{1/2}\right\} = \int_0^\infty -\frac{1}{(2\pi\alpha')^2 g_{YM}^2}\int d^{10}xTr\left\{\left[-\det\left(\eta_{\mu\nu}+2\pi\alpha' F_{\mu\nu}\right)\right]^{1/2}\right\}$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] = \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\cdot$$

$$\cdot\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{k_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right], \quad (19)$$

$$\frac{1}{g^2}\frac{p^2}{p-1}\left(\frac{2\pi}{\sqrt{2}}\right)^{d-q-1}\int d^{q+1}z\left[-\frac{1}{2}\tilde{\psi}p^{-\frac{1}{2}\Box_z}\tilde{\psi}+\frac{1}{p+1}\tilde{\psi}^{p+1}-C\left\{\frac{1}{2}\tilde{\xi}^i p^{-\frac{1}{2}\Box_z}\tilde{\xi}^i-\frac{1}{2}\tilde{\psi}^{p-1}\tilde{\xi}^i\tilde{\xi}^i\right\}+O\left(\tilde{\xi}^3\right)+...\right]\Rightarrow$$

$$\Rightarrow \frac{1}{g^2}\frac{p^2}{p-1}\left\{\frac{2\left\{2\Phi-\frac{3}{20}\left[R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q\frac{f^5(-t)}{f\left(-t^{1/5}\right)}\frac{dt}{t^{4/5}}\right)}\right]\right\}}{\sqrt{2}}\right\}^{d-q-1}.$$

$$\cdot\int d^{q+1}z\left[-\frac{1}{2}\tilde{\psi}p^{-\frac{1}{2}\Box_z}\tilde{\psi}+\frac{1}{p+1}\tilde{\psi}^{p+1}-C\left\{\frac{1}{2}\tilde{\xi}^i p^{-\frac{1}{2}\Box_z}\tilde{\xi}^i-\frac{1}{2}\tilde{\psi}^{p-1}\tilde{\xi}^i\tilde{\xi}^i\right\}+O\left(\tilde{\xi}^3\right)+...\right]\Rightarrow$$

$$\Rightarrow -\mu_{25}\int d^{26}\xi Tr\left\{e^{-\Phi}\left[-\det\left(G_{ab}+B_{ab}+2\pi\alpha' F_{ab}\right)\right]^{1/2}\right\} = \int_0^\infty -\frac{1}{(2\pi\alpha')^2 g_{YM}^2}\int d^{10}xTr\left\{\left[-\det\left(\eta_{\mu\nu}+2\pi\alpha' F_{\mu\nu}\right)\right]^{1/2}\right\}$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] = \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\cdot$$

$$\cdot\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{k_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right], \quad (20)$$

$$\zeta\,(\Box/2)\,\phi = \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} \Rightarrow$$

$$\Rightarrow \frac{1}{\left\{2\left\{2\Phi - \frac{3}{20}\left[R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\right\}\right\}^D} \int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi}$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right], \quad (21)$$

$$\zeta\,(\Box/4)\,\theta = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk = \sum_{n\geq 1}\left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right] \Rightarrow$$

$$\Rightarrow \frac{1}{\left\{2\left\{2\Phi - \frac{3}{20}\left[R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\right\}\right\}^D}$$

$$\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk = \sum_{n\geq 1}\left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right] \Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right] =$$

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]. \quad (22)$$

Furthermore, we can see easily that the equations described in the **Chapter 5** and **6** can be connected also among them.

Conclusion

Hence, in conclusion, also for some mathematical sectors concerning the Fermat's Last Theorem, can be obtained interesting and new connections with other sectors of Number Theory and String Theory, principally the p-adic and adelic numbers, the Ramanujan's modular equations, some formulae related to the Riemann zeta functions and p-adic and adelic strings.

Furthermore, also the fundamental relationship concerning the Palumbo-Nardelli model, a general relationship that links bosonic string action and superstring action (i.e. bosonic and fermionic strings acting in all natural systems), can be related with some equations regarding the p-adic (adelic) string sector.

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