On the link between the structure of A-branes observed in the homological mirror symmetry and the classical theory of automorphic forms: mathematical connections with the modular elliptic curves, p-adic and adelic numbers and p-adic and adelic strings.

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Abstract

This paper is a review of some interesting results that has been obtained in the study of the categories of A-branes on the dual Hitchin fibers and some interesting phenomena associated with the endoscopy in the geometric Langlands correspondence of various authoritative theoretical physicists and mathematicians.
The geometric Langlands correspondence has been interpreted as the mirror symmetry of the Hitchin fibrations for two dual reductive groups. This mirror symmetry reduces to T-duality on the generic Hitchin fibers. Also from this work we've showed that can be obtained interesting and new mathematical connections with some sectors of Number Theory and String Theory, principally with the modular elliptic curves, p-adic and adelic numbers and p-adic and adelic strings.
In the **Section 1**, we have described some equations regarding the Galois group and Abelian class field theory, automorphic representations of $GL_2(A_Q)$ and modular forms, adèles and vector bundles. In the **Section 2**, we have showed some equations regarding the moduli spaces of SL_2 and SO_3 Higgs bundles on an elliptic curve with tame ramification at one point. In the **Section 3**, we have showed some equations regarding the action of the Wilson and 't Hooft/Hecke operators on the electric and magnetic branes relevant to geometric endoscopy. In the **Section 4**, we have described the Hecke eigensheaves and the notion of fractional Hecke eigensheaves. In the **Section 5**, we have described some equations concerning the local and global Langlands correspondence. In the **Section 6**, we have described some equations regarding the automorphic functions associated to the fractional Hecke eigensheaves. In the **Section 7**, we have showed some equations concerning the modular elliptic curves belonging at the proof of Fermat's Last Theorem. In the **Section 8**, we have showed some equations concerning the p-adic and adelic numbers and the p-adic and adelic strings. In the **Section 9**, we have described the P-N Model (Palumbo-Nardelli model) and the Ramanujan identities, solution applied to ten dimensional IIB supergravity (uplifted 10-dimensional solution) and connections with some equations concerning the Riemann zeta function.

In conclusion, in the **Section 10**, we have described the possible mathematical connections obtained between some equations regarding the various sections.

1. Galois group and Abelian class field theory, automorphic representations of $GL_2(A_Q)$ and modular forms, adèles and vector bundles. [1]

With regard the entire group $Gal(\overline{F}/F)$, it has been known for some time what is the maximal abelian quotient of $Gal(\overline{F}/F)$. This quotient is naturally identified with the Galois group of the maximal abelian extension F^{ab} of F. By definition, F^{ab} is the largest of all subfields of $\overline{F}$ whose Galois group is abelian. For $F = Q$, the classical Kronecker-Weber theorem says that the maximal abelian extension Q^{ab} is obtained by adjoining to Q all roots of unity. In other words, Q^{ab} is the union of all cyclotomic fields $Q(\zeta_N)$. Therefore we obtain that the Galois group $Gal(Q^{ab}/Q)$ is isomorphic to the inverse limit of the groups $(Z/NZ)^{\times}$ with respect to the system of surjections $p_{N,M} : (Z/NZ)^{\times} \to (Z/MZ)^{\times}$ for M dividing N:

$$Gal(Q^{ab}/Q) \cong \lim_{\leftarrow}(Z/NZ)^{\times}. \quad (1.1)$$

By definition, an element of this inverse limit is a collection (x_N), $N > 1$, of elements of $(Z/NZ)^{\times}$ such that $p_{N,M}(x_N) = x_M$ for all pairs N, M such that M divides N. This inverse limit may be described more concretely using the notion of **p-adic numbers**. Recall that if p is a prime, the a **p-adic number** is an infinite series of the form

$$a_k p^k + a_{k+1} p^{k+1} + a_{k+2} p^{k+2} + \ldots, \quad (1.2)$$

where each a_k is an integer between 0 and $p-1$, and we choose $k \in Z$ in such a way that $a_k \neq 0$. One defines addition and multiplication of such expressions by "carrying" the result of powerwise addition and multiplication to the next power. One checks that with respect to these operations **the p-adic numbers form a field denoted by Q_p. It contains the subring Z_p of p-adic integers** which consists of the above expressions with $k \geq 0$. It is clear that Q_p is the field of fractions of Z_p. Note that the subring of Z_p consisting of all finite series of the form (1.2) with $k \geq 0$ is just the ring of integers Z. The resulting embedding $Z \mapsto Z_p$ gives rise to the embedding $Q \mapsto Q_p$. It is important to observe that Q_p is in fact a completion of Q. To see that, define a norm $|\cdot|_p$ on Q by the formula $|p^k a/b|_p = p^{-k}$, where a, b are integers relatively prime to p. With respect to this norm p^k becomes smaller and smaller as $k \to +\infty$. That is why the completion of Q with respect to this norm is the set of all infinite series of the form (1.2), going in the wrong direction. This is precisely the field Q_p. This norm extends uniquely to Q_p, with the norm of the p-adic number (1.2) being equal to p^{-k}. In fact, according to Ostrowski's theorem, any completion of Q is isomorphic to either Q_p or to the field R of real numbers.

Now observe that if $N = \Pi_p p^{m_p}$ is the prime factorization of N, then $Z/NZ \cong \Pi_p Z/p^{m_p}Z$. Let $\hat{Z}$ be the inverse limit of the rings Z/NZ with respect to the natural surjections $Z/NZ \to Z/MZ$ for M dividing N:

$$\hat{Z} = \lim_{\leftarrow} Z/NZ \cong \prod_p Z_p . \quad (1.3)$$

It follows that

$$\hat{Z} \cong \prod_p \left(\lim_{\leftarrow} Z/p^r Z\right), \text{ hence } \quad \hat{Z} = \lim_{\leftarrow} Z/NZ \cong \prod_p Z_p \cong \prod_p \left(\lim_{\leftarrow} Z/p^r Z\right),$$

where the inverse limit in the brackets is taken with respect to the natural surjective homomorphism $Z/p^r Z \to Z/p^s Z$, $r > s$. So we find that

$$\hat{Z} \cong \prod_p Z_p . \quad (1.4)$$

Note that $\hat{Z}$ defined above is actually a ring. The Kronecker-Weber theorem (1.1) implies that $Gal(Q^{ab}/Q)$ is isomorphic to the multiplicative group $\hat{Z}^\times$ of invertible elements of the ring $\hat{Z}$. But we find from (1.4) that $\hat{Z}^\times$ is nothing but **the direct product of the multiplicative group $Z_p^\times$ of the rings of p-adic integers where p runs over the set of all primes**. We thus conclude that

$$Gal(Q^{ab}/Q) \cong \hat{Z}^\times \cong \prod_p Z_p^\times . \quad (1.5)$$

The *abelian class field theory* (ACFT) describes its Galois group $Gal(F^{ab}/F)$, which is the maximal abelian quotient of $Gal(\overline{F}/F)$. It states that $Gal(F^{ab}/F)$ is isomorphic to the group of connected components of the quotient $F^\times \backslash A_F^\times$. **Here $A_F^\times$ is the multiplicative group of invertible elements in the ring A_F of adèles of F, which is a subring in the direct product of all completions of F. We define the adèles first in the case when $F = Q$. In this case, the completions of Q are the fields Q_p of p-adic numbers, where p runs over the set of all primes p, and the field R of real numbers. Hence the ring A_Q is a subring of the direct product of the fields Q_p**. More precisely, elements of A_Q are the collections $\left((f_p)_{p\in P}, f_\infty\right)$, where $f_p \in Q_p$ and $f_\infty \in R$, satisfying the condition that $f_p \in Z_p$ for all but finitely many p's. It follows from the definition that

$$A_Q \cong (\hat{Z} \otimes_Z Q) \times R .$$

We give the ring $\hat{Z}$ defined by (1.3) the topology of direct product, Q the discrete topology and R its usual topology. This defines A_Q the structure of topological ring on A_Q. Note that we have a diagonal embedding $Q \mapsto A_Q$ and the quotient

$$Q \backslash A_Q \cong \hat{Z} \times (R/Z)$$

is compact. This is in fact the reason for the above condition that almost all f_p 's belong to Z_p. **We also have the multiplicative group $A_Q^\times$ of invertible adèles (also called idèles)** and a natural diagonal embedding of groups $Q^\times \mapsto A_Q^\times$. In the case when $F = Q$, the statement of ACFT is that $Gal(Q^{ab}/Q)$ is isomorphic to the group of connected components of the quotient $Q^\times \setminus A_Q^\times$. It is not difficult to see that

$$Q^\times \setminus A_Q^\times \cong \prod_p Z_p^\times \times R_{>0}. \quad (1.6)$$

Hence the group of its connected components is isomorphic to $\prod_p Z_p^\times$, in agreement with the Kronecker-Weber theorem.

Now we discuss cuspidal automorphic representations of $GL_2(A) = GL_2(A_Q)$ and how to relate them to **classical modular forms** on the upper half-plane. We will then consider the two-dimensional representations of $Gal(\overline{Q}/Q)$ arising from **elliptic curves defined over** Q and look at what the Langlands correspondence means for such representations. **In this special case the Langlands correspondence becomes the statement of the Taniyama-Shimura conjecture which implies Fermat's last theorem**. The cuspidal automorphic representations of $GL_2(A)$ are those irreducible representations of this group which occur in the discrete spectrum of a certain space of functions on the quotient $GL_2(Q) \setminus GL_2(A)$.

We start by introducing the maximal compact subgroup $K \subset GL_2(A)$ which is equal to $\Pi_p GL_2(Z_p) \times O_2$. Let z be the center of the universal enveloping algebra of the Lie algebra gl_2. Then z is the polynomial algebra in the central element $I \in gl_2$ and the Casimir operator

$$C = \frac{1}{4} X_0^2 + \frac{1}{2} X_+ X_- + \frac{1}{2} X_- X_+, \quad (1.7)$$

where

$$X_0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad X_\pm = \frac{1}{2}\begin{pmatrix} 1 & \mp i \\ \mp i & -1 \end{pmatrix}$$

are basis elements of $sl_2 \subset gl_2$.

Consider the space of functions of $GL_2(Q) \setminus GL_2(A)$ which are locally constant as functions on $GL_2(A^f)$, where $A^f = \Pi'_p Q_p$, and smooth as functions on $GL_2(R)$. Such functions are called smooth. The group $GL_2(A)$ acts on this space by right translations:

$$(g \cdot f)(h) = f(hg), \qquad g \in GL_2(A).$$

In particular, the subgroup $GL_2(R) \subset GL_2(A)$, and hence its complexified Lie algebra gl_2 and the universal enveloping algebra of the latter also act. The group $GL_2(A)$ has the center $Z(A) \cong A^\times$ which consists of all diagonal matrices. For a character $\chi : Z(A) \to C^\times$ and a complex number ρ let

$$\mathcal{C}_{\chi,\rho}(GL_2(Q) \setminus GL_2(A))$$

be the space of smooth functions $f : GL_2(Q) \backslash GL_2(A) \to C$ satisfying the following additional requirements:

(i) (K-finiteness) the (right) translates of f under the action of elements of the compact subgroup K span a finite-dimensional vector space;

(ii) (central character) $f(gz) = \chi(z) f(g)$ for all $g \in GL_2(A)$, $z \in Z(A)$, and $C \cdot f = \rho f$, where C is the Casimir element (1.7):

(iii) (growth) f is bounded on $GL_n(A)$;

(iv) (cuspidality) $\int_{Q \backslash NA} f\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) du = 0$.

The space $\mathcal{C}_{\chi,\rho}(GL_2(Q) \backslash GL_2(A))$ is a representation of the group

$$GL_2(A^f) = \prod_{p(prime)}{}' GL_2(Q_p) \quad (1.8)$$

and the Lie algebra gl_2, whose actions commute with each other. In addition, the subgroup O_2 of $GL_2(R)$ acts on it, and the action of O_2 is compatible with the action of gl_2 making it into a module over the so-called Harish-Chandra pair (gl_2, O_2). It is known that $\mathcal{C}_{\chi,\rho}(GL_2(Q) \backslash GL_2(A))$ is a direct sum of irreducible representations of $GL_2(A^f) \times gl_2$, each occurring with multiplicity one. The irreducible representations occurring in these spaces are called the cuspidal automorphic representations of $GL_2(A)$. An irreducible cuspidal automorphic representation π may be written as a restricted infinite tensor product

$$\pi = \bigotimes_{p(prime)}{}' \pi_p \otimes \pi_\infty, \quad (1.9)$$

where π_p is an irreducible representation of $GL_2(Q_p)$ and π_∞ is a gl_2-module. For all but finitely many **primes** p, the representation π_p is unramified, which means that it contains a non-zero vector invariant under the maximal compact subgroup $GL_2(Z_p)$ of $GL_2(Q_p)$. This vector is then unique up to a scalar. Let us choose $GL_2(Z_p)$-invariant vectors v_p at all unramified **primes** p. Then the vector space (1.9) is the restricted infinite tensor product in the sense that it consists of finite linear combinations of vectors of the form $\otimes_p w_p \otimes w_\infty$, where $w_p = v_p$ for all but finitely many **prime numbers** p. It is clear from the definition of $A^f = \Pi'_p Q_p$ that the group $GL_2(A^f)$ acts on it. Suppose now that p is one of the **primes** at which π_p is ramified, so π_p does not contain $GL_2(Z_p)$-invariant vectors. Then it contains vectors invariant under smaller compact subgroups of $GL_2(Z_p)$.

Let us assume for simplicity that $\chi \equiv 1$. Then one shows that there is a unique, up to a scalar, non-zero vector in π_p invariant under the compact subgroup

$$K'_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \bmod p^{n_p} Z_p \right\} \quad (1.10)$$

for some positive integer n_p. Let us choose such a vector v_p at all **primes** where π is ramified. In order to have uniform notation, we will set $n_p = 0$ at those **primes** at which π_p is unramified, so at such primes we have $K_p^{'} = GL_2(Z_p)$. Let $K' = \prod_p K_p^{'}$. Thus, we obtain that the space of K'-invariants in π is the subspace

$$\tilde{\pi}_\infty = \otimes_p v_p \otimes \pi_\infty, \quad (1.11)$$

which carries an action of (gl_2, O_2). The space $\tilde{\pi}_\infty$ of K'-invariant vectors in π is realized in the space of functions on the double quotient $GL_2(Q) \backslash GL_2(A)/K'$.

Now we use the strong approximation theorem to obtain the following useful statement. Let us set $N = \prod_p p^{n_p}$ and consider the subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | \, c \equiv 0 \bmod NZ \right\}$$

of $SL_2(Z)$. Then

$$GL_2(Q) \backslash GL_2(A)/K' \cong \Gamma_0(N) \backslash GL_2^+(R), \quad (1.12)$$

where $GL_2^+(R)$ consists of matrices with positive determinant. Thus, the smooth functions on $GL_2(Q) \backslash GL_2(A)$ corresponding to vectors in the space $\tilde{\pi}_\infty$ given by (1.11) are completely determined by their restrictions to the subgroup $GL_2^+(R)$ of $GL_2(R) \subset GL_2(A)$.

Consider the case when π_∞ is a representation of the discrete series of $(gl_2(C), O(2))$. In this case $\rho = k(k-2)/4$, where k is an integer greater than 1. Then, as an sl_2-module, π_∞ is the direct sum of the irreducible Verma module of highest weight $-k$ and the irreducible Verma module with lowest weight k. The former is generated by a unique, up to a scalar, highest weight vector v_∞ such that

$$X_0 \cdot v_\infty = -k v_\infty, \qquad X_+ \cdot v_\infty = 0,$$

and the latter is generated by the lowest weight vector $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot v_\infty$.

Thus, the entire $gl_2(R)$-module π_∞ is generated by the vector v_∞, and so we focus on the function on $\Gamma_0(N) \backslash SL_2(R)$ corresponding to this vector. Let ϕ_π be the corresponding function on $SL_2(R)$. By construction, it satisfies

$$\phi_\pi(\gamma g) = \phi_\pi(g), \qquad \gamma \in \Gamma_0(N),$$
$$\phi_\pi\left(g \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right) = e^{ik\theta} \phi_\pi(g) \qquad 0 \le \theta \le 2\pi.$$

We assign to ϕ_π a function f_π on the upper half-plane

$$H = \{\tau \in C \,|\, \operatorname{Im} \tau > 0\}.$$

Recall that $H \cong SL_2(R)/SO_2$ under the correspondence

$$SL_2(R) \in g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{a+bi}{c+di} \in H .$$

We define a function f_π on $SL_2(R)/SO_2$ by the formula

$$f_\pi(g) = \phi(g)(ci+d)^k .$$

When written as a function of τ, the function f satisfies the conditions

$$f_\pi\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f_\pi(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N). \quad (1.13)$$

In addition, the "highest weight condition" $X_+ \cdot v_\infty = 0$ is equivalent to f_π being a holomorphic function of τ. Such functions are called *modular forms of weight* k *and level* N.
Thus, we have attached to an automorphic representation π of $GL_2(A)$ a holomorphic modular form f_π of weight k and level N on the upper half-plane. We expand it in the Fourier series

$$f_\pi(q) = \sum_{n=0}^{\infty} a_n q^n , \qquad q = e^{2\pi i \tau} . \quad (1.14)$$

The cuspidality condition on π means that f_π vanishes at the cusps of the fundamental domain of the action of $\Gamma_0(N)$ on H. Such modular forms are called "cusp forms". In particular, it vanishes at $q=0$ which corresponds to the cusp $\tau = i\infty$, and so we have $a_0 = 0$. Further, it can shown that $a_1 \neq 0$, and we will normalize f_π by setting $a_1 = 1$. **The eq.** (1.14) **in related also to the following conjecture: for each elliptic curve E over Q there should exist a modular cusp form**

$$f_E(q) = \sum_{n=1}^{\infty} a_n q^n ,$$

with $a_1 = 1$ and

$$a_p = p + 1 - \# E(F_p) \quad (1.14b)$$

for all but finitely many primes p. This is in fact the statement of the celebrated Taniyama-Shimura conjecture that has recently proved by A. Wiles and others. It implies Fermat's Last Theorem.
The normalized modular cusp form $f_\pi(q)$ contains all the information about the automorphic representation π. In particular, it "knows" about the Hecke eigenvalues of π.
Let us give the definition the Hecke operators. This is a local question that has to do with the local factor π_p in the decomposition (1.9) of π **at a prime** p, which is a representation of $GL_2(Q_p)$. Suppose that π_p is unramified, i.e., it contains a unique, up to a scalar, vector v_p that is invariant under the subgroup $GL_2(Z_p)$. Then it is an eigenvector of the spherical Hecke algebra H_p which is the algebra of compactly supported $GL_2(Z_p)$ bi-invariant functions on $GL_2(Q_p)$, with respect to the

convolution product. This algebra is isomorphic to the polynomial algebra in two generators $H_{1,p}$ and $H_{2,p}$, whose action on v_p is given by the formulas

$$H_{1,p}\cdot v_p=\int_{M_2^1(Z_p)}\rho_p(g)\cdot v_p dg\,,\quad (1.15)\qquad H_{2,p}\cdot v_p=\int_{M_2^2(Z_p)}\rho_p(g)\cdot \rho_p dg\,,\quad (1.16)$$

where $\rho_p: GL_2(Z_p)\to \mathrm{End}\ \pi_p$ is the representation homomorphism, $M_2^i(Z_p), i=1,2$ are the double cosets

$$M_2^1(Z_p)=GL_2(Z_p)\begin{pmatrix} p & 0\\ 0 & 1\end{pmatrix}GL_2(Z_p),\qquad M_2^2(Z_p)=GL_2(Z_p)\begin{pmatrix} p & 0\\ 0 & p\end{pmatrix}GL_2(Z_p)$$

in $GL_2(Q_p)$, and we use the Haar measure on $GL_2(Q_p)$ normalized so that the volume of the compact subgroup $GL_2(Z_p)$ is equal to 1.

Since the integrals are over $GL_2(Z_p)$-cosets, $H_{1,p}\cdot v_p$ and $H_{2,p}\cdot v_p$ are $GL_2(Z_p)$-invariant vectors, hence proportional to v_p. Under our assumption that the center $Z(A)$ acts trivially on $\pi(\chi\equiv 1)$ we have $H_2\cdot v_p=v_p$. But the eigenvalue $h_{1,p}$ of $H_{1,p}$ on v_p is an important invariant of π_p. This invariant is defined **for all primes** p at which π is unramified. These are precisely the Hecke eigenvalues.

Now we should consider automorphic representations of the **adèlic group** $GL_n(A)$. Here $A=A_F$ is **the ring of adèles** of F, defined in the same way as in the number field case. For any closed point x of X, we denote by F_x the completion of F at x and by O_x its ring of integers. If we pick a rational function t_x on X which vanishes at x to order one, then we obtain isomorphisms $F_x\cong k_x((t_x))$ and $\mathrm{O}_x\cong k_x[[t_x]]$, where k_x is the residue field of x (the quotient of the local ring O_x by its maximal ideal). This field is a finite extension of the base field k and hence is isomorphic to F_{q_x}, where $q_x=q^{\deg x}$. **The ring** A **of adèles of** F is by definition the *restricted* product of the fields F_x, where x runs over the set of all closed points of X.

Note that $GL_n(F)$ is naturally a subgroup of $GL_n(A)$. Let K be the maximal compact subgroup $K=\prod_{x\in X}GL_n(\mathrm{O}_x)$ of $GL_n(A)$. The group $GL_n(A)$ has the center $Z(A)\cong A^\times$ which consists of the diagonal matrices. Let $\chi: Z(A)\to C^\times$ be a character of $Z(A)$ which factors through a finite quotient of $Z(A)$. Denote by $\mathcal{C}_\chi(GL_n(F)\setminus GL_n(A))$ the space of locally constant functions $f: GL_n(F)\setminus GL_n(A)\to C$ satisfying the following additional requirements:

(i) (K-finiteness) the (right) translates of f under the action of elements of the compact subgroup K span a finite-dimensional vector space;

(ii) (central character) $f(gz)=\chi(z)f(g)$ for all $g\in GL_n(A),\ z\in Z(A)$;

(iii) (cuspidality) let $N_{n1,n2}$ be the unipotent radical of the standard parabolic subgroup $P_{n1,n2}$ of GL_n corresponding to the partition $n=n_1+n_2$ with $n_1,n_2>0$. Then

$$\int_{N_{n_1,n_2}(F)\setminus N_{n_1,n_2}(A)}\varphi(ug)du=0\,,\qquad \forall g\in GL_n(A).\quad (1.17)$$

The group $GL_n(A)$ acts on $\mathcal{C}_\chi(GL_n(F)\setminus GL_n(A))$ from the right: for

$$f \in \mathcal{C}_\chi(GL_n(F)\backslash GL_n(A)), \qquad g \in GL_n(A)$$

we have

$$(g \cdot f)(h) = f(hg), \qquad h \in GL_n(F)\backslash GL_n(A). \quad (1.18)$$

Under this action $\mathcal{C}_\chi(GL_n(F)\backslash GL_n(A))$ decomposes into a direct sum of irreducible representations. These representations are called *irreducible cuspidal automorphic representations* of $GL_n(A)$. We denote the set of equivalence classes of these representations by A_n.

Now let π be an irreducible cuspidal automorphic representation of $GL_n(A)$. One can show that it decomposes into a tensor product

$$\pi = \mathop{\otimes'}_{x \in X} \pi_x, \quad (1.19)$$

where each π_x is an irreducible representation of $GL_n(F_x)$. Furthermore, there is a finite subset S of X such that each π_x with $x \in X \setminus S$ is *unramified*, i.e., contains a non-zero vector v_x stable under the maximal compact subgroup $GL_n(\mathrm{O}_x)$ of $GL_n(F_x)$. The space $\otimes'_{x\in X} \pi_x$ is by definition the span of all vectors of the form $\otimes_{x\in X} w_x$, where $w_x \in \pi_x$ and $w_x = v_x$ for all but finitely many $x \in X \setminus S$. Therefore the action of $GL_n(A)$ on π is well-defined. Let x be a point of X with residue field F_{q_x}. By definition, H_x be the space of compactly supported functions on $GL_n(F_x)$ which are bi-invariant with respect to the subgroup $GL_n(\mathrm{O}_x)$. This is an algebra with respect to the convolution product

$$(f_1 * f_2)(g) = \int_{GL_n(F_x)} f_1(gh^{-1}) f_2(h) dh, \quad (1.20)$$

where dh is the Haar measure on $GL_n(F_x)$ normalized in such a way that the volume of the subgroup $GL_n(\mathrm{O}_x)$ is equal to 1. It is called the *spherical Hecke algebra* corresponding to the point x.

Let $H_{i,x}$ be the characteristic function of the $GL_n(\mathrm{O}_x)$ double coset

$$M_n^i(\mathrm{O}_x) = GL_n(\mathrm{O}_x) \cdot diag(t_x,\dots,t_x,1,\dots,1) \cdot GL_n(\mathrm{O}_x) \subset GL_n(F_x) \quad (1.21)$$

of the diagonal matrix whose first i entries are equal to t_x and the remaining $n-i$ entries are equal to 1. Note that this double coset does not depend on the choice of the coordinate t_x. Then H_x is the commutative algebra freely generated by $H_{1,x},\dots,H_{n-1,x},H_{n,x}^{\pm 1}$:

$$\mathrm{H}_x \cong C[H_{1,x},\dots,H_{n-1,x},H_{n,x}^{\pm 1}]. \quad (1.22)$$

Define an action of $f_x \in \mathrm{H}_x$ on $v \in \pi_x$ by the formula

$$f_x * v = \int f_x(g)(g \cdot v) dg. \quad (1.23)$$

Since f_x is left $GL_n(\mathrm{O}_x)$-invariant, the result is again a $GL_n(\mathrm{O}_x)$-invariant vector. If π_x is irreducible, then the space of $GL_n(\mathrm{O}_x)$-invariant vectors in π_x is one-dimensional. Let $v_x \in \pi_x$ be a generator of this one-dimensional vector space. Then

$$f_x * v_x = \phi(f_x)v_x$$

for some $\phi(f_x) \in C$. Thus, we obtain a linear functional $\phi : \mathrm{H}_x \to C$. In view of the isomorphism (1.22), a homomorphism $\mathrm{H}_x \to C$ is completely determined by its values on $H_{1,x},...,H_{n-1,x}$, which could be arbitrary complex numbers, and its value on $H_{n,x}$, which has to be a non-zero complex number. These values are the eigenvalues on v_x of the operators (1.23) of the action of $f_x = H_{i,x}$. These operators are called the *Hecke operators*. It is convenient to package these eigenvalues as an n-tuple of *unordered* non-zero complex numbers $z_1,...,z_n$, so that

$$H_{i,x} * v_x = q_x^{i(n-i)/2} s_i(z_1,...,z_n)v_x, \qquad i = 1,...,n, \quad (1.24)$$

where s_i is the ith elementary symmetric polynomial. In other words, the above formulas may be used to identify

$$\mathrm{H}_x \cong C\left[z_1^{\pm 1},...,z_n^{\pm 1}\right]^{S_n}. \quad (1.25)$$

Now, we show the interpretation of automorphic representations in terms of the moduli spaces of rank n vector bundles.

We will restrict ourselves from now on to the irreducible automorphic representations of $GL_n(A)$ that are unramified at all points of X. Suppose that we are given such a representation π of $GL_n(A)$. Then the space of $GL_n(\mathrm{O})$-invariants in π, where $\mathrm{O} = \prod_{x \in X} \mathrm{O}_x$, is one-dimensional, spanned by the vector

$$v = \underset{x \in X}{\otimes} v_x \in \underset{x \in X}{\otimes'} \pi_x = \pi, \quad (1.26)$$

Hence v gives rise to a $GL_n(\mathrm{O})$-invariant function on $GL_n(F) \backslash GL_n(A)$, or equivalently, a function f_π on the double quotient

$$GL_n(F) \backslash GL_n(A) / GL_n(\mathrm{O}). \quad (1.27)$$

This function is an eigenfunction of the spherical Hecke algebras H_x for all $x \in X$.

Let X be a smooth projective curve over any field k and $F = k(X)$ the function field of X. **We define the ring A of adèles and its subring O of integer adèles** in the same way as in the case when $k = F_q$. Then we have the following:

Lemma

There is a bijection between the set $GL_n(F) \backslash GL_n(A) / GL_n(\mathrm{O})$ and the set of isomorphism classes of rank n vector bundles on X.

We consider this statement in the case when X is a complex curve. Any rank n vector bundle V on X can be trivialized over the complement of finitely many points. This is equivalent to the

existence of n meromorphic sections of V whose values are linearly independent away from finitely many points of X.
Let $x_1,...,x_N$ be the set of points such that V is trivialized over $X \setminus \{x_1,...,x_N\}$. The bundle V can also be trivialized over the small discs D_{x_i} around those points. Thus, we consider the covering of X by the open subsets $X \setminus \{x_1,...,x_N\}$ and $D_{x_i}, i = 1,...,N$. The overlaps are the punctured discs $D_{x_i}^{\times}$, and our vector bundle is determined by the transition functions on the overlaps, which are GL_n-valued functions g_i on $D_{x_i}^{\times}, i = 1,...,N$. The difference between two trivializations of V on D_{x_i} amounts to a GL_n-valued function h_i on D_{x_i}. If we consider a new trivialization on D_{x_i} that differs from the old one by h_i, then the ith transition function g_i will get multiplied on the right by h_i: $g_i \mapsto g_i h_i |_{D_{x_i}^{\times}}$, whereas the other transition functions will remain the same. Likewise, the difference between two trivializations of V on $X \setminus \{x_1,...,x_N\}$ amounts to a GL_n-valued function h on $X \setminus \{x_1,...,x_N\}$. If we consider a new trivialization on $X \setminus \{x_1,...,x_N\}$ that differs from the old one by h, then the ith transition function g_i will get multiplied on the left by h: $g_i \mapsto h|_{D_{x_i}^{\times}} g_i$ for all $i = 1,...,N$. We obtain that the set of isomorphism classes of rank n vector bundles on X which become trivial when restricted to $X \setminus \{x_1,...,x_N\}$ is the same as the quotient

$$GL_n\left(X \setminus \{x_1,...,x_N\}\right) \setminus \prod_{i=1}^{N} GL_n\left(D_{x_i}^{\times}\right) / \prod_{i=1}^{N} GL_n\left(D_{x_i}\right). \quad (1.28)$$

Here for an open set U we denote by $GL_n(U)$ the group of GL_n-valued function on U, with pointwise multiplication. If we replace each D_{x_i} by the formal disc at x_i, then $GL_n\left(D_{x_i}^{\times}\right)$ will become $GL_n(F_x)$, where $F_x \cong C((t_x))$ is the algebra of formal Laurent series with respect to a local coordinate t_x at x, and $GL_n\left(D_{x_i}\right)$ will become $GL_n(O_x)$, where $O_x \cong C[[t_x]]$ is the ring of formal Taylor series. Then, if we also allow the set $x_1,...,x_N$ to be an arbitrary finite subset of X, we will obtain instead of (1.28) the double quotient

$$GL_n(F) \setminus {\prod_{x \in X}}' GL_n(F_x) / \prod_{x \in X} GL_n(O_x), \quad (1.29)$$

where $F = C(X)$ and the prime means the restricted product. But this is exactly the double quotient in the statement of the Lemma. This completes the proof.
Thus, when X is a curve over F_q, irreducible unramified automorphic representations π are encoded by the automorphic functions f_π, which are functions on $GL_n(F) \setminus GL_n(A) / GL_n(O)$. This double quotient makes perfect sense when X is defined over C and is in fact the set of isomorphism classes of rank n bundles on X. But what should replace the notion of an automorphic function f_π in this case? We will argue that the proper analogue is not a function, as one might naively expect, but a *sheaf* on the corresponding algebro-geometric object: the moduli stack Bun_n of rank n bundles on X. Let V be an algebraic variety over F_q. Then, **the "correct" geometric counterpart of the notion of a ($\overline{Q}_\ell$-valued) function on the set of F_q-points of V is the notion of a *complex of ℓ-adic sheaves* on V. Let us just say the simplest example of an ℓ-adic sheaf is an ℓ-adic local system, which is a locally constant $\overline{Q}_\ell$-sheaf on V. For a general**

ℓ**-adic sheaf there exists a stratification of V by locally closed subvarieties V_i such that the sheaves $\mathcal{F}|_{V_i}$ are locally constant**.

The important property of the notion of an ℓ-adic sheaf $\mathcal{F}$ on V is that for any morphism $f: V' \to V$ from another variety V' to V the group of symmetries of this morphism will act on the pull-back of $\mathcal{F}$ to V'. In particular, let x be an F_q-point of V and $\bar{x}$ the $\overline{F}_q$-point corresponding to an inclusion $F_q \mapsto \overline{F}_q$. Then the pull-back of $\mathcal{F}$ with respect to the composition $\bar{x} \to x \to V$ is a sheaf on $\bar{x}$, which is nothing but the fiber $\mathcal{F}_{\bar{x}}$ of $\mathcal{F}$ at $\bar{x}$, a $\overline{Q}_\ell$-vector space. But the Galois group $Gal(\overline{F}_q / F_q)$ is the symmetry of the map $\bar{x} \to x$, and therefore it acts on $\mathcal{F}_{\bar{x}}$. In particular, the (geometric) Frobenius element $Fr_{\bar{x}}$, which is the generator of this group acts on $\mathcal{F}_{\bar{x}}$. Taking the trace of $Fr_{\bar{x}}$ on $\mathcal{F}_{\bar{x}}$, we obtain a number $Tr(Fr_{\bar{x}}, \mathcal{F}_{\bar{x}}) \in \overline{Q}_\ell$. Hence we obtain a function $f_{\mathcal{F}}$ on the set of F_q-points of V, whose value at x is

$$f_{\mathcal{F}}(x) = Tr(Fr_{\bar{x}}, \mathcal{F}_{\bar{x}}). \quad (1.30)$$

More generally, **if K is a complex of ℓ-adic sheaves, one defines a function $f(\mathrm{K})$ on $V(F_q)$ by taking the alternating sums of the traces of $Fr_{\bar{x}}$ on the stalk cohomologies of K at $\bar{x}$:**

$$f_{\mathrm{K}}(x) = \sum_i (-1)^i Tr\left(Fr_{\bar{x}}, H^i_{\bar{x}}(\mathrm{K})\right). \quad (1.31)$$

The map $\mathrm{K} \to f_{\mathrm{K}}$ intertwines the natural operations on complexes sheaves with natural operations on functions.

Thus, because of the existence of the Frobenius automorphism in the Galois group $Gal(\overline{F}_q / F_q)$ (which is the group of symmetries of an F_q-point) we can pass from ℓ-adic sheaves to functions on any algebraic variety over F_q. **This suggests that the proper geometrization of the notion of a function in this setting is the notion of ℓ-adic sheaves**.

2. On some equations concerning the moduli spaces of SL_2 and SO_3 Higgs bundles on an elliptic curve with tame ramification at one point. [2]

We consider a Higgs field φ that is a section of

$$K \otimes \mathrm{O}(p) \otimes W = \oplus_{i=1}^{3} K \otimes \mathrm{O}(p)^2 \otimes \mathrm{O}(q_i)^{-1}.$$

For each i, we can pick a section u_i of $K \otimes \mathrm{O}(p)^2 \otimes \mathrm{O}(q_i)^{-1}$, namely $u_i = (dx/y)(x - e_i)$, with $Tru_i^2 = (dx/y)^2 (x - e_i)$. The general form of the Higgs field is

$$\varphi = \sum_{i=1}^{3} a_i u_i, \quad (2.1)$$

with complex constants a_i. This gives

$$Tr\varphi^2 = \left(\frac{dx}{y}\right)^2 \sum_{i=1}^{3} a_i^2 (x - e_i). \quad (2.2)$$

Letting $z \approx x^{-1/2}$ be a local parameter at infinity, the polar part of $Tr\varphi^2$ is $4(dz/z)^2 \sum_i a_i^2$. Setting this to $2\sigma_0^2 (dz/z)^2$, we require

$$\sum_i a_i^2 = \frac{\sigma_0^2}{2}. \quad (2.3)$$

This affine quadric describes a Zariski open set in $\mathrm{M}_H(C; SL_2^*)$. The constant term multiplying $(dx/y)^2$ on the right hand side of (2.2) is $-\sum_i e_i a_i^2$. This enables us to describe the Hitchin fibration; it is the map from (a_1, a_2, a_3) to

$$w_0 = -\sum_i e_i a_i^2. \quad (2.4)$$

A fiber of the Hitchin fibration is given by the intersection of the two quadrics (2.3) and (2.4). The parameter σ_0 can be scaled out of these equations, assuming that it is nonzero. We set $w_0 = -\sigma_0^2 w/2$, and $b_i = a_i(\sqrt{2}/\sigma_0)$ to put the two quadrics in the form

$$b_1^2 + b_2^2 + b_3^2 = 1; \qquad e_1 b_1^2 + e_2 b_2^2 + e_3 b_3^2 = w. \quad (2.5)$$

For generic w, this intersection is a smooth curve of genus 1, with some points omitted because we have assumed W to be stable. Now, if $f = \sum_i b_i^2 - 1$, $g = \sum_i e_i b_i^2 - w$, then a singularity of the fiber is a point with $f = g = df \wedge dg = 0$. A short calculation shows that $df \wedge dg = 0$ precisely if two of b_1, b_2 and b_3 vanish. If b_i is non-vanishing for some i and $b_j = 0$ for $j \neq i$, then we must have

$$w = e_i \quad (2.6)$$

and

$$b_i = \pm 1. \quad (2.7)$$

Moreover, each singular fiber F_w contains two singular points, given in eqn. (2.7). The singular fibers consist of two components of genus 0 joined at two double points. To see this, take $i = 1$. If $w = e_1$, then a linear combination of the equations $f = 0$ and $g = 0$ gives

$$(e_2 - e_1) b_2^2 + (e_3 - e_1) b_3^2 = 0,$$

or

$$b_2 = \pm b_3 \sqrt{-(e_3 - e_1)/(e_2 - e_1)}. \quad (2.8)$$

This describes a curve $F_{w,0}$ that is a union of two genus zero components meeting at one point, $b_2 = b_3 = 0$. Now solving for b_1 via $b_1^2 = 1 - b_3^2 (e_2 - e_3)/(e_2 - e_1)$ gives a double cover of $F_{w,0}$. The double cover, which is the fiber F_w of the Hitchin fibration, is branched over two points in each component of $F_{w,0}$. A double cover of a curve of genus zero branched at two points is still of genus zero. So F_w consists of two components of genus zero, meeting at the two points $b_2 = b_3 = 0$, $b_1 = \pm 1$. On of the most important properties of the moduli space M_H of stable Higgs bundles is that it can be approximated as $T^*\mathrm{M}$, where M is the moduli space of stable bundles (M is a Zariski open set the moduli space that parametrizes stable and semi-stable bundles). The reason for this is that the cotangent space to M, at a point corresponding to a stable bundle E, is $H^0(C, K \otimes ad(E))$. So a point in $T^*\mathrm{M}$ is a pair (E, φ), or in other words a Higgs bundle. This gives an embedding of $T^*\mathrm{M}$ as a Zariski open set in M_H. This has an analog for ramified Higgs bundles. In this case, one takes M to be the moduli space of stable bundles with parabolic structure at a point p, and M_H to be the moduli space of stable ramified Higgs bundles. Then M_H has a Zariski open set that is not quite $T^*\mathrm{M}$ but is an affine symplectic deformation of $T^*\mathrm{M}$. We denote such an affine symplectic deformation as $\tilde{T}^*\mathrm{M}$. Here $\tilde{T}^*\mathrm{M}$ denotes a complex symplectic variety with a map to M, such that locally in M, $\tilde{T}^*\mathrm{M}$ is symplectically isomorphic to $T^*\mathrm{M}$. For applications to geometric Langlands, it is important to restrict the fibers of the Hitchin fibration from M_H to $T^*\mathrm{M}$ or $\tilde{T}^*\mathrm{M}$, since this is an essential step in interpreting A-branes in terms of $\mathcal{D}$-modules.

Now let us consider the fibers of the Hitchin fibration. Their intersection with the quadric $\tilde{T}^*\mathrm{M}$ is obtained by supplementing the defining equation of the quadric with the equation

$$e_1 b_1^2 + e_2 b_2^2 + e_3 b_3^2 = w, \quad (2.9)$$

giving an algebraic curve F_w. This curve can be projected to M, and gives a double cover of M. So the fiber F_w of the Hitchin fibration of M_H is really, for generic w, the smooth projective curve that corresponds to the affine curve just described.

We can describe F_w as a projective curve by simply adding another variable b_4, where $b_1, ..., b_4$ are understood as homogeneous coordinates on CP^3 and obey

$$\sum_{i=1}^{3} b_i^2 = b_4^2 , \qquad \sum_{i=1}^{3} e_i b_i^2 = w b_4^2 . \quad (2.10)$$

Missing when one approximates M_H by an affine deformation of the cotangent bundle are the four points with $b_4 = 0$. These points correspond to stable Higgs bundles (E, φ) where the parabolic bundle E is unstable. They form a single orbit of the group $Q = Z_2 \times Z_2$ of pairwise sign changes of b_1, b_2, b_3. Explicitly, the values of z corresponding to these four points are

$$z = \pm\sqrt{\frac{e_2 - e_1}{e_2 - e_3}} \pm \sqrt{\frac{e_3 - e_1}{e_2 - e_3}} . \quad (2.11)$$

To gain some insight about the $\mathcal{D}$-modules arising in the geometric Langlands program, we must describe F_w as a curve in $\tilde{T}^*\mathrm{M}$. For this, we use the coordinates $z, \tilde{v}$, and find, after some algebra, that we can describe the fiber F_w by an explicit quadratic equation for $\tilde{v}$, of the form

$$A(z)\tilde{v}^2 + B(z)\tilde{v} + C(z) = 0 , \quad (2.12)$$

with

$$A(z)=(e_2-e_3)z^4+(4e_1-2e_2-2e_3)z^2+(e_2-e_3)$$
$$B(z)=-4z((e_2-e_3)z^2+2e_1-e_2-e_3)$$
$$C(z)=4((e_2-e_3)z^2+e_1-w). \quad (2.13)$$

Note that

$$B=-\frac{dA}{dz}. \quad (2.14)$$

Then, the eq. (2.12) can be rewritten also

$$[(e_2-e_3)z^4+(4e_1-2e_2-2e_3)z^2+(e_2-e_3)]\tilde{v}^2+[-4z((e_2-e_3)z^2+2e_1-e_2-e_3)]\tilde{v}+$$
$$+4((e_2-e_3)z^2+e_1-w)=0. \quad (2.14b)$$

If we let z approach one of the four values in eq. (2.11), then one of the roots of the quadratic equation for $\tilde{v}$ goes to infinity. So at any of those values of z, a point in the Hitchin fiber is "missing" if we restrict to $\tilde{T}^*\mathrm{M}$. In fact, the four critical values of z are precisely the zeroes of the polynomial $A(z)$. Let z^* be any one of the zeroes of $A(z)$.The behaviour of the polar branch of v near z^* is $v\approx -\sigma_0 B(z)/A(z)$, which using eq. (2.14) reduces to

$$v\approx \sigma_0\frac{1}{z-z^*}. \quad (2.15)$$

For the geometric endoscopy, we must examine in a similar way the singular fibers of the Hitchin fibration. For example, we take $w=e_1$, so that the fiber F_{e_1} splits into components $F_{e_1}^{\pm}$ defined by the ratio of b_2/b_3, as in eq. (2.8). Compactifying the two components in projective space, we see that of the four points at $b_4=0$, two lie on $F_{e_1}^{+}$ and two on $F_{e_1}^{-}$. If we restrict to $\tilde{T}^*\mathrm{M}$, the two curves $F_{e_1}^{\pm}$ behave near the two relevant critical values of z. So each fractional A-brane has two points with this sort of behaviour.

3. On some equations concerning the action of the Wilson and 't Hooft/Hecke operators on the electric and magnetic branes relevant to geometric endoscopy. [2]

A Wilson operator in ${}^LG$ gauge theory is associated to the choice of a point $p\in C$ and a representation ${}^LR$ of ${}^LG$. For simplicity, we will take ${}^LR$ to be the three-dimensional representation of ${}^LG=SO_3$, and we write W_p for the corresponding Wilson operator. The action of W_p on B-branes can be described as follows. Let $\mathcal{B}$ be a B-brane associated with a coherent sheaf $\mathrm{K}\to\mathrm{M}_H$. Then $W_p\cdot\mathcal{B}$ is the B-brane associated with the sheaf $\mathrm{K}\otimes W\big|_p$, where $W\big|_p$ is the restriction of W to $\mathrm{M}_H\times p$. (We understand W as a rank 3 vector bundle with structure group SO_3). Thus, the action of W_p on coherent sheaves is

$$\mathrm{K}\to\mathrm{K}\otimes W\big|_p. \quad (3.1)$$

We take the case of a brane supported at a Z_2 orbifold singularity $r \in \mathrm{M}_H$. Such a singularity is associated with an SO_3 local system whose structure group reduces to O_2. We recall that O_2 is embedded in SO_3 as the subgroup

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \qquad (3.2)$$

and that any SO_3 local system whose structure group reduces to O_2 has symmetry group Z_2, generated by the central element of O_2:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (3.3)$$

We will consider a generic local systems of this type whose group of automorphism is precisely this Z_2. In the present context, $W\big|_r$, the restriction of W to $r \times C$, is a local system whose structure group reduces to O_2, so it has a decomposition

$$W\big|_r = U \oplus \det U\,, \qquad (3.4)$$

where U is a rank 2 local system, with structure group O_2, and $\det U$ is its determinant. The central generator of Z_2 acts as -1 on U and as $+1$ on $\det U$.
The category of branes supported at the orbifold singularity r is generated by two irreducible objects $\mathcal{B}_+$ and $\mathcal{B}_-$. Each is associated with a skyscraper sheaf supported at r. They differ by whether the non-trivial element of Z_2 acts on this sheaf as multiplication by +1 or by – 1.
Since $\mathcal{B}_+$ and $\mathcal{B}_-$ both have skyscraper support at r, W_p acts on either of them by tensor product with the three-dimensional vector space $\mathrm{W}\big|_{r\times p}$, the fiber of W at $r \times p$. In view of eq. (3.4), there is a decomposition $\mathrm{W}\big|_{r\times p} = U\big|_p \oplus \det U\big|_p$, where the non-trivial element of Z_2 acts as – 1 on the first summand and as +1 on the second summand. So we have

$$\begin{aligned} W_p \cdot \mathcal{B}_+ &= \left(\mathcal{B}_- \otimes U\big|_p\right) \oplus \left(\mathcal{B}_+ \otimes \det U\big|_p\right) \\ W_p \cdot \mathcal{B}_- &= \left(\mathcal{B}_+ \otimes U\big|_p\right) \oplus \left(\mathcal{B}_- \otimes \det U\big|_p\right). \end{aligned} \qquad (3.5)$$

Note that $\det U\big|_p$ is a one-dimensional vector space on which Z_2 acts trivially, so $\mathcal{B}_\pm \otimes \det U\big|_p$ is isomorphic, non-canonically, to $\mathcal{B}_\pm$. And $U\big|_p$ is a two-dimensional vector space on which the non-trivial element of Z_2 acts as multiplication by – 1. So $\mathcal{B}_\pm \otimes U\big|_p$ is isomorphic, non-canonically, to the sum of two copies of $\mathcal{B}_\mp$. Thus up to isomorphism we have

$$W_p \cdot \mathcal{B}_+ = \mathcal{B}_+ + 2\mathcal{B}_-\,; \qquad W_p \cdot \mathcal{B}_- = \mathcal{B}_- + 2\mathcal{B}_+\,. \qquad (3.6)$$

The magnetic dual of a Wilson operator W_p is an 't Hooft operator T_p. An A-brane A that is an eigenbrane for the 't Hooft operators, in the sense that, for every 't Hooft operator T_p,

$$T_p \cdot \mathrm{A} = \mathrm{A} \otimes V_p \qquad (3.7)$$

for some vector space V_p, is known as a magnetic eigenbrane. Wilson operators of ${}^L G$ gauge theory are classified by a choice of representation of ${}^L G$, and 't Hooft operators of G gauge theory are likewise classified by representations of ${}^L G$. Electric-magnetic duality is expected to map Wilson operators to 't Hooft operators and electric eigenbranes to magnetic eigenbranes.

Let us review the action of an 't Hooft operator T_p on a Higgs bundle (E, φ). In case $\varphi = 0$, the possible Hecke modifications are the usual ones considered in the geometric Langlands program; they are parametrized by a subvariety of the affine Grassmannian known as a Schubert variety $\mathcal{V}$, which depends on a choice of representation ${}^L R$ of the dual group ${}^L G$. For instance, if $G = SL_2$ and ${}^L R$ is the three-dimensional representation of ${}^L G = SO_3$, then a generic point in $\mathcal{V}$ corresponds to a Hecke modification of an SL_2 bundle E of the following sort: for some local decomposition of E as a sum of line bundles $\mathrm{N}_1 \oplus \mathrm{N}_2$, E is mapped to $\mathrm{N}_1(p) \oplus \mathrm{N}_2(-p)$.

Letting N_1 and N_2 vary, this gives a two-parameter family of Hecke modifications of E. A family of modifications of E of this type can degenerate to a trivial modification, and $\mathcal{V}$ contains a point corresponding to the trivial Hecke transformation. If instead $\varphi \neq 0$, one must restrict to Hecke modifications that are in a certain sense φ-invariant. For $G = SL_2$, and assuming φ to be regular semi-simple at the point p, φ-invariance means that the decomposition $E = \mathrm{N}_1 \oplus \mathrm{N}_2$ must be compatible with the action of φ, in the sense that $\varphi : E \to E \otimes K$ maps N_1 to $\mathrm{N}_1 \otimes K$ and N_2 to $\mathrm{N}_2 \otimes K$. These are precisely two possible choices of N_1 and N_2: locally, as $\varphi(p)$ is regular semi-simple, we can diagonalize φ

$$\varphi = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \qquad (3.8)$$

and N_1 and N_2 must equal, up to permutation, the two "eigenspaces".

Now let us see what the φ-invariant Hecke modifications look like from the point of view of the spectral curve $\pi : D \to C$. We consider first the case of a generic spectral curve, given by an equation $\det(z - \varphi) = 0$. A φ-invariant Hecke modification leaves fixed the characteristic polynomial of φ and hence maps each fiber F of the Hitchin fibration to itself.

A point $p \in C$ at which φ is regular semi-simple lies under two distinct points $p', p'' \in D$. The bundle E is $\pi_*(\mathcal{L})$ for some line bundle $\mathcal{L} \to D$, and $\varphi = \pi_*(z)$. The latter condition means that the eigenspaces of $\varphi(p)$ correspond to the two distinct values of z lying above p, or in other words to the two points p' and p''. This being so, a non-trivial φ-invariant Hecke modification of (E, φ) at the point p comes from a transformation of $\mathcal{L}$ of the specific form

$$\mathcal{L} \to \mathcal{L} \otimes \mathrm{O}(p' - p'') \qquad (3.9)$$

for one or another of the two possible labellings of the two points p', p'' lying above p.

Now we can see why an A-brane A_F supported on a fiber F of the Hitchin fibration and endowed with a flat line bundle $\mathcal{R}$ is a magnetic eigenbrane, that is an eigenbrane for the 't Hooft operator

T_p. First of all, T_p maps F to itself, since it preserves the characteristic polynomial of φ. Since T_p preserves the support of A_F, it is conceivable for A_F to be an eigenbrane for T_p. Now, assuming that we choose p so that $\varphi(p)$ is regular semi-simple, the evaluation of $T_p \cdot \mathrm{A}_F$ comes from a sum of contributions from the three φ-invariant Hecke modifications that were just described. One of them is the trivial Hecke modification, and this leaves A_F invariant. The other two come from transformations $\mathcal{L} \to \mathcal{L} \otimes \mathrm{O}(p'-p'')$. Such a transformation can be interpreted as an isomorphism $\Phi : F \to F$ of the Hitchin fiber. If the labelling of the two points p' and p'' is reversed, then Φ is replaced by Φ^{-1}. F is a complex torus, and Φ is a "translation" of F by a constant vector. In general, if $\mathcal{R} \to F$ is a flat line bundle over a complex torus and $\Phi : F \to F$ is a translation, then $\Phi^*(\mathcal{R}) = \mathcal{R} \otimes \mathcal{V}$ for some one-dimensional vector space $\mathcal{V}$. From this it follows that A is an eigenbrane for T_p. In fact, we have

$$T_p \cdot \mathrm{A}_F = \mathrm{A}_F \otimes \left(C \oplus \mathcal{V} \oplus \mathcal{V}^{-1}\right), \qquad (3.10)$$

where the three contributions on the right come respectively from the trivial Hecke modification and the non-trivial modifications that involve Φ and Φ^{-1}.

Now, we consider a special fiber F of the Hitchin fibration that is a union of two irreducible components F_1 and F_2 that intersect each other on a divisor. This being so, we can construct rank 1 A-branes A_1 and A_2 supported on F_1 or F_2. These branes are unique if F_1 and F_2 are simply-connected, as in the case of a curve of genus 1 with 1 point of ramification. In the derivation of eq. (3.10) describing the action of T_p, a key ingredient was the map $\Phi : F \to F$ by $\mathcal{L} \to \mathcal{L} \otimes \mathrm{O}(p'-p'')$. The essential new fact in the case that F is reducible is simply that Φ exchanges the two component of F. Likewise Φ^{-1} exchanges the two components. Hence Φ or Φ^{-1} exchange A_1 and A_2. Since T_p acts by $1 + \Phi + \Phi^{-1}$, it follows that we have up to isomorphism

$$T_p \cdot \mathrm{A}_1 = \mathrm{A}_1 + 2\mathrm{A}_2\,; \qquad T_p \cdot \mathrm{A}_2 = \mathrm{A}_2 + 2\mathrm{A}_1\,. \qquad (3.11)$$

This is in perfect parallel with the formula (3.6) for the electric case.

If A_1 and A_2 have moduli, this should be described a little more precisely. A_1 depends on the choice of a suitable line bundle $\mathcal{L} \to F_1$, and we should take A_2 to be the brane associated with the line bundle $\Phi^*(\mathcal{L}) \to F_2$. Note that $\Phi^*(\mathcal{L})$ and $\left(\Phi^{-1}\right)^*(\mathcal{L})$ are isomorphic, though not canonically so.

One expects to get the more precise result analogous to (3.5). One uses standard methods of algebraic geometry to construct $T_p \cdot \mathrm{A}_1$ and $T_p \cdot \mathrm{A}_2$ as B-branes in complex structure I. This will give a result more precise than (3.11):

$$T_p \cdot \mathrm{A}_1 = \left(\mathrm{A}_1 \otimes \mathcal{J}_1\right) \oplus \left(\mathrm{A}_2 \otimes \mathcal{J}_2\right); \qquad T_p \cdot \mathrm{A}_2 = \left(\mathrm{A}_2 \otimes \mathcal{J}_1^{'}\right) \oplus \left(\mathrm{A}_1 \otimes \mathcal{J}_2^{'}\right), \qquad (3.12)$$

with vector spaces $\mathcal{J}_1, \mathcal{J}_2$, etc., of dimensions indicated by the subscripts. All these admit natural K-valued endomorphisms θ_1, θ_2, etc., coming from the Higgs field, and $(\mathcal{J}_1, \theta_1)$, etc., are Higgs bundles over C. Relating these Higgs bundles to local systems via Hitchin's equations, one expects to arrive at the analog of (3.5),

$$T_p \cdot \mathrm{A}_1 = \left(\mathrm{A}_2 \otimes U\big|_p\right) \oplus \left(\mathrm{A}_1 \otimes \det U\big|_p\right); \qquad T_p \cdot \mathrm{A}_2 = \left(\mathrm{A}_1 \otimes U\big|_p\right) \oplus \left(\mathrm{A}_2 \otimes \det U\big|_p\right). \qquad (3.13)$$

For gauge group SL_2, the basic Wilson operator to consider is the operator $\tilde{W}_p$ associated with the two-dimensional representation. Roughly speaking, it acts by the obvious analog of eq. (3.1). Letting $(\mathrm{E}, \hat{\varphi})$ denote the universal Higgs bundle over $\mathrm{M}_H(SL_2) \times C$, $\tilde{W}_p$ acts on the sheaf K defining a B-brane B by

$$\mathrm{K} \to \mathrm{K} \otimes \mathrm{E}\big|_p \qquad (3.14)$$

where $\mathrm{E}\big|_p$ is the restriction to $\mathrm{M}_H \times p$ of the universal rank two bundle $\mathrm{E} \to \mathrm{M}_H \times C$. $\tilde{W}_p$ obeys

$$\tilde{W}_p^2 = 1 + W_p, \qquad (3.15)$$

expressing the fact that the tensor product of the two-dimensional representation with itself is a direct sum of the trivial representation and the three-dimensional representation; they correspond to the terms 1 and W_p on the right hand side of eq. (3.15).
If we write B and B' for the ordinary and twisted versions of the brane related to the skyscraper sheaf, then the action of the Wilson operator is

$$\tilde{W}_p \cdot \mathrm{B} = \mathrm{B}' \otimes \mathrm{E}\big|_{r \times p}; \qquad \tilde{W}_p \cdot \mathrm{B}' = \mathrm{B} \otimes \mathrm{E}\big|_{r \times p}. \qquad (3.16)$$

The sum $\hat{\mathrm{B}} = \mathrm{B} \oplus \mathrm{B}'$ is therefore an electric eigenbrane in the usual sense:

$$\tilde{W}_p \cdot \hat{\mathrm{B}} = \hat{\mathrm{B}} \otimes \mathrm{E}\big|_{r \times p}. \qquad (3.17)$$

The action of the ‘t Hooft operator $\tilde{T}_p$ on branes $\mathrm{A}_{1,2}$ and $\mathrm{A}^*_{1,2}$ is schematically

$$\tilde{T}_p \cdot \mathrm{A}_1 = \mathrm{A}^*_1 + \mathrm{A}^*_2; \qquad \tilde{T}_p \cdot \mathrm{A}_2 = \mathrm{A}^*_1 + \mathrm{A}^*_2, \qquad (3.18)$$

and similar formulas with A_i and A^*_i exchanged. These formulas and the analogous ones for the action of the ‘t Hooft operator T_p dual to the three-dimensional representation are compatible with the relation

$$\tilde{T}_p^2 = 1 + T_p. \qquad (3.19)$$

This relation is dual to eq. (3.15).

4. On the Hecke eigensheaves and on the notion of “fractional Hecke eigensheaves”. [2]

Let us recall the traditional definition of Hecke eigensheaves used in the geometric Langlands Program.
These are $\mathcal{D}$-modules on Bun_G, the moduli stack of G-bundles on a curve C, satisfying the Hecke eigenobject property. Recall that for each finite-dimensional representation V of the dual group ${}^L G$ we have a Hecke functor H_V acting from the category of $\mathcal{D}$-modules on Bun_G to the category of

$\mathcal{D}$-modules on $C \times \mathrm{Bun}_G$. Let ε be a flat ${}^L G$-bundle on C. A Hecke eigensheaf with "eigenvalue" ε is by definition a collection of data

$$\left(\mathcal{F}, (\alpha_V)_{V \in \mathrm{Rep}({}^L G)}\right), \qquad (4.1)$$

where $\mathcal{F}$ is a $\mathcal{D}$-module on Bun_G and (α_V) is a collection of isomorphisms

$$\alpha_V : H_V(\mathcal{F}) \xrightarrow{\approx} V_\varepsilon \boxtimes \mathcal{F}, \qquad (4.2)$$

where

$$V_\varepsilon = \varepsilon \underset{{}^L G}{\times} V$$

is the flat vector bundle on C associated to V. For a Hecke eigensheaf (4.1), by restricting the isomorphisms α_V to x, we obtain a compatible collection of isomorphisms

$$\alpha_{V,x} : H_{V,x}(\mathcal{F}) \xrightarrow{\approx} V_{\varepsilon,x} \otimes \mathcal{F}. \qquad (4.3)$$

Here

$$V_{\varepsilon,x} = \varepsilon_x \underset{{}^L G}{\times} V,$$

where ε_x is the fiber of ε at x, is a vector space isomorphic to V.

Let us discuss the category of Hecke eigensheaves in our endoscopic example, when $G = SL_2$, ${}^L G = SO_3$, and $\Gamma = Z_2$. We expect that in this case any $\mathcal{D}$-module satisfying the Hecke eigensheaf property is a direct sum of copies of a $\mathcal{D}$-module, which we will denote by $\mathcal{F}$. The $\mathcal{D}$-module $\mathcal{F}$ corresponds to an A-brane A on a singular fiber of $\mathrm{M}_H(G)$, which is a magnetic eigenbrane with respect to the 't Hooft operators. This A-branes is reducible:

$$\mathrm{A} = \mathrm{A}_+ \oplus \mathrm{A}_-,$$

where the A-branes $\mathrm{A}_\pm$ are irreducible. Furthermore, there is not a preferred one among them. Therefore we expect that the $\mathcal{D}$-module $\mathcal{F}$ is also reducible:

$$\mathcal{F} = \mathcal{F}_+ \oplus \mathcal{F}_-,$$

and each $\mathcal{F}_\pm$ is irreducible. We also expect that neither of them is preferred over the other one.

Recall that the notion of an eigensheaf, includes the isomorphisms α_V for all representations V of SO_3. By using the compatibility with the tensor product structure, we find that everything is determined by the adjoint representation of SO_3, which we denote by W, as before. A Hecke eigensheaf may therefore be viewed as a pair $(\mathcal{F}, \alpha)$, where

$$\alpha : H_W(\mathcal{F}) \xrightarrow{\approx} W_\varepsilon \boxtimes \mathcal{F}. \qquad (4.4)$$

In the endoscopic case the structure group of our SO_3-local system ε is reduced to the subgroup $O_2 = Z_2 \propto C^\times$. Denote by U the defining two-dimensional representation of O_2. Then $\det U$ is the one-dimensional sign representation induced by the homomorphism $O_2 \to Z_2$. We have

$$W = (\det U \otimes I) \oplus (U \otimes S),$$

as a representation of $O_2 \times Z_2$, where Z_2 is the centralizer of O_2 in SO_3, and S is the sign representation of Z_2, and I is the trivial representation of Z_2. Therefore we have the following decomposition of the corresponding local system:

$$W_\varepsilon = (\det U_\varepsilon \otimes I) \oplus (U_\varepsilon \otimes S). \qquad (4.5)$$

Now we suppose that we are given an ${}^LG$-local system ε on a curve C, and let Γ be the group of its automorphisms. We will identify Γ with a subgroup of ${}^LG$ by picking a point $x \in C$ and choosing a trivialization of the fiber ε_x of ε at x.
Suppose that we are given an abelian subcategory $\mathcal{C}$ of the category of $\mathcal{D}$-modules on Bun_G equipped with an action of the tensor category $\mathrm{Rep}(\Gamma)$. In other words, for each $R \in \mathrm{Rep}(\Gamma)$ we have a functor

$$M \mapsto R * M,$$

and these functors compose in a way compatible with the tensor product structure on $\mathrm{Rep}(\Gamma)$. The category of Hecke eigensheaves with eigenvalue ε will have as objects the following data:

$$(\mathcal{F}, (\alpha_V)_{V \in \mathrm{Rep}({}^LG)}), \qquad (4.6)$$

where $\mathcal{F}$ is an object of $\mathcal{C}$, and the α_V are isomorphisms defined below. Denote by $\mathrm{Res}_\Gamma(V)$ the restriction of a representation V of ${}^LG$ to Γ. If $\mathrm{Rep}(\Gamma)$ is a semi-simple category, then we obtain a decomposition

$$\mathrm{Res}_\Gamma(V) = \bigoplus_i F_i \otimes R_i,$$

where the R_i are irreducible representations of Γ and F_i is the corresponding representation of the centralizer of Γ in ${}^LG$. Twisting by ε, we obtain a local system $(\mathrm{Res}_\Gamma(V))_\varepsilon$ on C with a commuting action of Γ, which decomposes as follows:

$$(\mathrm{Res}_\Gamma(V))_\varepsilon = \bigoplus_i (F_i)_\varepsilon \otimes R_i.$$

Note that since Γ is the group of automorphisms of ε, the structure group of ε is reduced to the centralizer of Γ in G, and F_i is a representation of this centralizer. Therefore F_i may be twisted by ε, and the resulting local system (or flat vector bundle) on C is denoted by $(F_i)_\varepsilon$. The isomorphisms α_V have the form

$$\alpha_V : H_V(M) \xrightarrow{\approx} (\mathrm{Res}_\Gamma(V))_\varepsilon * M = \bigoplus_i (F_i)_\varepsilon \boxtimes (R_i * M), \qquad M \in \mathcal{C}, \qquad (4.7)$$

and they have to be compatible in the obvious sense. We will denote the category with objects (4.6) satisfying the above conditions by $\mathrm{A}ut'_{\varepsilon}$. The category of Hecke eigensheaves of this type matches more closely the structure of the categories of A- and B-branes. What does the category $\mathrm{A}ut'_{\varepsilon}$ look like in our main example of geometric endoscopy? In this case the category $\mathcal{C}$ should have two irreducible objects, $\mathcal{F}_{+}$ and $\mathcal{F}_{-}$, which are the $\mathcal{D}$-modules on Bun_G corresponding to the fractional A-branes A_{+} and A_{-}. The category $\mathrm{Rep}(Z_2)$ acts on them as follows: the sign representation S of $\Gamma = Z_2$ permutes them,

$$S * \mathcal{F}_{\pm} = \mathcal{F}_{\mp},$$

while the trivial representation I acts identically.

Since the category of representations of SO_3 is generated by the adjoint representation W, it is sufficient to formulate the Hecke property (4.7) only for the adjoint representation W of SO_3. It reads

$$H_W(\mathcal{F}_{+}) \cong (\det U_{\varepsilon} \boxtimes \mathcal{F}_{+}) \oplus (U_{\varepsilon} \boxtimes \mathcal{F}_{-}), \qquad H_W(\mathcal{F}_{-}) \cong (\det U_{\varepsilon} \boxtimes \mathcal{F}_{-}) \oplus (U_{\varepsilon} \boxtimes \mathcal{F}_{+}), \qquad (4.8)$$

where $\det U_{\varepsilon}$ and U_{ε} are the summands of W_{ε} defined in formula (4.5). This matches the action of the ‘t Hooft operators on the A-branes given by formula (3.13). Since that formula describes the behaviour of the fractional branes $\mathrm{A}_{\pm}$, we will call the property expressed by formulas (4.7) and (4.8) the *fractional Hecke property*, and the corresponding $\mathcal{D}$-modules *fractional Hecke eigensheaves*.

5. On some equations concerning the local and global Langlands correspondence. [2]

The ring A_F of **adéles** of F is by definition the restricted product

$$A_F = \prod_{x \in C}{}' F_x, \qquad (5.1)$$

where the word “restricted” refers to fact that **elements of A_F are collections $(f_x)_{x \in C}$, where $f_x \in \mathrm{O}_x$ for all but finitely many $x \in C$. Let $Gal(\overline{F}/F)$ be the Galois group of F, the group of automorphisms of the separable closure $\overline{F}$ of F, which preserve F pointwise**. We have a natural homomorphism $Gal(\overline{F}/F) \to Gal(\overline{k}/k)$. The group $Gal(\overline{k}/k)$ is topologically generated by the Frobenius automorphism $Fr : y \mapsto y^q$, and is isomorphic to the pro-finite completion $\hat{Z}$ of the group of integers Z. The preimage of $Z \subset \hat{Z}$ in $Gal(\overline{F}/F)$ is the Weil group W_F of F. The Weil group is the arithmetic analogue of the fundamental group of C. Therefore the arithmetic analogue of a ${}^LG$-local system on C is a homomorphism

$$\sigma : W_F \to {}^LG. \qquad (5.2)$$

The global Langlands conjecture predicts, roughly speaking, that to each σ corresponds an automorphic representation $\pi(\sigma)$ of the group $G(A_F)$. This means that it may be realized in a certain space of functions on the quotient $G(F) \backslash G(A_F)$. The group W_{F_x} may be realized as a subgroup of the global Weil group W_F, but non-canonically. However, its conjugacy class in W_F is canonical. Hence the equivalence class of $\sigma : W_F \to {}^L G$ as above gives rise to an equivalence class of homomorphisms

$$\sigma_x : W_{F_x} \to {}^L G . \qquad (5.3)$$

We have the infinite-dimensional representation $\pi = \bigotimes'_{x \in C} \pi_x$ of $G(A_F)$. (**We note that** A_F **is the ring of adéles of** F). We take K to be the product

$$K = \prod_{x \in C} K_x$$

of compact subgroups $K_x \subset G(F_x) \cong G((t))$. A typical example is the subgroup $G(\mathrm{O}_x) = G[[t]]$. Any vector in π is invariant under the subgroup that is the product of $G(\mathrm{O}_x)$ for all but finitely many x. If π is automorphic, then π^K is realized in the space of functions on the double quotient $G(F) \backslash G(A_F) / K$, which are Hecke eigenfunctions for all $x \in C$ for which $K_x = G(\mathrm{O}_x)$.

THEOREM 1

The representation

$$\bigotimes_{x \in C}{}' \pi_x \qquad (5.4)$$

of $SL_2(A_F)$, ***where*** A_F ***is the ring of adéles of*** F *, is an automorphic representation if and only if* $\# S$ *is even.*

Denote by S_σ the group of automorphisms of our homomorphism $\sigma : W_F \to PGL_2$, that is, the centralizer of the image of σ in PGL_2. Let S_σ^0 be its connected component. Likewise, for each $x \in C$, let S_{σ_x} be the group of automorphisms of $\sigma_x : W_F \to PGL_2$ and $S_{\sigma_x}^0$ its connected component. We have natural homomorphisms $S_\sigma \to S_{\sigma_x}$ and $S_\sigma^0 \to S_{\sigma_x}^0$, and hence a homomorphism

$$S_\sigma / S_\sigma^0 \to S_{\sigma_x} / S_{\sigma_x}^0 . \qquad (5.5)$$

In our case, for generic σ in the class that we are considering here we have $S_\sigma = S_\sigma / S_\sigma^0 = Z_2$, generated by the element

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (5.6)$$

of PGL_2 (this is the centralizer of $O_2 \subset PGL_2$).
The Theorem 1 may then be reformulated as saying that (5.4) is automorphic if and only if S_σ / S_σ^0 acts trivially on the corresponding representation of $\prod_{x \in C} S_{\sigma_x} / S_{\sigma_x}^0$, via the diagonal homomorphism

$$S_\sigma / S_\sigma^0 \to \prod_{x \in C} S_{\sigma_x} / S_{\sigma_x}^0 . \qquad (5.7)$$

Note that, according to the above discussion, if x is split, then the homomorphism (5.5) has trivial image, even if the group $S_{\sigma_x}/S^0_{\sigma_x}$ is non-trivial. Therefore we may choose either of the two irreducible representations of $SL_2(F_x)$ from the local L-packet associated to such a point as π_x, and in both cases the corresponding representations (5.4) will simultaneously be automorphic or not. In this sense, the split points do not affect the automorphy of the representation (5.4), unlike the non-split points, for which it is crucial which one of the two members of the L-packet we choose as the local factor of (5.4).
Suppose now that $\# S$ is even and so the representation (5.4) is automorphic. Then the one-dimensional vector space

$$\bigotimes_{x\in C}(\pi_x)^{K_x} \qquad (5.8)$$

may be realized in the space of functions on

$$SL_2(F)\backslash SL_2(A_F)/\prod_{x\in C}K_x , \qquad (5.9)$$

where A_F is the ring of adéles of F and, for the eq. (5.1) the eq. (5.9) can be written also

$$SL_2(F)\backslash SL_2\left(\prod_{x\in C}{}'F_x\right)/\prod_{x\in C}K_x . \qquad (5.9b)$$

Moreover, any non-zero vector in (5.8) gives rise to a Hecke eigenfunction f on (5.9) with the eigenvalues prescribed by the conjugacy class $\sigma_x(Fr_x)$. This means that it is an eigenfunction of the Hecke operator $T_{W,x}$ corresponding to the adjoint representation W of PGL_2 and a point $x\in C$, that is,

$$T_{W,x}\cdot f = Tr(\sigma_x(Fr_x),W)f , \qquad (5.10)$$

where Fr_x is the Frobenius conjugacy class corresponding to x in W_F. Here $T_{W,x}$ is a generator of the spherical Hecke algebra of K_x bi-invariant compactly supported functions on $SL_2(F_x)$. For either choice of K_x this algebra is canonically isomorphic to $\mathrm{Rep}(PGL_2)$, and under this isomorphism $T_{W,x}$ corresponds to the class of the adjoint representation of PGL_2.

6. On some equations concerning the automorphic functions associated to the fractional Hecke eigensheaves. [2]

Now we replace a complex curve by a curve C defined over a finite field $k = F_q$. We have the moduli stack Bun_G of G-bundles on our curve C defined over k. This is an algebraic stack over k. Therefore we have the notion of a Hecke eigensheaf on Bun_G corresponding to an unramified homomorphism $\sigma : W_F \rightarrow {}^L G$. **We view σ as an ℓ-adic ${}^L G$-local system ε on C. Hence, for each representation V of ${}^L G$ the corresponding twist**

$$V_\varepsilon = \varepsilon \underset{{}^L G}{\times} V$$

is a locally constant ℓ-adic sheaf on C, and these sheaves are compatible with respect to the tensor product structure on representations of ${}^LG$. We also have Hecke functors H_V, $V \in \mathrm{Rep}({}^LG)$, defined in the same way as over C. **A Hecke eigensheaf with "eigenvalue" ε (or σ) is, by definition, a perverse (ℓ-adic) sheaf $\mathcal{F}$ on Bun$_G$ together with the additional data of isomorphisms**

$$\alpha_V : H_V(\mathcal{F}) \xrightarrow{\approx} V_\varepsilon \boxtimes \mathcal{F}. \qquad (6.1)$$

We recall that for any algebraic variety Y over F_q, we may assign a function on the set of F_q-points of Y to any **ℓ-adic sheaf** (or a complex) **$\mathcal{F}$ on** Y.
Indeed, let y be an F_q-point of Y and $\bar{y}$ the $\overline{F}_q$-point corresponding to an inclusion $F_q \mapsto \overline{F}_q$. **Then the pull-back of $\mathcal{F}$ with respect to the composition $\bar{y} \to y \to Y$ is a ℓ-adic sheaf on a point Spec $\overline{F}_q$**. The data of such a sheaf is the same as the data of a $\overline{Q}_\ell$-vector space, which we may think of as the stalk $\mathcal{F}_{\bar{y}}$ of $\mathcal{F}$ at $\bar{y}$. There is an additional piece of data on this vector space. Indeed, the Galois group $Gal(\overline{F}_q / F_q)$ is the symmetry group of the morphism $\bar{y} \to y$, and therefore it acts on $\mathcal{F}_{\bar{y}}$. In particular, we have an action of the (geometric) Frobenius element Fr_y, corresponding (the inverse of) the generator of the Galois group of F_q, acting as $x \mapsto x^q$. This automorphism depends on the choice of the morphism $\bar{y} \to y$, but its conjugacy class is independent of any choices. Thus, we obtain a conjugacy class of automorphisms of the stalk $\mathcal{F}_{\bar{y}}$. Therefore the trace of the geometric Frobenius automorphism is canonically assigned to $\mathcal{F}$ and y. We will denote it by $Tr(Fr_y, \mathcal{F})$. **If $\mathcal{F}$ is a complex of ℓ-adic sheaves, we take the alternating sum of the traces of Fr_y on the stalk cohomologies of $\mathcal{F}$ at $\bar{y}$**. Hence we obtain a function $\mathrm{f}_{\mathcal{F},F_q}$ on the set of F_q-points of Y, whose value at $y \in Y(F_q)$ is

$$\mathrm{f}_{\mathcal{F},F_q}(y) = \sum_i (-1)^i Tr\left(Fr_y, H^i_{\bar{y}}(\mathcal{F})\right).$$

Similarly, for each $n > 1$ we define a function $\mathrm{f}_{\mathcal{F},F_{q^n}}$ on the set of F_{q^n}-points of Y by the formula

$$\mathrm{f}_{\mathcal{F},F_{q^n}}(y) = \sum_i (-1) Tr\left(Fr_y, H^i_{\bar{y}}(\mathcal{F})\right), \qquad y \in Y(F_{q^n}). \qquad (6.2)$$

If $Y = \mathrm{Bun}_G$, then the set of F_q-points of Y is naturally isomorphic to the double quotient

$$G(F) \backslash G(A_F) / G\left(\prod_{x \in C} \mathrm{O}_x\right). \qquad (6.3)$$

(Also here A_F **is the ring of adéles of** F). Therefore any perverse sheaf $\mathcal{F}$ on Bun$_G$ gives rise to a function $\mathrm{f}_{\mathcal{F},F_q}$ on the double quotient (6.3). Suppose now that $(\mathcal{F},(\alpha_V))$ is a Hecke eigensheaf on Bun$_G$. Consider the corresponding function $\mathrm{f}_{\mathcal{F},F_q}$ on the set Bun$_G(F_q)$, isomorphic to the double quotient (6.3), and its transform under the Hecke functor H_V, restricted to

$$\left(C\times Bun_G\right)\left(F_q\right)=C\left(F_q\right)\times Bun_G\left(F_q\right).$$

The action of the Hecke functor H_V on sheaves becomes the action of the corresponding Hecke operators $T_{V,x}$ on functions. Hence for each $x\in C\left(F_q\right)$ the left hand side of (6.1) gives rise to the function $T_{V,x}\cdot \mathrm{f}_{\mathcal{F},F_q}$, whereas the right hand side becomes $Tr\left(Fr_x,V_\varepsilon\right)\mathrm{f}_{\mathcal{F},F_q}$. Hence the isomorphism (6.1) implies that

$$T_{V,x}\cdot \mathrm{f}_{\mathcal{F},F_q}=Tr\left(Fr_x,V_\varepsilon\right)\mathrm{f}_{\mathcal{F},F_q}=Tr\left(\sigma_x\left(Fr_x\right),V\right)\mathrm{f}_{\mathcal{F},F_q},\quad \forall x\in C\left(F_q\right). \qquad (6.4)$$

We have that the A-brane A corresponding to $\mathcal{F}$, which is represented by a rank one unitary local system on the singular Hitchin fiber, which has two irreducible components. We have that A splits into two A-branes, A_+ and A_- supported on the two irreducible components of the Hitchin fiber. Therefore we expect that the $\mathcal{D}$-module $\mathcal{F}$ also splits into a direct sum,

$$\mathcal{F}=\mathcal{F}_+\oplus\mathcal{F}_-, \qquad (6.5)$$

of two irreducible $\mathcal{D}$-modules on Bun_{SL_2} corresponding to the two A-branes on the singular Hitchin fiber. Moreover, since the A-branes $\mathrm{A}_\pm$ are fractional eigenbranes with respect to the ‘t Hooft operators, we expect that the sheaves $\mathcal{F}_\pm$ satisfy the fractional Hecke property introduced precedently.
This leads us to postulate that *the same phenomenon should also occur for curves over a finite field* F_q. Namely, **the regular Hecke eigensheaf $\mathcal{F}$ corresponding to an ℓ-adic local system ε on a curve C defined over F_q, should also split as a direct sum (6.5)**. Moreover, these sheaves should satisfy the fractional Hecke property and hence give rise to a category of fractional Hecke eigensheaves. **In the setting of curves over finite fields we can pass from ℓ-adic perverse sheaves on Bun_{SL_2}, to functions**. Then, we started with A-brines and ended up with automorphic functions satisfying the fractional Hecke property. Schematically, this passage looks as follows :

$$A\text{-branes}\overset{overC}{\rightarrow}\mathcal{D}\text{-modules}\overset{overC}{\rightarrow}\text{perverses sheaves}\overset{overF_q}{\rightarrow}\text{functions.}$$

Let C be a curve over F_q and ε and endoscopic ℓ-adic PGL_2-local system on C (corresponding to an unramified homomorphism $\sigma:W_F\rightarrow PGL_2$). This means that its structure group is reduced to O_2, but not to a proper subgroup. Then the group of automorphisms of ε is Z_2. Let D be a finite set of closed points of C. Denote by $\mathcal{F}^D$ a regular Hecke eigensheaf on $Bun_{SL_2}^{\mathrm{O}(D)}$ with the “eigenvalue” ε. Motivated by our results on A-branes in the analogous situation for curves over C, we conjecture that $\mathcal{F}^D$ splits as a direct sum

$$\mathcal{F}^D=\mathcal{F}_+^D\oplus\mathcal{F}_-^D \qquad (6.6)$$

of perverse sheaves $\mathcal{F}_\pm^D$ which satisfy the following fractional Hecke property with respect to ε (and so we also call them the *fractional Hecke eigensheaves*):

$$\alpha_+:H_W\left(\mathcal{F}_+^D\right)\overset{\approx}{\rightarrow}\left(\det U_\varepsilon\boxtimes\mathcal{F}_+^D\right)\oplus\left(U_\varepsilon\boxtimes\mathcal{F}_-^D\right), \qquad (6.7)$$

$$\alpha_- : H_W\left(\mathcal{F}_-^D\right) \xrightarrow{\approx} \left(U_\varepsilon \boxtimes \mathcal{F}_+^D\right) \oplus \left(\det U_\varepsilon \boxtimes \mathcal{F}_-^D\right). \qquad (6.8)$$

Here W is the adjoint representation of PGL_2 and we use the decomposition of the rank three local system W_ε on C with respect to the action of its group Z_2 of automorphisms as in formula (4.5),

$$W_\varepsilon = \left(\det U_\varepsilon \otimes I\right) \oplus \left(U_\varepsilon \otimes S\right), \qquad (6.9)$$

where I and S are trivial and sign representations of Z_2, respectively, and $\det U_\varepsilon$ and U_ε are the rank one and two local systems on C defined as follows.
Recall that by our assumption the PGL_2-local system ε is reduced to O_2, so we view it as an O_2-local system. We then set

$$U_\varepsilon = \varepsilon \underset{O_2}{\times} U,$$

where U is the defining two-dimensional representation of O_2. The formula (6.6) implies that

$$f^D = f_+^D + f_-^D, \qquad (6.10)$$

where $f^D = \mathrm{f}_{\mathcal{F}^D, F_q}$ is the function on $Bun_{SL_2}^{\mathrm{O}(D)}$ associated to the regular Hecke eigensheaf $\mathcal{F}^D$.
Furthermore, the functions corresponding to the fractional Hecke eigensheaves $\mathcal{F}_\pm$ are:

$$f_\pm = \frac{1}{2}\left(f \pm f'\right). \qquad (6.11)$$

In addition to the "proper" Hecke functors H_W acting on the categories of $\mathcal{D}$-modules on $Bun_{SL_2}^{\mathrm{O}(D)}$, there are also "improper" Hecke functors $\widetilde{H}_x$ acting from the category of $\mathcal{D}$-modules on $Bun_{SL_2}^{\mathrm{O}(D)}$ to the category of $\mathcal{D}$-modules on $Bun_{SL_2}^{\mathrm{O}(D+x)}$. They are defined via the Hecke correspondence between the two moduli stacks consisting of pairs of rank two bundles $\mathrm{M} \in Bun_{SL_2}^{\mathrm{O}(D)}$ and $\mathrm{M}' \in Bun_{SL_2}^{\mathrm{O}(D+x)}$ such that $\mathrm{M} \subset \mathrm{M}'$ as a coherent sheaf.
In formula (3.18) we have computed the action of the improper 't Hooft operators on the branes $\mathrm{A}_\pm$. Based in this formula, we conjecture that the improper Hecke operators should act on the fractional Hecke eigensheaves $\mathcal{F}_\pm^D$ as follows:

$$\widetilde{H}_x\left(\mathcal{F}_+^D\right) \cong \mathcal{F}_+^{D+x} \oplus \mathcal{F}_-^{D+x}, \qquad \widetilde{H}_x\left(\mathcal{F}_-^D\right) \cong \mathcal{F}_+^{D+x} \oplus \mathcal{F}_-^{D+x}. \qquad (6.12)$$

This should hold for all points $x \in C$ if C is defined over C, and all split points, if C is defined over F_q. This formula indicates that the improper Hecke functor fails to establish an equivalence between the categories of fractional Hecke eigensheaves on $Bun_{SL_2}^{\mathrm{O}(D)}$ and $Bun_{SL_2}^{\mathrm{O}(D+x)}$ for the endoscopic local systems. This may be viewed as a geometric counterpart of the vanishing of the improper Hecke operator acting on functions which is closely related to the structure of the global L-packets of automorphic representations associated to endoscopic $\sigma : W_F \to PGL_2$.
For a regular Hecke eigensheaf $\mathcal{F}^D = \mathcal{F}_+^D \oplus \mathcal{F}_-^D$ we have

$$\widetilde{H}_x\left(\mathcal{F}^D\right) \cong V \otimes \mathcal{F}^{D+x}, \qquad (6.13)$$

where V is a two-dimensional vector space.
Recall that the functions $f^D_{\sigma,\pm}$, corresponding to the sheaves $\mathcal{F}^D_{\sigma,\pm}$ on $Bun^{O(D)}_{SL_2}$, are given by formula (6.11),

$$f^D_{\sigma,\pm} = \frac{1}{2}\left(f^D_{\sigma} \pm f^D_{\sigma'}\right). \qquad (6.14)$$

With regard the fractional Hecke eigensheaves, we now revisit them in the case when the underlying curve C is defined over F_q. It is instructive to look at the corresponding functions on the sets of F_q-points of Bun_G and to express them in terms of the ordinary Hecke eigenfunctions, the way we did in the endoscopic example for $G = SL_2$ above (see formula (6.14)).
Consider first the case when G is a one-dimensional torus. The corresponding moduli space, the Picard variety Pic, breaks into connected components $\mathrm{Pic}_n, n \in Z$, and **the Hecke eigensheaf $\mathcal{F}_\sigma$ corresponding to a one-dimensional ℓ-adic representation σ of the Weil group W_F, breaks into a direct sum**

$$\mathcal{F}_\sigma = \bigoplus_{n \in Z} \mathcal{F}_{\sigma,n}, \qquad (6.15)$$

where $\mathcal{F}_{\sigma,n}$ is supported on Pic_n. This is an analogue of the decomposition (6.6). Let f_σ (resp., $f_{\sigma,n}$) be the function on $\mathrm{Pic}\left(F_q\right)$ (resp., $\mathrm{Pic}_n\left(F_q\right)$) corresponding to $\mathcal{F}_\sigma$ (resp., $\mathcal{F}_{\sigma,n}$). Then we have

$$f_\sigma = \sum_{n \in Z} f_{\sigma,n}.$$

This is analogous to formula (6.10). We now wish to express the functions $f_{\sigma,n}$ in terms of (ordinary) Hecke eigenfunctions $f_{\sigma'}$, similarly to formula (6.14).
This is achieved by a simple Fourier transform. Namely, for each non-zero number $\gamma \in C^\times$ (in what follows we identify $\overline{Q}_\ell$ with C) we define a one-dimensional representation α_γ of W_F as the composition of the homomorphism

$$res : W_f \to W_{F_q} = Z, \qquad (6.16)$$

obtained by restricting to the scalars $F_q \subset F$, and the homomorphism

$$Z \to C^\times, \qquad 1 \mapsto \gamma.$$

Now let $\sigma_\gamma = \sigma \otimes \alpha_\gamma$ be the twist of σ by α_γ. Then we have the following obvious formula

$$f_{\sigma,n} = \frac{1}{2\pi i}\int_{|\gamma|=1} f_{\sigma_\gamma} \gamma^{-n-1} d\gamma, \qquad (6.17)$$

expressing the functions $f_{\sigma,n}$ as integrals of the ordinary Hecke eigenfunctions corresponding to the twists σ_γ of σ by $\alpha_\gamma, |\gamma| = 1$.

In our most detailed example of A-branes corresponding to the elliptic curves in Section 2, we have considered some equations concerning a slightly different moduli space corresponding to ramified Higgs bundles. In this case the relevant moduli stack is $Bun^{\mathrm{O}(D)}_{SL_2,I_p}$ which parametrizes rank two vector bundles on C with determinant $\mathrm{O}(D)$ and a parabolic structure at a fixed point p of C. It is instructive to look at how the story with L-packets plays out in this case.

Let C be again defined over F_q. Then the set of F_q-points of $Bun^{\mathrm{O}(D)}_{SL_2,I_p}$ is isomorphic to the double quotient

$$SL_2(F)\setminus SL_2(A_F)/\left(\prod_{x\neq p} K_x \times I_p\right). \qquad (6.18)$$

(Note that also here A_F **is the ring of adéles of** F). Here $I_p = K_p^{'} \cap K_p^{''}$ is the Iwahori subgroup of $SL_2(F_p)$, and $K_x = K_x^{''}$ for $x \in D$, $K_x = K_x^{'}$, otherwise. Let us suppose that p is a non-split point of C, with respect to the unramified covering $C' \to C$ affiliated with an unramified homomorphism $\sigma : W_F \to O_2$. Then the local L-packet corresponding to p and a homomorphism $\sigma : W_F \to PGL_2$ constructed as above consists of two irreducible representations, $\pi_p^{'}$ and $\pi_p^{''}$, but now both $(\pi_p^{'})^{I_p}$ and $(\pi_p^{''})^{I_p}$ are one-dimensional.

Let us fix the local factors $\pi_x, x \neq p$. Then we have two non-isomorphic irreducible representations of $SL_2(A_F)$,

$$\bigotimes_{x\neq p} \pi_x \otimes \pi_p^{'} \qquad \text{and} \qquad \bigotimes_{x\neq p} \pi_x \otimes \pi_p^{''}.$$

According to Theorem 1, only one of them is automorphic; that is, may be realized as a constituent of an appropriate space of functions on $SL_2(F)\setminus SL_2(A_F)$. However, their spaces of invariants with respect to the subgroup

$$\prod_{x\neq p} K_x \times I_p$$

are both one-dimensional. Therefore no matter which one of them is automorphic, we will have a one-dimensional space of Hecke eigenfunctions on the double quotient (6.18). Thus, the function on the set of F_q-points of $Bun^{\mathrm{O}(D)}_{SL_2,I_p}$ associated to a regular Hecke eigensheaf will be non-zero. Then, we obtain that the functions $f_{\pm}^{D}$ associated to fractional Hecke eigensheaves are also non-zero in this case.

7. On some equations concerning the modular elliptic curves belonging the proof of Fermat's Last Theorem. [3]

Let $T_1(N,q)$ be the ring of endomorphism of $J_1(N,q)$ generated by the standard Hecke operators. One can check that U_p preserves B either by an explicit calculation or by noting that B is the maximal abelian subvariety of $J_1(N,q)$ with multiplicative reduction at q. We set $J_2 = J_1(N)\times J_1(N)$. More generally, one can consider $J_H(N)$ and $J_H(N,q)$ in place of $J_1(N)$ and $J_1(N,q)$ (where $J_H(N,q)$ corresponds to $X_1(N,q)/H$) and we write $T_H(N)$ and $T_H(N,q)$ for the associated Hecke rings.

In the following lemma if m is a maximal ideal of $\mathrm{T}_1(Nq^{r-1})$ or $\mathrm{T}_1(Nq^r)$ we use $m^{(q)}$ to denote the maximal ideal of $\mathrm{T}_1^{(q)}(Nq^r, q^{r+1})$ compatible with m, the ring $\mathrm{T}_1^{(q)}(Nq^r, q^{r+1}) \subset \mathrm{T}_1(Nq^r, q^{r+1})$ being the sub-ring obtained by omitting U_q from the list of generators.

LEMMA 1.

If $q \neq p$ is a prime and $r \geq 1$ then the sequence of abelian varieties

$$0 \to J_1(Nq^{r-1}) \xrightarrow{\xi_1} J_1(Nq^r) \times J_1(Nq^r) \xrightarrow{\xi_2} J_1(Nq^r, q^{r+1}) \quad (7.1)$$

where $\xi_1 = \left((\pi_{1,r} \circ \pi)^*, -(\pi_{2,r} \circ \pi)^*\right)$ and $\xi_2 = \left(\pi_{4,r}^*, \pi_{3,r}^*\right)$ induces a corresponding sequence of p-divisible groups which becomes exact when localized at any $m^{(q)}$ for which ρ_m is irreducible.

Now, we have the following theorem:

THEOREM 2.

Assume that ρ_0 is modular and absolutely irreducible when restricted to $Q\left(\sqrt{(-1)^{\frac{p-1}{2}} p}\right)$. Assume also that ρ_0 is of type (A), (B) or (C) at each $q \neq p$ in Σ. Then the map $\varphi_D : R_D \to \mathrm{T}_D$ (remember that φ_D is an isomorphism) is an isomorphism for all D associated to ρ_0, i.e., where $D = (\cdot, \Sigma, \mathrm{O}, \mathrm{M})$ with $\cdot =$Se, str, fl or ord. In particular if $\cdot =$Se, str or fl and f is any newform for which $\rho_{f,\lambda}$ is a deformation of ρ_0 of type D then

$$\# H_D^1(Q_\Sigma / Q, V_f) = \#(\mathrm{O}/\eta_{D,f}) < \infty \quad (7.2)$$

where $\eta_{D,f}$ is the invariant defined in the following equation $(\eta) = (\eta_{D,f}) = (\hat{\pi}(1))$.

We assume that

$$\rho = Ind_L^Q \kappa : Gal(\overline{Q}/Q) \to GL_2(\mathrm{O}) \quad (7.3)$$

is the p-adic representation associated to a character $\kappa : Gal(\overline{L}/L) \to \mathrm{O}^\times$ of an imaginary quadratic field L.

Let M_∞ be the maximal abelian p-extension of $L(\nu)$ unramified outside p.

PROPOSITION 1.

There is an isomorphism

$$H_{unr}^1(Q_\Sigma / Q, Y^*) \xrightarrow{\approx} Hom(Gal(M_\infty / L(\nu)), (K/\mathrm{O})(\nu))^{Gal(L(\nu)/L)} \quad (7.4)$$

where H_{unr}^1 denotes the subgroup of classes which are Selmer at p and unramified everywhere else.

Now we write $H^1_{str}\left(Q_\Sigma/Q,Y_n^*\right)$ (where $Y_n^*=Y^*_{\lambda^n}$ and similarly for Y_n) for the subgroup of $H^1_{unr}\left(Q_\Sigma/Q,Y_n^*\right)=\left\{\alpha\in H^1_{unr}\left(Q_\Sigma/Q,Y_n^*\right):\alpha_p=0 in H^1\left(Q_p,Y_n^*/\left(Y_n^*\right)^0\right)\right\}$ where $\left(Y_n^*\right)^0$ is the first step in the filtration under D_p, thus equal to $\left(Y_n/Y_n^0\right)^*$ or equivalently to $\left(Y^*\right)^0_{\lambda^n}$ where $\left(Y^*\right)^0$ is the divisible submodule of Y^* on which the action of I_p is via ε^2. It follows from an examination of the action I_p on Y_λ that

$$H^1_{str}\left(Q_\Sigma/Q,Y_n\right)=H^1_{unr}\left(Q_\Sigma/Q,Y_n\right). \quad (7.5)$$

In the case of Y^* we will use the inequality

$$\#H^1_{str}\left(Q_\Sigma/Q,Y^*\right)\leq\#H^1_{unr}\left(Q_\Sigma/Q,Y^*\right). \quad (7.6)$$

Furthermore, for n sufficiently large the map

$$H^1_{str}\left(Q_\Sigma/Q,Y_n^*\right)\rightarrow H^1_{str}\left(Q_\Sigma/Q,Y^*\right) \quad (7.7)$$

is injective.
The above map is then injective whenever the connecting homomorphism

$$H^0\left(L_{p^*},(K/\mathrm{O})(\nu)\right)\rightarrow H^1\left(L_{p^*},(K/\mathrm{O})(\nu)_{\lambda^n}\right)$$

is injective, which holds for sufficiently large n. Furthermore, we have

$$\frac{\#H^1_{str}\left(Q_\Sigma/Q,Y_n\right)}{\#H^1_{str}\left(Q_\Sigma/Q,Y_n^*\right)}=\#H^0\left(Q_p,\left(Y_n^0\right)^*\right)\frac{\#H^0\left(Q,Y_n\right)}{\#H^0\left(Q,Y_n^*\right)}. \quad (7.8)$$

Thence, setting $t=\inf_q\#\left(\mathrm{O}/\left(1-\nu(q)\right)\right)$ if $\nu \bmod \lambda=1$ or $t=1$ if $\nu \bmod \lambda\neq1$ (7.8b), we get

$$\#H^1_{Se}\left(Q_\Sigma/Q,Y\right)\leq\frac{1}{t}\cdot\prod_{\in\Sigma}\ell_q\cdot\#Hom\left(Gal\left(M_\infty/L(\nu)\right),(K/\mathrm{O})(\nu)\right)^{Gal(L(\nu)/L)} \quad (7.9)$$

where $\ell_q=\#H^0\left(Q_q,Y^*\right)$ for $q\neq p$, $\ell_p=\lim_{n\to\infty}\#H^0\left(Q_p,\left(Y_n^0\right)^*\right)$. This follows from Proposition 1, (7.5)-(7.8) and the elementary estimate

$$\#\left(H^1_{Se}\left(Q_\Sigma/Q,Y\right)/H^1_{unr}\left(Q_\Sigma/Q,Y\right)\right)\leq\prod_{q\in\Sigma-\{p\}}\ell_q, \quad (7.10)$$

which follows from the fact that $\#H^1\left(Q_q^{unr},Y\right)^{Gal\left(Q_q^{unr}/Q_q\right)}=\ell_q$. (Remember that ℓ is the ℓ-adic representation).
Let w_f denote the number of roots of unity ζ of L such that $\zeta\equiv1 \bmod f$ (f an integral ideal of O_L). We choose an f prime to p such that $w_f=1$. Then there is a grossencharacter φ of L satisfying $\varphi((\alpha))=\alpha$ for $\alpha\equiv1 \bmod f$. According to Weil, after fixing an embedding $\overline{Q}\mapsto\overline{Q}_p$ we

can associate a p-adic character φ_p to φ. We choose an embedding corresponding to a prime above p and then we find $\varphi_p = \kappa \cdot \chi$ for some χ of finite order and conductor prime to p.
The grossencharacter φ (or more precisely $\varphi \circ N_{F/L}$) is associated to a (unique) elliptic curve E defined over $F = L(f)$, the ray class field of conductor f, with complex multiplication by O_L and isomorphic over C to C/O_L. We may even fix a Weierstrass model of E over O_F which has good reduction at all primes above p. For each prime B of F above p we have a formal group $\hat{E}_B$, and this is a relative Lubin-Tate group with respect to F_B over L_p. We let $\lambda = \lambda_{\hat{E}_B}$ be the logarithm of this formal group.
Let U_∞ be the product of the principal local units at the primes above p of $L(fp^\infty)$; i.e.,

$$U_\infty = \prod_{B|p} U_{\infty,B} \quad \text{where} \quad U_{\infty,B} = \lim_{\leftarrow} U_n, B .$$

To an element $u = \lim_{\leftarrow} u_n \in U_\infty$ we can associate a power series $f_{u,B}(T) \in O_B[T]^\times$ where O_B is the ring of integers of F_B. For B we will choose the prime above p corresponding to our chosen embedding $\overline{Q} \mapsto \overline{Q}_p$. This power series satisfies $u_{n,B} = (f_{u,B})(\omega_n)$ for all $n > 0, n \equiv 0(d)$ where $d = [F_B : L_p]$ and $\{\omega_n\}$ is chosen as an inverse system of π^n division points of $\hat{E}_B$. We define a homomorphism $\delta_k : U_\infty \to O_B$ by

$$\delta_k(u) := \delta_{k,B}(u) = \left(\frac{1}{\lambda'_{\hat{E}_B}(T)} \frac{d}{dT} \right)^k \log f_{u,B}(T)\Big|_{T=0} . \quad (7.11)$$

Then

$$\delta_k(u^\tau) = \theta(\tau)^k \delta_k(u) \quad (7.12) \quad \text{for} \quad \tau \in Gal(\overline{F}/F)$$

where θ denotes the action on $E[p^\infty]$. Now $\theta = \varphi_p$ on $Gal(\overline{F}/F)$. We want a homomorphism on u_∞ with a transformation property corresponding to ν on all of $Gal(\overline{L}/L)$. We observe that $\nu = \varphi_p^2$ on $Gal(\overline{F}/F)$.
Let S be a set of coset representatives for $Gal(\overline{L}/L)/Gal(\overline{L}/F)$ and define

$$\Phi_2(u) = \sum_{\sigma \in S} \nu^{-1}(\sigma) \delta_2(u^\sigma) \in O_B[\nu]. \quad (7.13)$$

Each term is independent of the choice of coset representative by (7.8b) and it is easily checked that

$$\Phi_2(u^\sigma) = \nu(\sigma) \Phi_2(u).$$

It takes integral values in $O_B[\nu]$. Let $U_\infty(\nu)$ denote the product of the groups of local principal units at the primes above p of the field $L(\nu)$. Then Φ_2 factors through $U_\infty(\nu)$ and thus defines a continuous homomorphism

$$\Phi_2 : U_\infty(\nu) \to C_p .$$

Let C_∞ be the group of projective limits of elliptic units in $L(\nu)$. Then we have a crucial theorem of Rubin:

THEOREM 3.

There is an equality of characteristic ideals as $\Lambda = Z_p[[Gal(L(\nu)/L)]]$*-modules*:

$$char \wedge (Gal(M_\infty / L(\nu))) = char \wedge (U_\infty(\nu)/\overline{C}_\infty).$$

Let $\nu_0 = \nu \bmod \lambda$. For any $Z_p[Gal(L(\nu_0)/L)]$-module X we write $X^{(\nu_0)}$ for the maximal quotient of $X \underset{Z_p}{\otimes} \mathrm{O}$ on which the action of $Gal(L(\nu_0)/L)$ is via the Teichmuller lift of ν_0. Since $Gal(L(\nu)/L)$ decomposes into a direct product of a pro-p group and a group of order prime to p,

$$Gal(L(\nu)/L) \cong Gal(L(\nu)/L(\nu_0)) \times Gal(L(\nu_0)/L),$$

we can also consider any $Z_p[[Gal(L(\nu)/L)]]$-module also as a $Z_p[Gal(L(\nu_0)/L)]$-module. In particular $X^{(\nu_0)}$ is a module over $Z_p[Gal(L(\nu_0)/L)]^{(\nu_0)} \cong \mathrm{O}$. Also $\Lambda^{(\nu_0)} \cong \mathrm{O}[[\mathrm{T}]]$.
Now according to results of Iwasawa, $U_\infty(\nu)^{(\nu_0)}$ is a free $\Lambda^{(\nu_0)}$-module of rank one. We extend Φ_2 O-linearly to $U_\infty(\nu) \otimes_{Z_p} \mathrm{O}$ and it then factors through $U_\infty(\nu)^{(\nu_0)}$. Suppose that u is a generator of $U_\infty(\nu)^{(\nu_0)}$ and β an element of $\overline{C}_\infty^{(\nu_0)}$. Then $f(\gamma - 1)u = \beta$ for some $f(\mathrm{T}) \in \mathrm{O}[[\mathrm{T}]]$ and γ a topological generator of $Gal(L(\nu)/L(\nu_0))$. Computing Φ_2 on both u and β gives

$$f(\nu(\gamma) - 1) = \phi_2(\beta)/\Phi_2(u). \quad (7.14)$$

We have that ν can be interpreted as the grossencharacter whose associated p-adic character , via the chosen embedding $\overline{Q} \mapsto \overline{Q}_p$, is ν, and $\overline{\nu}$ is the complex conjugate of ν.
Furthermore, we can compute $\Phi_2(u)$ by choosing a special local unit and showing that $\Phi_2(u)$ is a p-adic unit.
Now, if we have that

$$\# H^1_{Se}(Q_\Sigma / Q, Y) \leq \# \left(\mathrm{O}/\Omega^{-2} L_{f_0}(2, \overline{\nu})\right) \cdot \prod_{q \in \Sigma} \ell_q,$$

and

$$\#(\mathrm{O}/h_L) \cdot \prod_{q \in \Sigma - \{p\}} \ell_q, \quad (7.15)$$

where $\ell_q = \# H^0\left(Q_q, ((K/\mathrm{O})(\psi) \oplus K/\mathrm{O})^*\right)$ and h_L is the class number of O_L, combining these we obtain the following relation:

$$\# H^1_{Se}(Q_\Sigma / Q, V) \leq \# \left(\mathrm{O}/\Omega^{-2} L_{f_0}(2, \overline{\nu})\right) \# (\mathrm{O}/h_L) \cdot \prod_{q \in \Sigma} \ell_q, \quad (7.16)$$

where $\ell_q = \# H^0(Q_q, V^*)$ (for $q \neq p$), $\ell_p = \# H^0\left(Q_p, (Y^0)^*\right)$. (Also here, we remember that ℓ is p-adic).

Let ρ_0 be an irreducible representation as in (5). Suppose that f is a newform of weight 2 and level N, λ a prime of O_f above p and $\rho_{f,\lambda}$ a deformation of ρ_0. Let m be the kernel of the homomorphism $\mathrm{T}_1(N) \to \mathrm{O}_f / \lambda$ arising from f.

We now give an explicit formula for η developed by Hida by interpreting $\langle , \rangle$ in terms of the cup product pairing on the cohomology of $X_1(N)$, and then in terms of the Petersson inner product of f with itself. Let

$$(,): H^1(X_1(N), \mathrm{O}_f) \times H^1(X_1(N), \mathrm{O}_f) \to \mathrm{O}_f \quad (7.17)$$

be the cup product pairing with O_f as coefficients. Let p_f be the minimal prime of $\mathrm{T}_1(N) \otimes O_f$ associated to f, and let

$$L_f = H^1(X_1(N), \mathrm{O}_f)[p_f].$$

If $f = \sum a_n q^n$ let $f^\rho = \sum \bar{a}_n q^n$. Then f^ρ is again a newform and we define L_{f^ρ} by replacing f by f^ρ in the definition of L_f. Then the pairing $(,)$ induces another by restriction

$$(,): L_f \times L_{f^\rho} \to \mathrm{O}_f. \quad (7.18)$$

Replacing O by the localization of O_f at p (if necessary) we can assume that L_f and L_{f^ρ} are free of rank 2 and direct summands as O_f-modules of the respective cohomology groups. Let δ_1, δ_2 be a basis of L_f. Then also $\bar{\delta}_1, \bar{\delta}_2$ is a basis of $L_{f^\rho} = \bar{L}_f$. Here complex conjugation acts on $H^1(X_1(N), \mathrm{O}_f)$ via its action on O_f. We can then verify that

$$(\delta, \bar{\delta}) := \det(\delta_i, \bar{\delta}_j)$$

is an element of O_f whose image in $\mathrm{O}_{f,\lambda}$ is given by $\pi(\eta^2)$ (unit).

To give a more useful expression for $(\delta, \bar{\delta})$ we observe that f and $\bar{f}^\rho$ can be viewed as elements of $H^1(X_1(N), C) \cong H^1_{DR}(X_1(N), C)$ via $f \mapsto f(z)dz$, $\bar{f}^\rho \mapsto \bar{f}^\rho d\bar{z}$. Then $\{f, \bar{f}^\rho\}$ form a basis for $L_f \otimes_{\mathrm{O}_f} C$. Similarly $\{\bar{f}, f^\rho\}$ form a basis for $L_{f^\rho} \otimes_{\mathrm{O}_f} C$. Define the vectors $\omega_1 = (f, \bar{f}^\rho)$, $\omega_2 = (\bar{f}, f^\rho)$ and write $\omega_1 = C\delta$ and $\omega_2 = \bar{C}\bar{\delta}$ with $C \in M_2(C)$. Then writing $f_1 = f, f_2 = \bar{f}^\rho$ we set

$$(\omega, \bar{\omega}) := \det((f_i, \bar{f}_j)) = (\delta, \bar{\delta}) \det(C\bar{C}).$$

Now $(\omega, \bar{\omega})$ is given explicitly in terms of the (non-normalized) Petersson inner product $\langle , \rangle$: $(\omega, \bar{\omega}) = -4\langle f, f \rangle^2$ where $\langle f, f \rangle = \int_{\Im / \Gamma_1(N)} f\bar{f} dxdy$. Hence, we have the following equation:

$$(\omega, \bar{\omega}) = -4\left(\int_{\Im / \Gamma_1(N)} f\bar{f} dxdy \right)^2. \quad (7.19)$$

To compute $\det(C)$ we consider integrals over classes in $H_1(X_1(N),\mathrm{O}_f)$. By Poincaré duality there exist classes c_1, c_2 in $H_1(X_1(N),\mathrm{O}_f)$ such that $\det\left(\int_{c_j}\delta_i\right)$ is a unit in O_f. Hence $\det C$ generates the same O_f-module as is generated by $\left\{\det\left(\int_{c_j} f_i\right)\right\}$ for all such choices of classes (c_1,c_2) and with $\{f_1, f_2\}=\{f, \bar{f}^{\rho}\}$. Letting u_f be a generator of the O_f-module $\left\{\det\left(\int_{c_j} f_i\right)\right\}$ we have the following formula of Hida:

$$\pi(\eta^2)=\langle f, f\rangle^2 / u_f\bar{u}_f \times (\text{unit in } \mathrm{O}_{f,\lambda}).$$

Now, we choose a (primitive) grossencharacter φ on L together with an embedding $\overline{Q}\mapsto\overline{Q}_p$ corresponding to the prime p above p such that the induced p-adic character φ_p has the properties:

(i) $\varphi_p \bmod \overline{p}=\kappa_0$ ($\overline{p}=$ maximal ideal of $\overline{Q}_p$).

(ii) φ_p factors through an abelian extension isomorphic to $Z_p\oplus T$ with T of finite order prime to p.

(iii) $\varphi((\alpha))=\alpha$ for $\alpha\equiv 1(f)$ for some integral ideal f prime to p.

Let $p_0=\ker\psi_f : \mathrm{T}_1(N)\to\mathrm{O}_f$ and let $A_f=J_1(N)/\,p_0J_1(N)$ be the abelian variety associated to f by Shimura. Over F^+ there is an isogeny $A_{f/F^+}\approx(E_{/F^+})^d$ where $d=[\mathrm{O}_f:\mathrm{Z}]$.

We have that the p-adic Galois representation associated to the Tate modules on each side are equivalent to $\left(Ind_F^{F^+}\varphi_0\right)\otimes_{\mathrm{Z}_p}K_{f,p}$ where $K_{f,p}=\mathrm{O}_f\otimes Q_p$ and where $\varphi_p : Gal(\overline{F}/F)\to\mathrm{Z}_p^{\times}$ is the p-adic character associated to φ and restricted to F .We now give an expression for $\langle f_\varphi, f_\varphi\rangle$ in terms of the L-function of φ. We note that $L_N(2,\overline{\nu})=L_N(2,\nu)=L_N(2,\varphi^2\overline{\hat{\chi}})$ and remember that ν is the p-adic character, and $\overline{\nu}$ is the complex conjugate of ν, we have that:

$$\langle f_\varphi, f_\varphi\rangle=\frac{1}{16\pi^3}N^2\left\{\prod_{\substack{q|N\\ q\notin S_\varphi}}\left(1-\frac{1}{q}\right)\right\}L_N(2,\varphi^2\overline{\hat{\chi}})L_N(1,\psi),\quad (7.20)$$

where χ is the character of f_φ and $\hat{\chi}$ its restriction to L; ψ is the quadratic character associated to L; $L_N(\ )$ denotes that the Euler factors for primes dividing N have been removed; S_φ is the set of primes $q|N$ such that $q = qq'$ with $q\,|$ cond φ and q,q' primes of L, not necessarily distinct.

THEOREM 4.

Suppose that ρ_0 is an irreducible representation of odd determinant such that $\rho_0=Ind_L^Q\kappa_0$ for a character κ_0 of an imaginary quadratic extension L of Q which is unramified at p. Assume also that:

(i) $\det\rho_0|_{I_p}=\omega$;

(ii) ρ_0 *is ordinary.*

Then for every $D=(\cdot,\Sigma,\mathrm{O},\phi)$ *such that* ρ_0 *is of type* D *with* $\cdot =$ *Se or ord,*

$$R_D \cong \mathrm{T}_D$$

and T_D *is a complete intersection.*

COROLLARY.

For any ρ_0 *as in the theorem suppose that*

$$\rho : Gal(\overline{Q}/Q) \to GL_2(\mathrm{O})$$

is a continuous representation with values in the ring of integers of a local field, unramified outside a finite set of primes, satisfying $\overline{\rho} \cong \rho_0$ *when viewed as representations to* $GL_2(\overline{F}_p)$*. Suppose further that:*

(i) $\rho|_{D_p}$ *is ordinary;*

(ii) $\det \rho|_{I_p} = \chi\varepsilon^{k-1}$ *with* χ *of finite order,* $k \geq 2$.

Then ρ *is associated to a modular form of weight* k.

THEOREM 5. (Langlands-Tunnell)

Suppose that $\rho : Gal(\overline{Q}/Q) \to GL_2(C)$ *is a continuous irreducible representation whose image is finite and solvable. Suppose further that* $\det \rho$ *is odd. Then there exists a weight one newform* f *such that* $L(s,f) = L(s,\rho)$ *up to finitely many Euler factors.*

Suppose then that

$$\rho_0 : Gal(\overline{Q}/Q) \to GL_2(F_3)$$

is an irreducible representation of odd determinant. This representation is modular in the sense that over $\overline{F}_3$, $\rho_0 \approx \rho_{g,\mu} \bmod \mu$ for some pair (g,μ) with g some newform of weight 2. There exists a representation

$$i : GL_2(F_3) \mapsto GL_2(\mathrm{Z}[\sqrt{-2}]) \subset GL_2(C).$$

By composing i with an automorphism of $GL_2(F_3)$ if necessary we can assume that i induces the identity on reduction $\bmod(1+\sqrt{-2})$. So if we consider $i \circ \rho_0 : Gal(\overline{Q}/Q) \to GL_2(C)$ we obtain an irreducible representation which is easily seen to be odd and whose image is solvable.
Now pick a modular form E of weight one such that $E \equiv 1(3)$. For example, we can take $E = 6E_{1,\chi}$ where $E_{1,\chi}$ is the Eisenstein series with Mellin transform given by $\zeta(s)\zeta(s,\chi)$ for χ the quadratic character associated to $Q(\sqrt{-3})$. Then $fE \equiv f \bmod 3$ and using the Deligne-Serre lemma we can find an eigenform g' of weight 2 with the same eigenvalues as f modulo a prime μ' above $(1+\sqrt{-2})$. There is a newform g of weight 2 which has the same eigenvalues as g' for almost all T_l's, and we replace (g',μ') by (g,μ) for some prime μ above $(1+\sqrt{-2})$. Then the pair (g,μ) satisfies our requirements for a suitable choice of μ (compatible with μ').

We can apply this to an elliptic curve E defined over Q, and we have the following fundamental theorems:

THEOREM 6.

All semistable elliptic curves over Q are modular.

THEOREM 7.

Suppose that E is an elliptic curve defined over Q with the following properties:
(i) *E has good or multiplicative reduction at 3, 5,*
(ii) *For p = 3, 5 and for any prime $q \equiv -1 \bmod p$ either $\overline{\rho}_{E,p}\big|D_q$ is reducible over $\overline{F}_p$ or $\overline{\rho}_{E,p}\big|I_q$ is irreducible over $\overline{F}_p$.*
Then E is modular.

8. On some equations concerning p-adic and adelic numbers, p-adic and adelic strings.

8.1 Measure and integration on the adelic space A concerning the adelic study of the zeta function. [4]

We take the case where $S = \varepsilon$ is a minimal regular model of elliptic curve E over a global field k. We will assume the set S^- of horizontal curves in S' contains the image of the zero section of $\varepsilon \to B$. To work with the zeta integral we will need measure and integration on $(A \times A)^\times$, and also on $B \times B$ and $(B \times B)^\times$. The central object of this subsection is an unramified zeta integral. The zeta integral will be an integral with respect to a measure on $(A \times A)^\times$.
The space $A_y^\times$ coincides with the preimage of its image with respect to the projection map p_y. Functions which we will integrate in the study of the zeta integral will all be constant on groups associated to A_y^1. Hence, it is sufficient to work with an R-valued measure on $(A_y \times A_y)^\times$ which is the pullback with respect to (p_y, p_y) of a normalized one dimensional adelic measure on $(A_{k(y)} \times A_{k(y)})^\times$, and with the measure on $(A \times A)^\times$ which is their tensor product.
From the definition of A we deduce that **the multiplicative group $A^\times$ is the restricted product of $A_y^\times$ with respect to $(OA_y)^\times, y \in S'$. Similarly to the definition of $A_y = A_y^0$ define an adelic space**

$$A_y \times A_y := \left\{ \left(\alpha_{x,z}^{(1)}, \alpha_{x,z}^{(2)} \right)_{x \in y} : \alpha_{x,z}^{(m)} \in K_{x,z}, \left(\alpha_{x,z}^{(m)} \right) \in A_y, m = 1,2 \right\}. \quad (8.1)$$

Define $(A \times A)^\times$ as the restricted product of $(A_y \times A_y)^\times$ with respect to $(A_y \times A_y \cap OA_y \times OA_y)^\times$.
We define $\mu_{(A_y \times A_y)^\times}$ as the tensor product of the normalized local measures $\mu_{(K_{x,z} \times K_{x,z})^\times}, x \in y$. The definition of $(A_y \times A_y)^\times$ implies that $\mu_{(A_y \times A_y)^\times}$ is a real valued measure. Define $\mu_{(A \times A)^\times}$ as the tensor product of $\mu_{(A_y \times A_y)^\times}, y \in S'$. Define a space of functions $R_{(A_y \times A_y)^\times}$ as the linear space generated by $g_y = \otimes_{x \in y} \left(f_{x,z}^{(1)}, f_{x,z}^{(2)} \right)$ with $g_y = h_y \circ (p_y, p_y)$ for an integrable function h_y on $(A_{k(y)} \times A_{k(y)})^\times$, and

such that $f_{x,z}^{(m)}$ is continuous on $K_{x,z}^{\times}, f_{x,z}^{(m)} char_{K_{x,z}^{\times}} \in R_{K_{x,z}}$ for all $x \in y$ and $f_{x,z}^{(m)}\Big|_{O_{x,z}^{\times}} = 1$ for almost all $x \in y, m = 1,2$. For $f_y = \otimes_{x\in y} f_{x,z} \in R_{(A_y \times A_y)^{\times}}$ define

$$\int f_y d\mu_{(A_y\times A_y)^{\times}} = \prod\nolimits_{x\in y} \int f_{x,z} d\mu_{(K_{x,z}\times K_{x,z})^{\times}} \quad (8.2)$$

and extend by linearity to $R_{(A_y\times A_y)^{\times}}$.

Define a space of functions $R_{(A\times A)^{\times}}$ as the linear space generated by $\otimes_{y\in S'} f_y$ with $f_y = \left(f_y^{(1)}, f_y^{(2)}\right) \in R_{(A_y\times A_y)^{\times}}$ such that $\otimes_{y\in S'} f_y$ induces a continuous map $(A\times A)^{\times} \to C$ and

$$\prod\nolimits_{y\in S'} \int f_y d\mu_{(A_y\times A_y)^{\times}} \quad (8.3)$$

absolutely converges in the compactified complex plane $C \cup \{\infty\}$.

For $f = \otimes f_y \in R_{(A\times A)^{\times}}$ with $f_y \in R_{(A_y\times A_y)^{\times}}$ define

$$\int f d\mu_{(A\times A)^{\times}} = \prod_{y\in S'} \int f_y d\mu_{(A_y\times A_y)^{\times}} \quad (8.4)$$

and extend by linearity to $R_{(A\times A)^{\times}}$.

Now we describe an exemple. Let $f = \otimes_{y\in S'} \otimes_{x\in y} f_{x,z}$ where for all non-archimedean x, z

$$f_{x,z} = |\ |_{x,z}^{s} char_{\left(t_{1x,z}^{c_{x,z,1}} O_{x,z}, t_{1x,z}^{c_{x,z,2}} O_{x,z}\right)}, \quad (8.5)$$

and for all $y \in S'$ $c_{x,z,m} = 0$ for almost all $x \in y$, $m = 1,2$, and for almost all $y \in S'$ $\prod_{x\in y} q_{x,z}^{c_{x,z,m}} = 1$, $m = 1,2$. Define the components of f over archimedean places as

$$f_{\omega,y}(\alpha,\beta) = |\ |_{\omega,y}^{s} \exp\left(-e_\omega \pi\left(|p_y(\alpha)|^2 + |p_y(\beta)|^2\right)\right), \quad (8.6)$$

for $(\alpha,\beta) \in \mathrm{O}_{\omega,y} \times \mathrm{O}_{\omega,y}$ where $|\ |$ is the usual absolute value, p_y is the projection map, $e_\omega = 1$ if ω is a real embedding and $e_\omega = 2$ if ω is a complex embedding. Then

$$\int f_y d\mu_{(A_y\times A_y)^{\times}} = \prod_{x\in y, na} q_{x,z}^{d_{x,z} - (c_{x,z,1} + c_{x,z,2})s} \left(\frac{1}{1 - q_{x,z}^{-s}}\right)^2 \prod_{\omega\in y} \Gamma_{\omega,y}(s), \quad (8.7)$$

where for $y \in S^-$ the factor $\Gamma_{\omega,y}(s) = \pi^{-s}\Gamma(s/2)^2$ if ω is a real embedding and $\Gamma_{\omega,y}(s) = (2\pi)^{2-2s}\Gamma(s)^2$ if ω is a complex embedding. So we get

$$\int f d\mu_{(A\times A)^{\times}} = \prod_{y\in S'} \prod_{x\in y, na} q_{x,z}^{d_{x,z} - (c_{x,z,1} + c_{x,z,2})s} \left(\frac{1}{1 - q_{x,z}^{-s}}\right)^2 \prod_{\omega\in y\in S^-} \Gamma_{\omega,y}(s). \quad (8.8)$$

The product of the Euler factors over $y \in S'$, up to finitely many removed, equals the square of the Hesse zeta function of ε.

Now we define an $R((X))$-valued translation invariant measure $\mu_{B_y \times B_y}$ on $B_y \times B_y$ which lifts the discrete counting measure on $k(y) \times k(y)$. Components of a measurable set with respect to this measure for almost all $y \in S'$ are sets $(p_y, p_y)^{-1}(pt)$. Define a measure $\mu_{B\times B} = \otimes_{y\in S'} \mu_{B_y \times B_y}$.

For a subset S_0 of S' and $f = \otimes f_y$, $f_y = \otimes_{x\in y}\left(f^{(1)}_{x,z}, f^{(2)}_{x,z}\right)$, $f^{(m)}_{x,z} \in Q_{K_{x,z}}$, $f_y = g_y \circ (p_y, p_y)$ where $g_y = \left(g^{(1)}_y, g^{(2)}_y\right)$, $g^{(m)}_y$ are integrable functions on $A_{k(y)}$, define $\int_{B_{S_0}\times B_{S_0}} f(\beta) d\mu_{B\times B}(\beta)$ as equal to $\prod_{y\in S_0}\int_{k(y)\times k(y)} g_y d\mu_{k(y)\times k(y)}$ and extend to the space generated by such functions. The right hand side can diverge if S_0 is infinite. Then, we have:

$$\int_{B_{S_0}\times B_{S_0}} f(\beta) d\mu_{B\times B}(\beta) = \prod_{y\in S_0}\int_{k(y)\times k(y)} g_y d\mu_{k(y)\times k(y)}. \quad (8.9)$$

Since the measure on $k(y)$ is discrete counting, we can take the induced by it measure on $k(y)^{\times}$. Define the measure on $(B_y \times B_y)^{\times}$ as induced from the measure on $B_y \times B_y$. So this measure is just the pullback with respect to (p_y, p_y) of the discrete measure on $(k(y)\times k(y))^{\times}$. Define the measure on $(B\times B)^{\times}$ as the induced from the measure on $B\times B$. For a subset $B = \prod (p_y, p_y)^{-1}(B_y)$ of $(B\times B)^{\times}$ and $f = \otimes f_y$ as above, define

$$\int_B f d\mu_{(B\times B)^{\times}} = \prod_y \int_{B_y} g_y d\mu_{k(y)\times k(y)}. \quad (8.10)$$

Using the local transforms $\mathcal{F}_{x,z}$ one easily gets an adelic transform $\mathcal{F}$. Define spaces $Q_A, Q_{A\times A}$ as adelic version of the local space Q_F. For a function $f \in Q_{A\times A}$ and a finite subset S_0 of S', $\alpha \in \left(A_{S_0} \times A_{S_0}\right)^{\times}$ we get a summation formula which follows from the one dimensional formula

$$\int_{B_{S_0}\times B_{S_0}} f(\alpha\beta) d\mu_{B_{S_0}\times B_{S_0}}(\beta) = \frac{1}{|\alpha|}\int_{B_{S_0}\times B_{S_0}} f\left(\alpha^{-1}\beta\right) d\mu_{B_{S_0}\times B_{S_0}}(\beta). \quad (8.11)$$

8.2 *Zeta integrals*.[4]

We will define zeta integrals in the local case and then in the adelic case. The general formula for the zeta integral has a shape similar to the dimension one zeta integral:

$$\zeta(g,\chi) = \int_{\mathcal{T}} \tilde{g}\chi_t d\mu \quad (8.12)$$

where g is a function in the spaces R or Q, χ is a quasi-character on the group which describes abelian extensions and χ_t is its pullback to a quasi-character on the group T, local or adelic, tildes and $\mathcal{T}$ stand for a certain rescaling of the original functions and groups, the need for which is dictated by dimension two theory needs. If one prefers to ignore the higher class field theory and K-delic objects, in the unramified theory without essential loss one can work with the zeta integral

$$\zeta\left(g, |\ |^s\right) = \int_{\mathcal{T}} \tilde{g}|\ |^{s/2} d\mu. \quad (8.13)$$

First, **we define rescaling local homomorphisms** o' **and an adelic homomorphism** O'. For a non-archimedean two dimensional local field F with local parameters t_2, t_1 define

$$o': T \to F \times F, \quad \left(t_1^j u_1, t_1^l u_2\right) \mapsto \left(t_1^{2j} u_1, t_1^{2l} u_2\right); \quad (8.14)$$

in the archimedean case define

$$o': T \to F \times F, \quad \left(\alpha_1 u_1, \alpha_2 u_2\right) \mapsto \left(\alpha_1 |\alpha_1| u_1, \alpha_2 |\alpha_2| u_2\right), \quad (8.15)$$

where $\alpha_i \in E^\times, u_i \in 1 + tE[[t]]$ and $|\ |$ is the usual absolute value on E. Denote by o the bijection $o'(T) \to T$.

On the adelic side define $o' = \otimes o'_{x,z}: T_S \to A_S \times A_S$, and the inverse bijection $o: o'(T_S) \to T_S$. For $\alpha \in T$ we will use the notation $\tilde{\alpha} = o'(\alpha)$. For a complex valued continuous function f whose domain includes T and is a subset of $F \times F$ form $f \circ o: o'(T) \to C$, then extend it by continuity to the closure of $o'(T)$ in $F \times F$ and by zero outside the closure, denote the result by $f_o: F \times F \to C$. Note that the closure of T in $F \times F$ is $\mathrm{O} \times \mathrm{O}$. Introduce an extension $\tilde{g}$ of a function $g = f_o: F \times F \to C$ as the continuous extension on $\mathrm{O} \times \mathrm{O}$ of

$$\tilde{g}(\alpha_1, \alpha_2) = g(\alpha_1, \alpha_2) + \sum_{1 \le i \le 3} g((\alpha_1, \alpha_2)\nu_i), \quad (\alpha_1, \alpha_2) \in F^\times \times F^\times, \quad (8.16)$$

in the non-archimedean case, where $\nu_1 = \left(t_1^{-1}, t_1^{-1}\right), \nu_2 = \left(t_1^{-1}, 1\right), \nu_3 = \left(1, t_1^{-1}\right)$ and as the continuous extension of f_o in the archimedean case. So, for example, if $f = char_{\left(t_1^j O, t_1^l O\right)}$ then $f_o(\alpha_1, \alpha_2) = 1$ for $(\alpha_1, \alpha_2) \in F^\times \times F^\times$ only if $\alpha_1 \in t_1^{2k} U, \alpha_2 \in t_1^{2m} O^\times$, $k \ge j, m \ge l$, hence f_o is not a continuous function; but $\tilde{f} = char_{\left(t_1^{2j} O, t_1^{2l} O\right)}$ is.

For a complex function g on the adelic T which is the tensor product of its local components $g_{x,z}$ define $\tilde{g}$ as the tensor product of the local $\tilde{g}_{x,z}$, and extend the definition to the space generated by such functions. We will have different rescaling on vertical and horizontal curves. For a curve y denote by $T_{1,y}$ the kernel of the module map on T_y. Choose a set of representatives $M_y \in T_y$ of $N_y = |T_y|$ which forms a group. So if $k(y)$ is of positive characteristic then N_y is a cyclic group generated by q_y, if $k(y)$ is of characteristic zero then N_y is the multiplicative group of positive real numbers. Define a map $O' = \otimes O'_y: T \to T$ curvewise : $O'_y(\alpha) = o'(\alpha)$ if y is vertical, $O'_y(m\gamma) = o'(m)\gamma$ for $m \in M_y$, $\gamma \in T_{1,y}$ if y is horizontal. Put

$$\mathcal{T} = O'(T), \quad \mathcal{T}_y = O'(T_y). \quad (8.17)$$

If $k(y)$ is of characteristic zero then $\mathcal{T}_y = T_y$. Using the homomorphism t define for a function f on $K_2^t(F)$ or on J_S the function f_t is the pullback with respect to t to the local or adelic T. So $|\ |_{2_t}$

is the module function $|\ |$ on T. Now, for a function $g \in R_{F\times F}$ or $g \in R_{(A\times A)^\times}$ and a quasi-character $\chi : K_2^t(F) \to C^\times$ or $\chi : J_S \to C^\times$, $\chi|P_S = 1$ define a generic local zeta integral as

$$\zeta(g,\chi) = \zeta(g,\chi,\mu) = \int_{o'(T)} \tilde{g}\tilde{\chi}_t d\mu_{(F\times F)^\times} \quad (8.18)$$

and an **adelic zeta integral** as

$$\zeta(g,\chi) = \zeta_{S,S'}(g,\chi,\mu) = \int_{\mathcal{J}} \tilde{g}\tilde{\chi}_t d\mu_{(A\times A)^\times}. \quad (8.19)$$

For $g \in Q_{F\times F}$ and $g \in R_{(A\times A)^\times}$ the zeta integrals take complex values.
For the local non-archimedean zeta integral in the case where $\mu(O) = 1$, we have that

$$\zeta(g,\chi) = \left(1-q^{-1}\right)^{-2} \sum_{j,l\in Z} \left(q^{-s}\right)^{j+l} \int_{O^\times\times O^\times} g\left(t_1^j u_1, t_1^l u_2\right)\chi_0\left(t(u_1,u_2)\right) d\mu_{F\times F}(u_1,u_2), \quad (8.20)$$

where $\chi = \chi_0|\ |_2^s$, χ_0 is of finite order and trivial on $\{t_1, t_2\}$. If, moreover, for fixed j,l the value $g\left(t_1^j u_1, t_1^l u_2\right)$ is constant $= g_0(j,l)$, then

$$\zeta(g,\chi) = \left(1-q^{-1}\right)^{-2} \sum_{j,l\in Z} \left(q^{-s}\right)^{j+l} g_0(j,l) \int_{O^\times\times O^\times} \chi_0\left(t(u_1,u_2)\right) d\mu_{F\times F}(u_1,u_2). \quad (8.21)$$

If $g = \otimes_{y\in S'} g_y$ then

$$\zeta_{S,S'}(g,\chi) = \prod_{y\in S'} \zeta_y(g_y,\chi), \qquad \zeta_y(g_y,\chi) = \int_{\mathcal{J}_y} \tilde{g}_y \tilde{\chi}_t d\mu_{(A\times A)^\times}, \quad (8.22)$$

thence

$$\zeta_{S,S'}(g,\chi) = \prod_{y\in S'} \int_{\mathcal{J}_y} \tilde{g}_y \tilde{\chi}_t d\mu_{(A\times A)^\times}. \quad (8.23)$$

Write the quasi-character χ as $\prod \chi_{x,z}$. If, furthermore, $g_y = \otimes_{x\in y} g_{x,z}$, then we have the following formulas. If y is a vertical curve then $\zeta_y(g_y,\chi) = \prod_{x\in y} \zeta_{x,z}(g_{x,z},\chi_{x,z})$ is the product of the generic local zeta integrals. If y is horizontal in characteristic zero then $\zeta_y(g_y,\chi) = \prod_{x\in y} \zeta_{x,z}(g_{x,z},\chi_{x,z})$ where the local factor equals

$$\int_{T_{x,z}} \tilde{g}_{x,z} \tilde{\chi}_{x,z_{t_{x,z}}} d\mu_{K_{x,z}^\times \times K_{x,z}^\times} \quad (8.24)$$

and therefore differs from the generic local zeta integral defined above, since $T_{x,z}$ which differs from $o'(T_{x,z})$ unless $F_{x,z}$ is archimedean. If y is horizontal in positive characteristic then introduce an auxiliary zeta integral

$$\zeta_y^a\left(g_y, |\ |_2^s\right) = \int_{T_y} \tilde{g}_y(\alpha)|\alpha|^{s/2} d\mu_{(A_y\times A_y)^\times}(\alpha). \quad (8.25)$$

The latter is the product of

$$\int_{T_{x,z}} \tilde{g}_y(\alpha)|\alpha|^{s/2} d\mu_{K_{x,z}^\times \times K_{x,z}^\times}(\alpha), \quad (8.26)$$

to calculate which one can use the formulas for horizontal y in characteristic zero. If

$$\zeta_y^a\left(g_y, |\ |_2^s\right) = \sum_{n\in N_y} \int_{T_{1,y}} \tilde{g}_y(m_n\gamma) d\mu_{(A_y\times A_y)^\times}(\gamma) n^{-s/2}, \quad (8.27)$$

$$m_n \in T_y, \quad |m_n|_y = n, \text{ then} \qquad \zeta_y\left(g_y, |\ |_2^s\right) = \sum_{n\in N_y} c_{2n} n^{-s}.$$

The adelic zeta integral diverges unless $S = \varepsilon$ corresponds to an elliptic curve over a global field, since otherwise ℓ_y, will take the same value different from 1 for infinitely many vertical curves. Now, we assume that $S = \varepsilon$ is a minimal regular model of elliptic curve E over a global field k, (see pg.9) and that the set S' of curves contains the image of the zero section $S \to B$. From Tate's algorithm for the special fibre of a regular model of elliptic curve we know that for every point x on ε there is a branch z of a vertical y passing through x such that the finite residue field $k_z(x)$ coincides with the residue field $k(x)$ of x.

Now we define a kind of a centrally normalized function f for which the calculation of the adelic zeta integral is straightforward. Put

$$f = \otimes_{y\in S'} f_y, \qquad f_y = \otimes_{x\in y} f_{x,z} \quad (8.28)$$

and define the local factors as follows. For non-archimedean (x,z) on vertical curves y in a non-singular fibre and horizontal curves in characteristic zero put $f_{x,z} = char_{(O_{x,z}, O_{x,z})}$. On a vertical curve y in a non-singular fibre $\tilde{f}_y = f_y$ and $\mathcal{F}(f_y)(\alpha) = f_y(\nu_y^{-1}\alpha)$ with $\nu_y \in T_{1,y}$. On a vertical curve y in a singular fibre define $f_{x,z}(\alpha) = char_{(O_{x,z}, O_{x,z})}(\varepsilon_{x,z}\alpha)$ with $(\varepsilon_{x,z}) \in T_y$, $\varepsilon_{x,z} = \left(t_{1,x,z}^{-c_{x,z}}, t_{1,x,z}^{-c'_{x,z}}\right)$, such that for $f_y = \otimes_{x\in y} f_{x,z}$ we have $\mathcal{F}(f_y)(\alpha) = f_y(\nu_y^{-1}\alpha)$ with $\nu_y \in T_{1,y}$. On a horizontal curve y in positive characteristic define $f_{x,z}(\alpha) = char_{(O_{x,z}, O_{x,z})}(\varepsilon_{x,z}\alpha)$ with $(\varepsilon_{x,z}) \in T_y$ such that for $f_y = \otimes_{x\in y} f_{x,z}$ we have $\mathcal{F}(\tilde{f}_y)(\alpha) = \tilde{f}_y(\rho_y^{-1}\alpha)$ with $\rho_y \in T_{1,y}$. Over archimedean places put

$$f_{\omega,y}^{pr}(\alpha,\beta) = \exp\left[-e_\omega \pi\left(|p_y(\alpha)|^4 + |p_y(\beta)|^4\right)\right], \quad (8.29)$$

for $(\alpha,\beta) \in \mathrm{O}_{\omega,y} \times \mathrm{O}_{\omega,y}$.

So then

$$\tilde{f}_{\omega,y}^{pr}(\alpha,\beta) = \exp\left[-e_\omega \pi\left(|p_y(\alpha)|^2 + |p_y(\beta)|^2\right)\right]. \quad (8.30)$$

For a fixed archimedean σ choose $\eta_{\omega,y} \in R_{>0}$ equal each other, such that

$$\prod_\omega \eta_{\omega,y}^{2e_\omega} = \eta_{\omega,y}^{2n} = \ell_y \quad (8.31)$$

where $n = |k : Q|$.

For a horizontal y in characteristic zero define f_y as having components $char_{(O_{x,z},O_{x,z})}$ at non-archimedean data and $f_{\omega,y}(\alpha)=f^{pr}_{\omega,y}\left[\left(\sqrt{\eta}_{\omega,y},\sqrt{\eta}_{\omega,y}\right)\alpha\right]$ at ω,y. Then on horizontal curves in characteristic zero we have $\mathcal{F}\left(\tilde{f}_y\right)(\alpha)=\tilde{f}_y\left(\rho_y^{-1}\alpha\right)$ with $\rho_y\in T_{1,y}$. On vertical fibres put $\rho_y=\tilde{\nu}_y$ and define

$$\rho=\otimes_{y\in S'}\rho_y. \quad (8.32)$$

Now, using the previous formulas it is easy to obtain the following theorem of $\zeta\left(f,|\ |_2^s\right)$.

THEOREM

For every vertical curve y we have

$$\zeta_y\left(f,|\ |_2^s\right)=\ell_y^{1-s}\prod_{x\in y}\left(\frac{1}{1-q_{x,z}^{-s}}\right)^2. \quad (8.33)$$

For every horizontal curve y the zeta integral $\zeta_y\left(f,|\ |_2^s\right)$ is a meromorphic function which satisfies the functional equation $\zeta_y\left(f,|\ |_2^s\right)=\zeta_y\left(f,|\ |_2^{2-s}\right)$ and which is holomorphic outside its poles of multiplicity two at $s=0,2$ in characteristic zero and at $q_y^s=1,q_y^2$ in positive characteristic. For a horizontal curve y in characteristic zero the zeta integral $\zeta_y\left(f,|\ |_2^s\right)$ is the square of a one dimensional integral at $s/2$ on $k(y)$.

Recall that the (unramified) zeta function of a scheme S is

$$\zeta_S(s)=\prod_{x\in S_{0'}}\left(1-|k(x)|^{-s}\right)^{-1}, \quad (8.34)$$

where x runs through the set of closed points on S. It is equal to the product $\prod_{b\in B_0}\zeta_{S_b}(s)$, where $S_b=S\times_B b$. It is easy to see that $\zeta_S(s)$ absolutely and normally converges on $\mathcal{R}(s)>2$. Classically we know even a stronger property: $\zeta_S(s)$ extends to a meromorphic function on $\mathcal{R}(s)>3/2$ with the only simple pole(s) at $s=2$ in characteristic zero and $q^s=q^2$ in positive characteristic. The previous theorem implies a comparison of the zeta integral and the square of the Hasse function of ε which, in particular, implies the convergence of the zeta integral on the half plane $\mathcal{R}(s)>2$.

COROLLARY

On $\mathcal{R}(s)>2$ $\quad \zeta_{\varepsilon,S'}\left(f,|\ |_2^s\right)=c_{\varepsilon,S'}\left(|\ |_2^s\right)\zeta_\varepsilon(s)^2, \quad c_{\varepsilon,S'}\left(|\ |_2^s\right)=c_{\varepsilon,S'}\left(|\ |_2^s\right)c_{\varepsilon,S'}-\left(|\ |_2^s\right).$

The first factor $c_{\varepsilon,S'}\left(|\ |_2^s\right)=\prod_{b\in B_0}\imath_b(s)$ *where*

$$\imath_b(s)=\prod_{y\subset\varepsilon_b}\left(\ell_y^{1-s}\prod_{x\in y}\left(1-|k_z(x)|^{-s}\right)^{-2}\right)\prod_{x\in\varepsilon_b}\left(1-|k(x)|^{-s}\right)^2, \quad (8.35)$$

where y *runs through irreducible components of* ε_b *without multiplicities. The factor* $\boldsymbol{\iota}_b(s)$ *equals 1 on all non-singular fibres* ε_b*. Furthermore, we have:*

$$\zeta_{\varepsilon,S'}\left(f,|\ |_2^s\right)=\prod_{b\in B_0}\prod_{y\subset\varepsilon_b}\left(\boldsymbol{\ell}_y^{1-s}\prod_{x\in y}\left(1-|k_z(x)|^{-s}\right)^{-2}\right)\prod_{x\in\varepsilon_b}\left(1-|k(x)|^{-s}\right)^2\zeta_\varepsilon(s)^2. \quad (8.36)$$

The second factor is the product of zeta integrals for horizontal curves and hence has a meromorphic continuation to the complex plane and satisfies the functional equation $c_{\varepsilon,S}-\left(|\ |_2^s\right)=c_{\varepsilon,S}-\left(|\ |_2^{2-s}\right)$ *and is holomorphic outside its poles at* $s=0,2$ *in characteristic zero and at* $q^s=1,q^2$ *in positive characteristic. The zeta integral* $\zeta_{\varepsilon,S'}\left(f,|\ |_2^s\right)$ *absolutely and normally converge on* $\mathcal{R}(s)>2$*.*

The general case of S introduce a (renormalized) zeta integral

$$\zeta_{S,S'}\left(g,|\ |_{2,S}^s\right)=\prod_{y\subset S_b,b\in B_0}\left(\zeta_{P^1(B),P^1(B)_b}\left(\boldsymbol{\ell},|\ |_{2,P^1(B)}^s\right)^{g-1}\zeta_{S,y}\left(g,|\ |_{2,S}^s\right)\right)\cdot\prod_{y\in S\setminus S'}\zeta_{S,y}\left(g,|\ |_{2,S}^s\right). \quad (8.37)$$

For y in a non-singular fibre S_b the y-factor $\boldsymbol{\ell}_y^{1-s}$ of $\zeta_{S,y}\left(g,|\ |_{2,S}^s\right)$ is cancelled out by the b-factor of $\zeta_{P^1(B)}\left(\boldsymbol{\ell},|\ |_2^s\right)^{g-1}$ which is equal to its inverse. Define f similar to the function f above. Similar to the previous calculation one deduces that for $\mathcal{R}(s)>2$ the zeta integral $\zeta_{S,S'}\left(f,|\ |_{2,S}^s\right)$ equals the product of $\zeta_{P^1(B)}(s)^{2g-2}\zeta_S(s)^2$ and of $c_{S,S'}\left(|\ |_2^s\right)$ which is the product of exponential and Euler factors for vertical curves in singular fibres and of factors for horizontal curves in S'.

8.3 Adelic strings and zeta strings. [5] [6] [7] [8]

Recall that the field of rational numbers Q plays an important role in physics. On Q there is the usual $\left(|\cdot|_\infty\right)$ and p-adic $\left(|\cdot|_p\right)$ absolute value, where p denotes a prime number. Completion of Q with respect to $|\cdot|_\infty$ and $|\cdot|_p$ yields the field of real $(R\equiv Q_\infty)$ and p-adic $\left(Q_p\right)$ numbers, respectively.
The set of all adeles A may be given in the form

$$A=\bigcup_S \mathrm{A}(S), \qquad \mathrm{A}(S)=R\times\prod_{p\in S}Q_p\times\prod_{p\notin S}Z_p. \quad (8.38)$$

A has the structure of a topological ring. Recall that quantum amplitudes defined by means of path integral may be symbolically presented as

$$A(K)=\int A(X)\chi\left(-\frac{1}{h}S[X]\right)\mathcal{D}X, \quad (8.39)$$

where K and X denote classical momenta and configuration space, respectively. We note that $\chi(a)$ is an additive character, $S[X]$ is a classical action and h is the Planck constant.
Now we consider the simple p-adic and adelic bosonic string amplitudes based on the functional integral (8.39). We know that the scattering of two real bosonic strings in 26-dimensional space-time at the tree level can be described in terms of the path integral in 2-dimensional quantum field theory formalism as follows:

$$A_\infty(k_1,\dots,k_4) = g_\infty^2 \int \mathcal{D}X \exp\left(\frac{2\pi i}{h} S_0[X]\right) \times \prod_{j=1}^{4} \int d^2\sigma_j \exp\left(\frac{2\pi i}{h} k_\mu^{(j)} X^\mu(\sigma_j,\tau_j)\right), \quad (8.40)$$

where $\mathcal{D}X = \mathcal{D}X^0(\sigma,\tau)\mathcal{D}X^1(\sigma,\tau)\dots\mathcal{D}X^{25}(\sigma,\tau)$, $d^2\sigma_j = d\sigma_j d\tau_j$ and

$$S_0[X] = -\frac{T}{2}\int d^2\sigma \partial_\alpha X^\mu \partial^\alpha X_\mu \quad (8.41)$$

with $\alpha = 0,1$ and $\mu = 0,1,\dots,25$. Hence, we obtain:

$$A_\infty(k_1,\dots,k_4) = g_\infty^2 \int \mathcal{D}X \exp\left(\frac{2\pi i}{h}\left(-\frac{T}{2}\int d^2\sigma \partial_\alpha X^\mu \partial^\alpha X_\mu\right)\right) \times \prod_{j=1}^{4} \int d^2\sigma_j \exp\left(\frac{2\pi i}{h} k_\mu^{(j)} X^\mu(\sigma_j,\tau_j)\right).$$
$$(8.41b)$$

It is possible to obtain the crossing symmetric Veneziano amplitude

$$A_\infty(k_1,\dots,k_4) = g_\infty^2 \int_R |x|_\infty^{k_1 k_2} |1-x|_\infty^{k_2 k_3} dx. \quad (8.42)$$

As p-adic Veneziano amplitude, the p-adic analogue of eq. (8.42) is

$$A_p(k_1,\dots,k_4) = g_p^2 \int_{Q_p} |x|_p^{k_1 k_2} |1-x|_p^{k_2 k_3} dx, \quad (8.43)$$

where only the string world sheet, parametrized by x, is p-adic. Expressions (8.42) and (8.43) are Gel'fand-Graev beta functions on R and Q_p, respectively.
Now we take p-adic analogue of (8.40), i.e.

$$A_p(k_1,\dots,k_4) = g_p^2 \int \mathcal{D}X \chi_p\left(-\frac{1}{h} S_0[X]\right) \times \prod_{j=1}^{4} \int d^2\sigma_j \chi_p\left(-\frac{1}{h} k_\mu^{(j)} X^\mu(\sigma_j,\tau_j)\right), \quad (8.44)$$

to be p-adic string amplitude, where $\chi_p(u) = \exp(2\pi i\{u\}_p)$ is p-adic additive character and $\{u\}_p$ is the fractional part of $u \in Q_p$. In (8.44), all space-time coordinates X_μ, momenta k_i and world sheet (σ,τ) are p-adic. Evaluation of (8.44) leads to the following equation:

$$A_p(k_1,\dots,k_4) = g_p^2 \prod_{j=1}^{4} \int d^2\sigma_j \chi_p\left[\frac{\sqrt{-1}}{2hT}\sum_{i<j} k_i k_j \log\left((\sigma_i-\sigma_j)^2 + (\tau_i-\tau_j)^2\right)\right]. \quad (8.45)$$

Adelic string amplitude is product of real and all p-adic amplitudes, i.e.

$$A_A(k_1,...,k_4) = A_\infty(k_1,...,k_4)\prod_p A_p(k_1,...,k_4). \quad (8.46)$$

In the case of the Veneziano amplitude and $(\sigma_i,\tau_j)\in \mathrm{A}(S)\times\mathrm{A}(S)$, where $\mathrm{A}(S)$ is defined in (8.38), we have

$$A_A(k_1,...,k_4) = g_\infty^2\int_R |x|_\infty^{k_1k_2}|1-x|_\infty^{k_2k_3}dx\times\prod_{p\in S}g_p^2\prod_{j=1}^4\int d^2\sigma_j\times\prod_{p\notin S}g_p^2. \quad (8.47)$$

Hence, we take the adelic coupling constant as

$$g_A^2 = |g|_\infty^2\prod_p|g|_p^2 = 1, \quad 0\neq g\in Q. \quad (8.48)$$

Furthermore, it follows that p-adic effects in the adelic Veneziano amplitude induce discreteness of string momenta and contribute to an effective coupling constant in the form

$$g_{ef}^2 = g_A^2\prod_{p\in S}\prod_{j=1}^4\int d^2\sigma_j \geq 1. \quad (8.49)$$

Like in ordinary string theory, the starting point of p-adic strings is a construction of the corresponding scattering amplitudes. Now, for the eq. (8.42) we can write also the following equations

$$A_\infty(a,b) = g^2\int_R|x|_\infty^{a-1}|1-x|_\infty^{b-1}dx \quad (8.50)$$

$$= g^2\left[\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}+\frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)}+\frac{\Gamma(c)\Gamma(a)}{\Gamma(c+a)}\right] \quad (8.51)$$

$$= g^2\frac{\zeta(1-a)}{\zeta(a)}\frac{\zeta(1-b)}{\zeta(b)}\frac{\zeta(1-c)}{\zeta(c)} \quad (8.52)$$

$$= g^2\int \mathcal{D}X\exp\left(-\frac{i}{2\pi}\int d^2\sigma\partial^\alpha X_\mu\partial_\alpha X^\mu\right)\prod_{j=1}^4\int d^2\sigma_j\exp\left(ik_\mu^{(j)}X^\mu\right), \quad (8.53)$$

where $\hbar = 1$, $T = 1/\pi$, and $a = -\alpha(s) = -1-\frac{s}{2}$, $b = -\alpha(t)$, $c = -\alpha(u)$ with the conditions $s+t+u=-8$, i.e. $a+b+c=1$.
The p-adic generalization of the expression (8.50) is

$$A_p(a,b) = g_p^2\int_{Q_p}|x|_p^{a-1}|1-x|_p^{b-1}dx, \quad (8.54)$$

where $|\cdot|_p$ denotes p-adic absolute value. In this case only string world-sheet parameter x is treated as p-adic variable, and all other quantities have their usual (real) valuation. A further adelic formula is

$$A_\infty(a,b)\prod_p A_p(a,b)=1 \quad (8.55)$$

where $A_\infty(a,b)$ denotes the usual Veneziano amplitude (8.50). Hence, we obtain the following equation

$$g^2\int_R |x|_\infty^{a-1}|1-x|_\infty^{b-1}dx\prod_p A_p(a,b)=1. \quad (8.55b)$$

Now for the unified path integral approach to ordinary and p-adic N-point bosonic string amplitudes at the tree level, we have that

$$A_v(k_1,...,k_N)=g_v^{N-2}\prod_{j=1}^{N}\int d^2\sigma_j\int\chi_v\left(-\frac{1}{h}\int \mathcal{L}\left(X^\mu,\partial_\alpha X^\mu\right)d^2\sigma\right)\mathcal{D}_v X\ , \quad (8.56)$$

where $v=\infty,2,...,p,...,\quad \mu=0,1,...,25,\quad \alpha=0,1,$ and $\chi_\infty(a)=\exp(-2\pi i a)$, $\chi_p(a)=\exp\left(2\pi i\{a\}_p\right)$. The above Lagrangian is

$$\mathcal{L}=-\frac{T}{2}\partial_\alpha X^\mu(\sigma,\tau)\partial^\alpha X_\mu(\sigma,\tau)+\sqrt{-1}\sum_{j=1}^{N}k_\mu^{(j)}X^\mu(\sigma,\tau)\delta\left(\sigma-\sigma_j\right)\delta\left(\tau-\tau_j\right). \quad (8.57)$$

Note that this approach is adelic and based on the following assumptions: (i) space-time and matter are adelic at the Planck (M-theory) scale, (ii) Feynman's path integral method is an inherent ingredient of quantum theory, and (iii) adelic quantum theory is a more complete theory than the ordinary one. Consequently, a string is an adelic object which has simultaneously real and all p-adic characteristics. The target space and world-sheet are adelic spaces. Adelic Feynman's path integral is an infinite product of the ordinary one and all p-adic counterparts. The corresponding adelic string amplitude is

$$A\left(k_A^{(1)},...,k_A^{(N)}\right)=A_\infty\left(k_\infty^{(1)},...,k_\infty^{(N)}\right)\prod_{p\in S}A_p\left(k_p^{(1)},...,k_p^{(N)}\right)\prod_{p\notin S}A_p\left(k_p^{(1)},...,k_p^{(N)}\right), \quad (8.58)$$

where $k_A^{(i)}$ is an adele, i.e.

$$k_A^{(i)}=\left(k_\infty^{(i)},k_2^{(i)},...,k_p^{(i)},...\right) \quad (8.59)$$

with the restriction that $k_p^{(i)}\in Z_p$ for all but a finite set S of primes p. The topological ring of adeles $k_A^{(i)}$ provides a framework for simultaneous and unified consideration of real and p-adic string momenta. Adelic string amplitude contains nontrivial p-adic modification of the ordinary one. Now we consider the case $A\to C$ with regard p-adic and adelic analysis. In this case functions are complex-valued while their arguments are adeles. The related analysis is used in adelic approach to quantum mechanics, quantum cosmology, quantum field theory and string theory. Many important complex-valued functions from real and p-adic analysis can be easily extended to this adelic case. Adelic multiplicative and additive characters are:

$$\pi_s(x)=|x|^s=|x_\infty|_\infty^s\prod_p|x_p|_p^s\ , \qquad x\in I\ ,\ s\in C\ , \quad (8.60)$$

$$\chi(x)=\chi_\infty(x_\infty)\prod_p \chi_p(x_p)=e^{-2\pi i x_\infty}\prod_p e^{2\pi i\{x_p\}_p}, \qquad x\in A. \quad (8.61)$$

Since all except finite number of factors in (8.60) and (8.61) are equal to unity, it is evident that these infinite products are convergent. One can show that $\pi_s(x)=1$ if x is a principal idele, and $\chi(x)=1$ if x is a principal adele, i.e.

$$|x|_\infty^s\prod_p |x|_p^s=1, \quad x\in Q^*, \quad s\in C, \quad (8.62)$$

$$\chi_\infty(x)\prod_p \chi_p(x)=e^{-2\pi i x}\prod_p e^{2\pi i\{x\}_p}=1, \quad x\in Q. \quad (8.63)$$

It is worth noting that expressions (8.62) and (8.63) for $s=1$ represent the simplest adelic product formulas, which clearly connect real and p-adic properties of the same rational number. In fact, the formula (8.62), for $s=1$, connects usual absolute value and p-adic norms at the multiplicative group of rational numbers Q^*. Maps $\varphi_{\mathcal{P}}: A\to C$, which have the form

$$\varphi_{\mathcal{P}}(x)=\varphi_\infty(x_\infty)\prod_{p\in\mathcal{P}}\varphi_p(x_p)\prod_{p\notin P}\Omega_p\left(|x_p|_p\right), \quad (8.64)$$

where $\varphi_\infty(x_\infty)$ are infinitely differentiable functions and fall to zero faster than any power of $|x_\infty|_\infty$ as $|x_\infty|_\infty\to\infty$, and $\varphi_p(x_p)$ are locally constant functions with compact support, are called elementary functions on A. All finite linear combinations of the elementary functions (8.64) make the set $S(A)$ of Schwartz-Bruhat functions $\varphi(x)$. A is a locally compact ring and therefore there is the corresponding Haar measure, which is product of the real and all p-adic additive Haar measures. The Fourier transform of the Schwartz-Bruhat functions $\varphi(x)$ is

$$\tilde{\varphi}(\xi)=\int_A \varphi(x)\chi(x\xi)dx \quad (8.65)$$

and it maps $S(A)$ onto $S(A)$. The Mellin transform of $\varphi(x)\in S(A)$ is defined using the multiplicative character $|x|^s$ in the following way:

$$\Phi(s)=\int_I \varphi(x)|x|^s d^*x=\int_R \varphi_\infty(x_\infty)|x_\infty|_\infty^{s-1}dx_\infty\times\prod_p\int_{Q_p}\varphi_p(x_p)|x_p|_p^{s-1}\frac{dx}{1-p^{-1}}, \quad \mathrm{Re}\,s>1. \quad (8.66)$$

$\Phi(s)$ may be analytically continued on the whole complex plane, except $s=0$ and $s=1$, where it has simple poles with residues $-\varphi(0)$ and $\tilde{\varphi}(0)$, respectively. Denoting by $\tilde{\Phi}$ the Mellin transform of $\tilde{\varphi}$ then there is place the Tate formula

$$\Phi(s)=\tilde{\Phi}(1-s). \quad (8.67)$$

If we take

$$\varphi(x)=\sqrt[4]{2}e^{-\pi x_\infty^2}\prod_p\Omega_p\left(|x_p|_p\right),$$

which is the simplest ground state of the adelic harmonic oscillator, then from the Tate formula (8.67) one gets the well-known functional relation for the Riemann zeta function, i.e.

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{\frac{s-1}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \quad (8.68)$$

Its interesting this connection between the harmonic oscillator and the Riemann zeta function.
With regard the zeta strings, hence the Riemann zeta function applied to the strings, the equation of motion for the zeta string ϕ is

$$\zeta\left(\frac{\square}{2}\right)\phi=\frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon}e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi} \quad (8.69)$$

which has an evident solution $\phi=0$. For the case of time dependent spatially homogeneous solutions one has to consider the equation of motion

$$\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t)=\frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon}e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0=\frac{\phi(t)}{1-\phi(t)}. \quad (8.70)$$

In the weak field approximation $(|\phi(t)|<<1)$ the above expression $\phi/(1-\phi)\approx\phi$ and (8.70) becomes a linear equation which can be written in the form

$$\int_R e^{-ik_0t}\left[\zeta\left(\frac{k_0^2}{2}\right)\theta\left(|k_0|-\sqrt{2}-\varepsilon\right)-1\right]\tilde{\phi}(k_0)dk_0=0, \quad (8.71)$$

where θ is the Heaviside function. Since $\zeta\left(\frac{k_0^2}{2}\right)>1$ when $|k_0|>\sqrt{2}$ the equation (8.71) has solution only for $\tilde{\phi}(k_0)=0$. This also means the absence of mass.
Furthermore, with regard the coupled zeta strings ϕ and θ which are open and closed respectively, the equations of motion are:

$$\zeta\left(\frac{\square}{2}\right)\phi=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\sum_{n\geq 1}\theta^{\frac{n(n-1)}{2}}\phi^n, \quad (8.72)$$

$$\zeta\left(\frac{\square}{4}\right)\theta=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk=\sum_{n\geq 1}\left[\theta^{n^2}+\frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right]. \quad (8.73)$$

9. The P-N Model (Palumbo-Nardelli model) and the Ramanujan identities, solution applied to ten dimensional IIB supergravity (uplifted 10-dimensional solution) and connections with some equations concerning the Riemann zeta function. [9]

Palumbo (2001) has proposed a simple model of the birth and of the evolution of the Universe. Palumbo and Nardelli (2005) have compared this model with the theory of the strings, and translated it in terms of the latter obtaining:

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi\right]=$$
$$=\int_{0}^{\infty}\frac{1}{2\kappa_{10}^{2}}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_{\mu}\Phi\partial^{\mu}\Phi-\frac{1}{2}\left|\tilde{H}_{3}\right|^{2}-\frac{\kappa_{10}^{2}}{g_{10}^{2}}Tr_{\nu}\left(\left|F_{2}\right|^{2}\right)\right], \qquad (9.1)$$

A general relationship that links bosonic and fermionic strings acting in all natural systems.
It is well-known that the series of Fibonacci's numbers exhibits a fractal character, where the forms repeat their similarity starting from the reduction factor $1/\phi$ = 0,618033 = $\frac{\sqrt{5}-1}{2}$ (Peitgen et al. 1986). Such a factor appears also in the famous fractal Ramanujan identity (Hardy 1927):

$$0{,}618033=1/\phi=\frac{\sqrt{5}-1}{2}=R(q)+\cfrac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_{0}^{q}\frac{f^{5}(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}, \qquad (9.2)$$

and $$\pi=2\Phi-\frac{3}{20}\left[R(q)+\cfrac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_{0}^{q}\frac{f^{5}(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right], \qquad (9.3)$$

where $$\Phi=\frac{\sqrt{5}+1}{2}.$$

Furthermore, we remember that π arises also from the following identity:

$$\pi=\frac{12}{\sqrt{130}}\log\left[\frac{\left(2+\sqrt{5}\right)\left(3+\sqrt{13}\right)}{\sqrt{2}}\right], \quad (9.3a) \quad \text{and} \quad \pi=\frac{24}{\sqrt{142}}\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right].$$
(9.3b)

The introduction of (9.2) and (9.3) in (9.1) provides:

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16G}\cdot\cfrac{1}{2\Phi-\frac{3}{20}\left(R(q)+\cfrac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_{0}^{q}\frac{f^{5}(-t)}{f\left(-t^{1/5}\right)}\frac{dt}{t^{4/5}}\right)}\right)}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)+\right.$$

$$-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\ \ ]=\int_0^\infty \frac{R}{\kappa_{11}^2}\cdot 2\Phi-\frac{3}{20}\left[R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\cdot$$

$$\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{11}^2}{2\Phi-\frac{3}{20}\left(R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right)2Rg_{10}^2}Tr_v\right.$$

$$\left.\left(\left|F_2\right|^2\right)\ \right], \qquad (9.4)$$

which is the translation of (30) in the terms of the Theory of the Numbers, specifically the possible connection between the Ramanujan identity and the relationship concerning the Palumbo-Nardelli model.

a. Solution applied to ten dimensional IIB supergravity (uplifted 10-dimensional solution).

This solution can be oxidized on a three sphere S^3 to give a solution to ten dimensional IIB supergravity. This 10D theory contains a graviton, a scalar field, and the NSNS 3-form among other fields, and has a ten dimensional action given by

$$S_{10}=\int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R-\frac{1}{2}(\partial\phi)^2-\frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right]. \quad (9.5)$$

We have a ten dimensional configuration given by

$$ds_{10}^2=\left(\frac{2}{r}\right)^{3/4}\left[-h(r)dt^2+r^2dx_{0,5}^2+\frac{r^2}{h(r)}dr^2\right]+\left(\frac{r}{2}\right)^{5/4}\left[d\theta^2+d\psi^2+d\varphi^2+\left(d\psi+\cos\theta d\varphi-\frac{Q}{5r^5}dt\right)^2\right]$$

$$\phi=-\frac{5}{4}\log\frac{r}{2},$$

$$H_3=-\frac{Q}{r^6}dr\wedge dt\wedge(d\psi+\cos\theta d\varphi)-\frac{g}{\sqrt{2}}\sin\theta d\theta\wedge d\varphi\wedge d\psi. \quad (9.6)$$

This uplifted 10-dimensional solution describes NS-5 branes intersecting with fundamental strings in the time direction.
Now we make the manipulation of the angular variables of the three sphere simpler by introducing the following left-invariant 1-forms of SU(2):

$$\sigma_1=\cos\psi d\theta+\sin\psi\sin\theta d\varphi,\quad \sigma_2=\sin\psi d\theta-\cos\psi\sin\theta d\varphi,\quad \sigma_3=d\psi+\cos\theta d\varphi, \quad (9.7)$$

and $$h_3=\sigma_3-\frac{Q}{5}\frac{1}{r^5}dt. \quad (9.8)$$

Next, we perform the following change of variables

$$\frac{r}{2}=\rho^{\frac{4}{5}}, \quad t=\frac{5}{32}\tilde{t}, \quad dx_4=\frac{1}{2\sqrt{2}}d\tilde{x}_4, \quad dx_5=\frac{1}{2}dZ, \quad g=\sqrt{2}\tilde{g}, \quad Q=\sqrt{2}2^7\tilde{Q}, \quad \sigma_i=\frac{1}{\tilde{g}}\tilde{\sigma}_i. \quad (9.9)$$

It is straightforward to check that the 10-dimensional solution (9.6) becomes, after these changes

$$d\tilde{s}_{10}^2=\frac{1}{2}\rho^{-1}\left[d\tilde{s}_6^2\right]+\frac{\rho}{\tilde{g}^2}\left[\tilde{\sigma}_1^2+\tilde{\sigma}_2^2+\left(\tilde{\sigma}_3-\frac{\tilde{g}\tilde{Q}}{4\sqrt{2}}\frac{1}{\rho^4}d\tilde{t}\right)^2\right]+\rho dZ^2,$$

$$\phi=-\ln\rho,$$

$$H_3=-\frac{1}{\tilde{g}^2}\tilde{\sigma}_1\wedge\tilde{\sigma}_2\wedge\tilde{h}_3+\frac{\tilde{Q}}{\sqrt{2}\tilde{g}\rho^5}d\tilde{t}\wedge d\rho\wedge\tilde{h}_3, \quad (9.10)$$

where we define

$$d\tilde{s}_6^2=-\tilde{h}(\rho)d\tilde{t}^2+\frac{\rho^2}{\tilde{h}(\rho)}d\rho^2+\rho^2 d\tilde{x}_{0,4}^2 \quad (9.11)$$

and, after re-scaling M,

$$\tilde{h}=-\frac{2\tilde{M}}{\rho^2}+\frac{\tilde{g}^2}{32}\rho^2+\frac{\tilde{Q}^2}{8}\frac{1}{\rho^6}. \quad (9.12)$$

We now transform the solution from the Einstein to the string frame. This leads to

$$d\bar{s}_{10}^2=\frac{1}{2}\rho^{-2}\left[d\tilde{s}_6^2\right]+\frac{1}{\tilde{g}^2}\left[\tilde{\sigma}_1^2+\tilde{\sigma}_2^2+\left(\tilde{\sigma}_3-\frac{\tilde{g}\tilde{Q}}{4\sqrt{2}}\frac{1}{\rho^4}d\tilde{t}\right)^2\right]+dZ^2,$$

$$\bar{\phi}=-2\ln\rho,$$

$$\bar{H}_3=H_3. \qquad (9.13)$$

We have a solution to 10-dimensional IIB supergravity with a nontrivial NSNS field. If we perform an S-duality transformation to this solution we again obtain a solution to type-IIB theory but with a nontrivial RR 3-form, F_3. The S-duality transformation acts only on the metric and on the dilaton, leaving invariant the three form. In this way we are led to the following configuration, which is S-dual to the one derived above

$$d\bar{s}_{10}^2=\frac{1}{2}\left[d\tilde{s}_6^2\right]+\frac{\rho^2}{\tilde{g}^2}\left[\tilde{\sigma}_1^2+\tilde{\sigma}_2^2+\left(\tilde{\sigma}_3-\frac{\tilde{g}\tilde{Q}}{4\sqrt{2}}\frac{1}{\rho^4}d\tilde{t}\right)^2\right]+\rho^2 dZ^2,$$

$$\bar{\phi}=2\ln\rho,$$

$$F_3 = H_3. \qquad (9.14)$$

With regard the T-duality, in the string frame we have

$$d\bar{s}_{10}^2 = \frac{1}{2}\left[ds_6^2\right] + \frac{r^2}{g^2}\left[\sigma_1^2 + \sigma_2^2 + \left(\sigma_3 - \frac{gQ}{4\sqrt{2}}\frac{1}{r^4}dt\right)^2\right] + r^{-2}dZ^2. \quad (9.15)$$

This gives a solution to IIA supergravity with excited RR 4-form, C_4. We proceed by performing a T-duality transformation, leading to a solution of IIB theory with nontrivial RR 3-form, C_3. The complete solution then becomes

$$d\bar{s}_{10}^2 = \frac{1}{2}\left[ds_6^2\right] + \frac{r^2}{g^2}\left[\sigma_1^2 + \sigma_2^2 + \left(\sigma_3 - \frac{gQ}{4\sqrt{2}}\frac{1}{r^4}dt\right)^2\right] + r^2 dZ^2 \ ,$$

$$\bar{\phi} = 2\ln r$$

$$C_3 = -\frac{1}{g^2}\sigma_1 \wedge \sigma_2 \wedge h_3 - \frac{Q}{\sqrt{2}g}\frac{1}{r^5}dt \wedge dr \wedge h_3. \quad (9.16)$$

We are led in this way to precisely the same 10D solution as we found earlier [see formula (9.14)].
With regard the Palumbo-Nardelli model, we have the following connection with the eq. (9.5):

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] =$$
$$= \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right] \to$$
$$\to \int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right]. \quad (9.17)$$

b. Connections with some equations concerning the Riemann zeta function.

We have obtained interesting connections between some solutions concerning ten dimensional IIB supergravity and some equations concerning the Riemann zeta function, specifying the Goldston-Montgomery theorem.
In the chapter "Goldbach's numbers in short intervals" of Languasco's paper "The Goldbach's conjecture", is described the Goldston-Montgomery theorem.

THEOREM 1

Assume the Riemann hypothesis. We have the following implications: (1) If $0 < B_1 \le B_2 \le 1$ and $F(X,T) \approx \frac{1}{2\pi}T\log T$ uniformly for $\frac{X^{B_1}}{\log^3 X} \le T \le X^{B_2}\log^3 X$, then

$$\int_1^X (\psi(1+\delta)x) - \psi(x) - \delta(x)^2\, dx \approx \frac{1}{2}\delta X^2 \log\frac{1}{\delta}, \quad (9.18)$$

uniformly for $\frac{1}{X^{B_2}} \le \delta \le \frac{1}{X^{B_1}}$.

(2) If $1 < A_1 \le A_2 < \infty$ and $\int_1^X (\psi((1+\delta)x) - \psi(x) - \delta x)^2\, dx \approx \frac{1}{2}\delta X^2 \log\frac{1}{\delta}$ uniformly for $\frac{1}{X^{1/A_1}\log^3 X} \le T \le \frac{1}{X^{1/A_2}}\log^3 X$, then $F(X,T) \approx \frac{1}{2\pi} T \log T$ uniformly for

$$T^{A_1} \le X \le T^{A_2}.$$

Now, for show this theorem, we must to obtain some preliminary results .

Preliminaries Lemma. (Goldston-Montgomery)

Lemma 1.

We have $f(y) \ge 0 \quad \forall y \in R$ and let $I(Y) = \int_{-\infty}^{+\infty} e^{-2|y|} f(Y+y)\,dy = 1 + \varepsilon(Y)$. If R(y) is a Riemann-integrable function, we have:

$$\int_a^b R(y) f(Y+y)\,dy = \left(\int_a^b R(y)\,dy\right)(1+\varepsilon'(y)).$$

Furthermore, fixed R, $|\varepsilon'(Y)|$ is little if $|\varepsilon(y)|$ is uniformly small for $Y + a - 1 \le y \le Y + b + 1$.

Lemma 2.

Let $f(t) \ge 0$ a continuous function defined on $[0,+\infty)$ such that $f(t) << \log^2(t+2)$.
If

$$J(T) = \int_0^T f(t)\,dt = (1+\varepsilon(T))T \log T,$$

then

$$\int_0^\infty \left(\frac{\sin ku}{u}\right)^2 f(u)\,du = \left(\frac{\pi}{2} + \varepsilon'(k)\right) k \log\frac{1}{k},$$

with $|\varepsilon'(k)|$ small for $k \to 0^+$ if $|\varepsilon(T)|$ is uniformly small for

$$\frac{1}{k\log^2 k} \le T \le \frac{1}{k}\log^2 k.$$

Lemma 3.

Let $f(t) \geq 0$ a continuous function defined on $[0,+\infty)$ such that $f(t) << \log^2(t+2)$. If

$$I(k) = \int_0^{\infty} \left(\frac{\sin ku}{u} \right)^2 f(u)du = \left(\frac{\pi}{2} + \varepsilon'(k) \right) k \log \frac{1}{k}, \quad (9.19) \qquad \text{then}$$

$$J(T) = \int_0^T f(t)dt = (1+\varepsilon')T \log T \quad , \quad (9.20)$$

with $|\varepsilon'|$ small if $|\varepsilon(k)| \leq \varepsilon$ uniformly for $\dfrac{1}{T \log T} \leq k \leq \dfrac{1}{T} \log^2 T$.

Lemma 4.

Let $F(X,T) := \sum_{0<\gamma,\gamma'<T} \dfrac{4X^{i}(\gamma-\gamma')}{4+(\gamma-\gamma')^2}$. Then (i) $F(X,T) \geq 0$; (ii) $F(X,T) = F(1/X,T)$; (iii) If

The Riemann hypothesis is preserved, then we have

$$F(X,T) = T\left(\frac{1}{X^2} \log^2 T + \log X \right)\left(\frac{1}{2\pi} + O\left(\sqrt{\frac{\log\log T}{\log T}} \right) \right)$$

uniformly for $1 \leq X \leq T$.

Lemma 5.

Let $\delta \in (0,1]$ and $a(s) = \dfrac{(1+\delta)^s - 1}{s}$. If $c(\gamma) \leq 1 \quad \forall y$ we have that

$$\int_{-\infty}^{+\infty} |a(it)|^2 \left| \sum_{\gamma} \frac{c(\gamma)}{1+(t-\gamma)^2} \right|^2 dt = \int_{-\infty}^{+\infty} \left| \sum_{|\gamma| \leq Z} a(1/2+i\gamma) \frac{c(\gamma)}{1+(t-\gamma)^2} \right|^2 dt + O\left(\delta^2 \log^3 \frac{2}{\delta} \right) + O\left(\frac{1}{Z} \log^3 Z \right)$$

for $Z > \dfrac{1}{\delta}$.

For to show the Theorem 1, there are two parts. We go to prove (1).
We define

$$J(X,T) = 4\int_0^T \left| \sum_{\gamma} \frac{X^{i\gamma}}{1+(t-\gamma)^2} \right|^2 dt .$$

Montgomery has proved that $J(X,T) = 2\pi F(X,T) + O(\log^3 T)$ and thence the hypothesis $F(X,T) \approx \dfrac{1}{2\pi} T \log T$ is equal to $J(X,T) = (1+o(1))T \log T$. Putting $k = \dfrac{1}{2}\log(1+\delta)$, we have

$$|a(it)|^2 = 4\left(\frac{\sin kt}{t}\right)^2 .$$

For the Lemma 2, we obtain that

$$\int_0^\infty |a(it)|^2 \left|\sum_\gamma \frac{X^{i\gamma}}{1+(t-\gamma)^2}\right|^2 dt = \left(\frac{\pi}{2}+o(1)\right)k\log\frac{1}{k} = \left(\frac{\pi}{4}+o(1)\right)\delta\log\frac{1}{\delta}$$

for $$\frac{1}{\delta\log^2\frac{1}{\delta}} \le T \le \frac{3}{\delta}\log^2\frac{1}{\delta} .$$

For the Lemma 5 and the parity of the integrand, we have that

$$\int_{-\infty}^{+\infty}\left|\sum_{|\gamma|\le Z} a(\rho)\frac{X^{i\gamma}}{1+(t-\gamma)^2}\right|^2 dt = \left(\frac{\pi}{2}+o(1)\right)\delta\log\frac{1}{\delta} \qquad \text{(a)}$$

if $Z \ge \frac{1}{\delta}\log^3\frac{1}{\delta}$.

From the $S(t) = \sum_{|\gamma|\le Z} a(\rho)\frac{X^{i\gamma}}{1+(t-\gamma)^2}$ we note that the Fourier's transformed verify that

$$\hat{S}(u) = \pi\sum_{|\gamma|\le Z} a(\rho)X^{i\gamma}e(-\gamma u)e^{-2\pi|u|} .$$

From the Plancherel identity, we have that

$$\int_{-\infty}^{+\infty}\left|\sum_{|\gamma|\le Z} a(\rho)X^{i\gamma}e(-\gamma u)\right|^2 e^{-4\pi|u|}du = \left(\frac{2}{\pi}+o(1)\right)\delta\log\frac{1}{\delta} .$$

For the substitution $Y = \log X$, $-2\pi u = y$ we obtain

$$\int_{-\infty}^{+\infty}\left|\sum_{|\gamma|\le Z} a(\rho)e^{i\gamma(Y+y)}\right|^2 e^{-2|y|}dy = (1+o(1))\delta\log\frac{1}{\delta} . \qquad \text{(b)}$$

Using the Lemma 1 with $R(y) = e^{2y}$ if $0 \le y \le \log 2$ and $R(y) = 0$ otherwise, and putting $x = e^{Y+y}$ we have that

$$\int_X^{2X}\left|\sum_{|\gamma|\le Z} a(\rho)x^\rho\right|^2 dx = \left(\frac{3}{2}+o(1)\right)\delta X^2\log\frac{1}{\delta} .$$

Substituting X with $X2^{-j}$, summarizing on j, $1 \le j \le K$, and using the explicit formula for $\psi(x)$ with $Z = X\log^3 X$ we obtain

$$\int_{X2^{-K}}^{X} \left(\psi((1+\delta)x)-\psi(x)-\delta x\right)^2 dx = \frac{1}{2}\left(1-2^{-2K}+o(1)\right)\delta X^2 \log\frac{1}{\delta}.$$

Furthermore, we put $K = [\log\log X]$ and we utilize, for the interval $1 \le x \le X2^{-K}$, the estimate of Lemma 4 (placing $X2^{-K}$ for X). Thus, we obtain (1).
Now, we prove (2).
We fix an real number X_1. Making an integration for parts between X_1 and $X_2 = X_1 \log^{2/3} X_1$ we obtain, remembering that for hypothesis we have

$$\int_{1}^{X} \left(\psi((1+\delta)x)-\psi(x)-\delta x\right)^2 dx \approx \frac{1}{2}\delta X^2 \log\frac{1}{\delta},$$

that
$$\int_{X_1}^{X_2} \left(\psi((1+\delta)x)-\psi(x)-\delta x\right)^2 x^{-4} dx = \left(\frac{1}{2}+o(1)\right)\delta X_1^{-2} \log\frac{1}{\delta}. \quad \text{(c)}$$

Utilizing the estimate, valid under the Riemann hypothesis

$$\int_{1}^{X} \left(\psi((1+\delta)x)-\psi(x)-\delta x\right)^2 dx << \delta X^2 \log^2\frac{2}{\delta},$$

we obtain analogously as before that

$$\int_{X_2}^{\infty} \left(\psi((1+\delta)x)-\psi(x)-\delta x\right)^2 x^{-4} dx << \delta X_2^{-2} \log^2\frac{1}{\delta} = o\left(\delta X_1^{-2} \log\frac{1}{\delta}\right). \quad \text{(d)}$$

Now, summarizing (c) and (d) and multiplying the sum for X_1^2 we obtain

$$\int_{1}^{\infty} \min\left(\frac{x^2}{X_1^2}, \frac{X_1^2}{x^2}\right)\left(\psi((1+\delta)x)-\psi(x)-\delta x\right)^2 x^{-2} dx = (1+o(1))\delta \log\frac{1}{\delta}.$$

Putting $X_1 = X$, $Y = \log X$, $x = e^{Y+y}$ and using the explicit formula for $\psi(x)$ with $Z = X\log^3 X$, we obtain the equation (b).
Now, we take the equations (9.6) and (9.14) and precisely $\phi = -\frac{5}{4}\log\frac{r}{2}$ and $\bar{\phi} = 2\ln\rho$. We note that from the equation (9.20) for $\varepsilon' = \frac{3}{2}$ and T = 1/2 , we have $J(T) = \int_0^T f(t)dt = (1+\varepsilon')T\log T = \frac{5}{4}\log\frac{1}{2}$.

Furthermore, for $\varepsilon' = 3$ and T = 1/2 , we have $J(T) = \int_0^T f(t)dt = (1+\varepsilon')T\log T = 2\log\frac{1}{2}$.

These results are related to $\phi = -\frac{5}{4}\log\frac{r}{2}$ putting r = 1 and to $\bar{\phi} = 2\ln\rho$ putting $\rho = 1/2$, hence with the Lemma 3 of Goldston-Montgomery theorem. Then, we have the following interesting relations:

$$\phi = -\frac{5}{4}\log\frac{r}{2} \Rightarrow -\int_0^T f(t)dt = -[(1+\varepsilon')T\log T], \ (9.21) \qquad \bar{\phi} = 2\ln\rho \Rightarrow \int_0^T f(t)dt = (1+\varepsilon')T\log T, \Rightarrow$$

$$\Rightarrow \int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right] \quad (9.22)$$

hence the connection between the 10-dimensional solutions (9.5) and some equations related to the Riemann zeta function.

From this the possible connection between cosmological solutions concerning string theory and some mathematical sectors concerning the zeta function, whose the Goldston-Montgomery Theorem and the related Goldbach's Conjecture.

10. Mathematical connections.

Now we take the eq. (7.11) of **Section 7**. We note that can be related with the Godston-Montgomery equation, the ten dimensional action (9.5) and the relationship of Palumbo-Nardelli model (9.1) of **Section 9**, hence we have the following connection:

$$\delta_k(u) := \delta_{k,\mathrm{B}}(u) = \left(\frac{1}{\lambda'_{\hat{E}_\mathrm{B}}(\mathrm{T})}\frac{d}{d\mathrm{T}}\right)^k \log f_{u,\mathrm{B}}(\mathrm{T})\Big|_{\mathrm{T}=0} \Rightarrow 2\ln\rho \Rightarrow \int_0^T f(t)dt = (1+\varepsilon')T\log T, \Rightarrow$$

$$\Rightarrow \int d^{10}x\sqrt{|g|}\left[\frac{1}{4}R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-2\phi}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right] \Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right] =$$

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]. \quad (10.1)$$

Now we take the eq. (7.20) of **Section 7**. We note that can be related with the equation regarding the Palumbo-Nardelli model and with the Ramanujan's identity concerning π. Hence, we have the following connections:

$$\langle f_\varphi, f_\varphi \rangle = \frac{1}{16\pi^3}N^2\left\{\prod_{\substack{q|N \\ q\notin S_\varphi}}\left(1-\frac{1}{q}\right)\right\}L_N\left(2,\varphi^2\bar{\bar{\chi}}\right)L_N(1,\psi) \Rightarrow$$

$$\Rightarrow \int_0^\infty \pi^2 \langle f_\varphi, f_\varphi \rangle \cdot \frac{1}{N^2 \left\{ \prod_{q|N} \left(1 - \frac{1}{q} \right) \right\} L_N \left(2, \varphi^2 \bar{\tilde{\chi}} \right) L_N (1, \psi) G_N} \int d^{10} x (-G)^{1/2} e^{-2\Phi} \cdot$$

$$\cdot \left[R + 4 \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} \left| \tilde{H}_3 \right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_v \left(\left| F_2 \right|^2 \right) \right] =$$

$$= -\int d^{26} x \sqrt{g} \left[\left(-R \cdot \pi^2 \langle f_\varphi, f_\varphi \rangle \cdot \frac{1}{N^2 \left\{ \prod_{q|N} \left(1 - \frac{1}{q} \right) \right\} L_N \left(2, \varphi^2 \bar{\tilde{\chi}} \right) L_N (1, \psi) G} \right) - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} Tr \left(G_{\mu\nu} G_{\rho\sigma} \right) f(\phi) + \right.$$

$$\left. - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right], \quad (10.2)$$

$$\langle f_\varphi, f_\varphi \rangle = \frac{1}{16\pi^3} N^2 \left\{ \prod_{\substack{q|N \\ q \notin S_\varphi}} \left(1 - \frac{1}{q} \right) \right\} L_N \left(2, \varphi^2 \bar{\tilde{\chi}} \right) L_N (1, \psi) \Rightarrow$$

$$\Rightarrow \frac{1}{16 \left\{ 2\Phi - \frac{3}{20} \left[R(q) + \frac{\sqrt{5}}{1 + \frac{3+\sqrt{5}}{2} \exp\left(\frac{1}{\sqrt{5}} \int_0^q \frac{f^5(-t)}{f(-t^{1/5})} \frac{dt}{t^{4/5}} \right)} \right] \right\}^3} N^2 \left\{ \prod_{\substack{q|N \\ q \notin S_\varphi}} \left(1 - \frac{1}{q} \right) \right\} L_N \left(2, \varphi^2 \bar{\tilde{\chi}} \right) L_N (1, \psi).$$

$$(10.3)$$

Now, we take the eqs. (8.44) and (8.47) of **Section 8**. We note that can be related with the Palumbo-Nardelli relationship. Thence, we have the following connections:

$$A_p(k_1, \ldots, k_4) = g_p^2 \int DX \chi_p \left(-\frac{1}{h} S_0[X] \right) \times \prod_{j=1}^4 \int d^2 \sigma_j \chi_p \left(-\frac{1}{h} k_\mu^{(j)} X^\mu (\sigma_j, \tau_j) \right) \Rightarrow$$

$$\Rightarrow -\int d^{26} x \sqrt{g} \left[-\frac{R}{16\pi G} - \frac{1}{8} g^{\mu\rho} g^{\nu\sigma} Tr \left(G_{\mu\nu} G_{\rho\sigma} \right) f(\phi) - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] =$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2} \int d^{10} x (-G)^{1/2} e^{-2\Phi} \left[R + 4 \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} \left| \tilde{H}_3 \right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_v \left(\left| F_2 \right|^2 \right) \right], \quad (10.4)$$

$$A_A(k_1, \ldots, k_4) = g_\infty^2 \int_R |x|_\infty^{k_1 k_2} |1 - x|_\infty^{k_2 k_3} dx \times \prod_{p \in S} g_p^2 \prod_{j=1}^4 \int d^2 \sigma_j \times \prod_{p \notin S} g_p^2 \Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$
$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right], \quad (10.5)$$

While, if we take the eqs. (8.40), (8.69) and (8.73) of **Section 8**, we note that can be related with the Ramanujan's identity concerning π and with Palumbo-Nardelli model. Then, we obtain the following connections:

$$A_\infty(k_1,...,k_4)=g_\infty^2\int DX\exp\left(\frac{2\pi i}{h}S_0[X]\right)\times\prod_{j=1}^4\int d^2\sigma_j\exp\left[\frac{2\pi i}{h}k_\mu^{(j)}X^\mu\left(\sigma_j,\tau_j\right)\right]\Rightarrow$$
$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$
$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right]\Rightarrow$$
$$\Rightarrow g_\infty^2\int DX\exp\left\{2\left\{2\Phi-\frac{3}{20}\left[R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q\frac{f^5(-t)}{f\left(-t^{1/5}\right)}\frac{dt}{t^{4/5}}\right)}\right]\right\}i\frac{1}{h}S_0X\right\}\times$$
$$\times\prod_{j=1}^4\int d^2\sigma_j\exp\left\{2\left\{2\Phi-\frac{3}{20}\left[R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q\frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\right\}\frac{i}{h}k_\mu^{(j)}X^\mu\left(\sigma_j,\tau_j\right)\right\},$$
$$(10.6)$$

$$\zeta\,(\Box/2)\phi=\frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon}e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}\Rightarrow$$
$$\Rightarrow\frac{1}{\left\{2\left\{2\Phi-\frac{3}{20}\left[R(q)+\frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q\frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\right\}\right\}^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon}e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$
$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right], \quad (10.7)$$

$$\zeta\,(\square/4)\,\theta = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk = \sum_{n\geq 1}\left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right] \Rightarrow$$

$$\Rightarrow \frac{1}{\left\{2\left\{2\Phi - \frac{3}{20}\left[R(q) + \frac{\sqrt{5}}{1+\frac{3+\sqrt{5}}{2}\exp\left(\frac{1}{\sqrt{5}}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}\right)}\right]\right\}\right\}^D}$$

$$\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk = \sum_{n\geq 1}\left[\theta^{n^2} + \frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right] \Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right] =$$

$$-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G} - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]. \quad (10.8)$$

Now, we take the eqs. (6.17) of **Section 6** and eq. (7.20) of **Section 7**. We have the following connections:

$$\frac{i\cdot f_{\sigma,n}}{8\pi^2\int_{|\gamma|=1} f_{\sigma_\gamma}\gamma^{-n-1}d\gamma} = \left\langle f_\varphi, f_\varphi\right\rangle \cdot \frac{1}{N^2\left\{\prod_{\substack{q|N \\ q\notin S_\varphi}}\left(1-\frac{1}{q}\right)\right\}L_N\left(2,\varphi^2\bar{\hat{\chi}}\right)L_N(1,\psi)} \Rightarrow$$

$$\Rightarrow i\cdot f_{\sigma,n} = 8\pi^2\int_{|\gamma|=1} f_{\sigma_\gamma}\gamma^{-n-1}d\gamma \cdot \frac{\left\langle f_\varphi, f_\varphi\right\rangle}{N^2\left\{\prod_{\substack{q|N \\ q\notin S_\varphi}}\left(1-\frac{1}{q}\right)\right\}L_N\left(2,\varphi^2\bar{\hat{\chi}}\right)L_N(1,\psi)} \Rightarrow$$

$$\Rightarrow \int_0^\infty \pi^2\left\langle f_\varphi, f_\varphi\right\rangle \cdot \frac{1}{N^2\left\{\prod_{q|N}\left(1-\frac{1}{q}\right)\right\}L_N\left(2,\varphi^2\bar{\hat{\chi}}\right)L_N(1,\psi)G_N}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\cdot$$

$$\cdot\left[R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right] =$$

$$= -\int d^{26}x\sqrt{g}\left[\left(-R\cdot\pi^2\left\langle f_\varphi, f_\varphi\right\rangle \cdot \frac{1}{N^2\left\{\prod_{q|N}\left(1-\frac{1}{q}\right)\right\}L_N\left(2,\varphi^2\bar{\hat{\chi}}\right)L_N(1,\psi)G}\right) - \frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi) + \right.$$

$$\left. -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] \quad (10.9)$$

With regard the eqs. (6.15) and (6.17) of **Section 6**, we have also the following connections with the eq. (7.20) of **Section 7** and the eqs. (8.44) and (8.47) of **Section 8**:

$$\mathcal{F}_\sigma = \bigoplus_{n\in Z} \mathcal{F}_{\sigma,n} \Rightarrow f_{\sigma,n} = \frac{1}{2\pi i}\int_{|\gamma|=1} f_{\sigma_\gamma}\gamma^{-n-1}d\gamma \Rightarrow$$

$$\Rightarrow i\cdot f_{\sigma,n} = 8\pi^2\int_{|\gamma|=1} f_{\sigma_\gamma}\gamma^{-n-1}d\gamma \cdot \frac{\langle f_\varphi, f_\varphi\rangle}{N^2\left\{\prod_{\substack{q|N \\ q\notin S_\varphi}}\left(1-\frac{1}{q}\right)\right\}L_N\left(2,\varphi^2\bar{\bar{\chi}}\right)L_N(1,\psi)} \Rightarrow$$

$$\Rightarrow A_p(k_1,\dots,k_4) = g_p^2\int \mathcal{D}X\chi_p\left(-\frac{1}{h}S_0[X]\right)\times\prod_{j=1}^{4}\int d^2\sigma_j\chi_p\left(-\frac{1}{h}k_\mu^{(j)}X^\mu(\sigma_j,\tau_j)\right)\Rightarrow$$

$$\Rightarrow A_A(k_1,\dots,k_4) = g_\infty^2\int_R |x|_\infty^{k_1k_2}|1-x|_\infty^{k_2k_3}dx\times\prod_{p\in S}g_p^2\prod_{j=1}^{4}\int d^2\sigma_j\times\prod_{p\notin S}g_p^2 . \quad (10.10)$$

Now, with regard the eqs. (5.1) and (5.9b) of **Section 5**, it is possible the following mathematical connections with the eq. (7.20) of **Section 7** and with the eq. (8.47) of **Section 8**, hence:

$$A_F = \prod_{x\in C}{}' F_x \Rightarrow \frac{1}{16\pi^3}N^2\left\{\prod_{\substack{q|N \\ q\notin S_\varphi}}\left(1-\frac{1}{q}\right)\right\}L_N\left(2,\varphi^2\bar{\bar{\chi}}\right)L_N(1,\psi)\Rightarrow$$

$$\Rightarrow A_A(k_1,\dots,k_4) = g_\infty^2\int_R |x|_\infty^{k_1k_2}|1-x|_\infty^{k_2k_3}dx\times\prod_{p\in S}g_p^2\prod_{j=1}^{4}\int d^2\sigma_j\times\prod_{p\notin S}g_p^2 , \quad (10.11)$$

$$SL_2(F)\backslash SL_2\left(\prod_{x\in C}{}' F_x\right)/\prod_{x\in C}K_x \Rightarrow \frac{1}{16\pi^3}N^2\left\{\prod_{\substack{q|N \\ q\notin S_\varphi}}\left(1-\frac{1}{q}\right)\right\}L_N\left(2,\varphi^2\bar{\bar{\chi}}\right)L_N(1,\psi)\Rightarrow$$

$$\Rightarrow A_A(k_1,\dots,k_4) = g_\infty^2\int_R |x|_\infty^{k_1k_2}|1-x|_\infty^{k_2k_3}dx\times\prod_{p\in S}g_p^2\prod_{j=1}^{4}\int d^2\sigma_j\times\prod_{p\notin S}g_p^2 . \quad (10.12)$$

Conclusion

Hence, in conclusion, also for some mathematical sectors concerning the link between the structure of A-branes observed in the homological mirror symmetry and the classical theory of automorphic forms, can be obtained interesting and new connections with other sectors of Number Theory and String Theory, principally the p-adic and adelic numbers, the Ramanujan's modular equations, some formulae related to the Riemann zeta functions, the modular elliptic curves and p-adic and adelic strings.
Furthermore, also the fundamental relationship concerning the Palumbo-Nardelli model, a general relationship that links bosonic string action and superstring action (i.e. bosonic and fermionic

strings acting in all natural systems), can be related with some equations regarding the p-adic (adelic) string sectors and some sectors of Number Theory.

Acknowledgments

I would like to thank Prof. **Branko Dragovich** of Institute of Physics of Belgrade (Serbia) for the important and fundamental advices and references that he has give me and his availability and friendship with regard me. Furthermore I would like to thank also Prof. M. Watkins for his availability and his excellent Number Theory & Physics archive:
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Finito di stampare nel mese di Marzo 2008
Presso DI. VI. Service – Via Miranda, 50 – 80131 Napoli