Block Notes of Math

The Landau's prime numbers and the Legendre's conjecture

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Abstract

In this work the authors will explain and prove an equivalent RH which is linked to the Landau's prime numbers and discuss on the Legendre's conjecture and on the infinity of Landau's prime numbers.

Acknowledgments

The authors thank all readers, if they will return a feedback on this paper.

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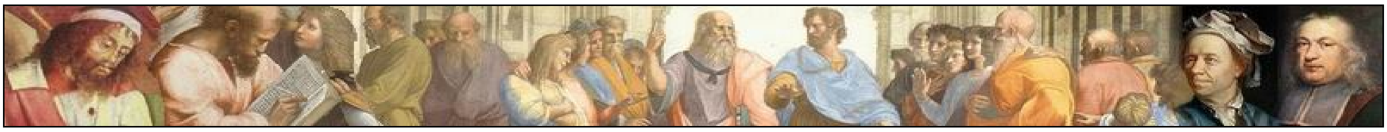

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The Landau's prime numbers – an equivalent RH

We propose to call "*Landau's prime numbers*" the prime numbers that we can obtain from the form:

$$Landau's\ prime\ numbers:\ p = n^2 + 1, \text{with } p \text{ prime number}$$

In order that p is prime, n must be even but this excepts n=1 which gives 2. So we can't obtain them, for example, from any prime numbers or from any square prime numbers (this excepts for n=2).

The Landau's prime numbers *La(x)* aren't many (see Table 1), but we can infinitely produce them with a simple program (see Appendix) up to 1E+14. The formulas n^2+1 is similar at the Euler's form x^2+x+41.

x	La(x)
1E+1	3
1E+2	5
1E+3	10
1E+4	19
1E+5	51
1E+6	112
1E+7	316
1E+8	841
1E+9	2378
1E+10	6656
1E+11	18822
1E+12	54110
1E+13	156081
1E+14	456362

Table 1 – Landau's prime numbers

We propose a close form that links La(x) and π(N) or La(x) and Li(x).

We say that:

$$\left|\frac{La(x)}{x} - \int_2^x \frac{dt}{t(\ln t)^2}\right| = O(x^{\frac{1}{2}+\varepsilon}) \quad (1)$$

This is an equivalent RH (**R. Turco, M. Colonnese, ERATOSTENE group**)".

The (1) is equivalent to:

$$\left|\frac{La(N)}{N} - \frac{\pi(N)}{N\ln N}\right| < KC(N),\ K=1,\ C(N)=\frac{1}{2*\ln N} \quad (2)$$

Introducing now the big O function, the previous expression becomes:

$$\left|\frac{La(N)}{N} - \frac{\pi(N)}{N\ln N}\right| = O((2*\ln N)^{-1}) \quad (3)$$

If instead of π(N) we introduce Li then it is:

$$\frac{La(x)}{x} - \int_2^x \frac{dt}{t(\ln t)^2}\Big| = O((2*\ln x)^{-1}) + O(x^{\frac{1}{2}+\varepsilon}) = O(2^{-1} * x^{-\varepsilon}) + O(x^{\frac{1}{2}+\varepsilon}) \approx O(x^{\frac{1}{2}+\varepsilon}) \quad (4)$$

We can do some calculation with the excel and set us also a rule that will automatically check the inequality ABS(A-B) < C(N). We will have "YES" if the rule is verified.

x	La(x)	La(x)/x	π(x)	C(x)=1/LN(x)	π(x)/x ln x	ABS(A-B)	ABS(A-B)<C(x)?
10	3	0,3	4	0,434294482	0,173718	0,126282	YES
100	5	0,05	25	0,217147241	0,054287	0,004287	YES
1.000	10	0,01	168	0,144764827	0,02432	0,01432	YES
10.000	19	0,0019	1.229	0,10857362	0,013344	0,011444	YES
100.000	51	0,00051	9.592	0,086858896	0,008332	0,007822	YES
1.000.000	112	0,000112	78.498	0,072382414	0,005682	0,00557	YES
10.000.000	316	3,16E-05	664.579	0,062042069	0,004123	0,004092	YES

Table 2 – La(N) and C(N)

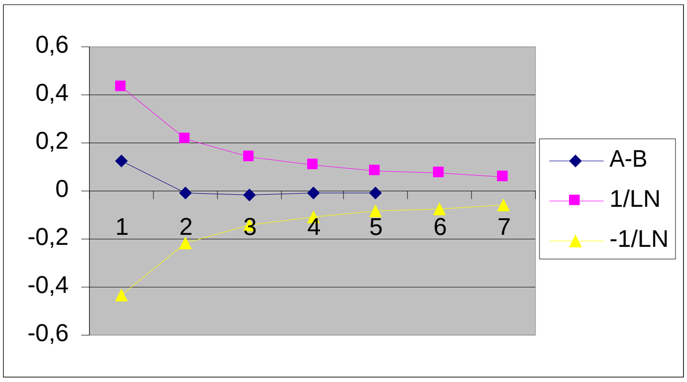


Figure 1 – bounds |1/lnN| of A-B

The term of error 1/(ln(N)) in (2), numerically, *works well*; while the term of error $1/\ln(N)^2$ has got an exception at N=100.000 (see table 3)

x	La(x)	La(x)/x	π(x)	C(x)=1/LN(x)^2	π(x)/x ln x	ABS(A-B)	ABS(A-B)<C(x)?
10	3	0,3	4	0,188611697	0,173718	0,126282	YES
100	5	0,05	25	0,047152924	0,054287	0,004287	YES
1.000	10	0,01	168	0,020956855	0,02432	0,01432	YES
10.000	19	0,0019	1.229	0,011788231	0,013344	0,011444	YES
100.000	51	0,00051	9.592	0,007544468	0,008332	0,007822	NO

Table 3 – 1/ln(N)^2

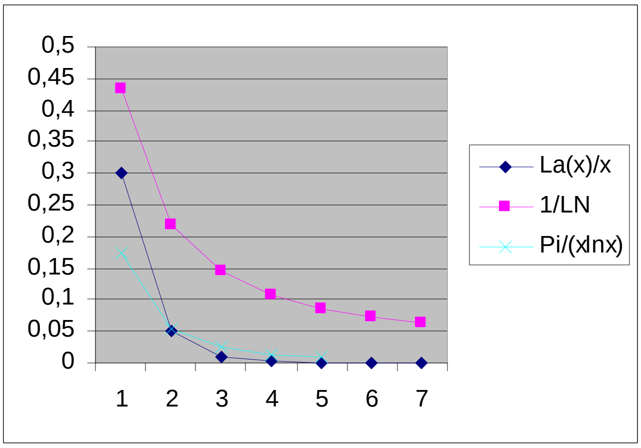


Figure 2 – behaviour of La(x)/x, π(x)/x ln x e 1/lnx

In figure 2 we see that the error is for low values of x. Another mode to see the same situation of the (2) is to consider that always A-B/C(N) < 1 (see figure 3).

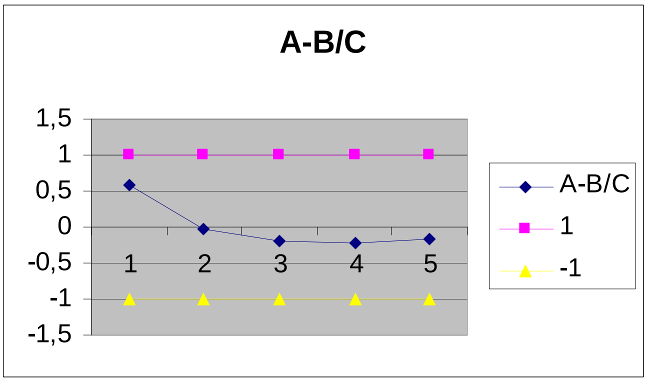


Figure 3 – A-B/C (N) < 1

Are the Landau's prime numbers infinite? Yes.

We have seen, through experimental way, that Landau's prime numbers are infinite. We can proof.

*In every arithmetic progression a, a + q, a + 2q, a + 3q, ... where the positive integers a and q are coprime, there are infinitely many primes (**Dirichlet's theorem on arithmetic progressions**).*

In fact if $a=n^2$ and q=1, con n even (with exception n=1), then gcd(a, q) = 1, so we can obtain infinite prime numbers. We say that: *"Landau's prime numbers n^2+1 are prime numbers if only if $\varphi(n^2+1)=n^2$"*, where φ(x) is the Euler's totient function.

Is true the Legendre's conjectures?

Let p be a prime number, then the Legendre's conjecture is:

$$\exists p : (2+n)^2 < p < (2+n+1)^2, \forall n \in N \quad (5)$$ [1]

In (5) the sign < and not ≤ derives from Lemma one.

Lemma one
A perfect square isn't a prime number.

It is a trivial lemma, which derives from *Fundamental Theorem of Arithmetic*..

Lemma two
Let x, a, d $\in$ *N, with* $x = (2+n+1)^2$, $a = (2+n)^2$ *and* $d=x-a$, *then* $d=2n+5$ *and it is always* $gcd(a, d)=1$.

<u>*Proof of the Lemma 2*</u>

For substitution from $d=x-a$ we obtain $d=2n+5$. Now for absurd we suppose *a* and *d* have got any common divisors such that gcd(a, d)≠1; then *a* is, for example, a multiple of *d*:

$$a = k\,d\,,\ k \in N \text{ (or k is integer).}$$

It is the same if we write:

$$a/d = k = [(2+n)^2]/(2n+5) = (n^2+4n+4)/2n+5$$

This is a division between two polynomials terms of 2.nd and 1.st grade.

We can use the Ruffini's rule and show that k isn't an integer, contrary to the hypothesis

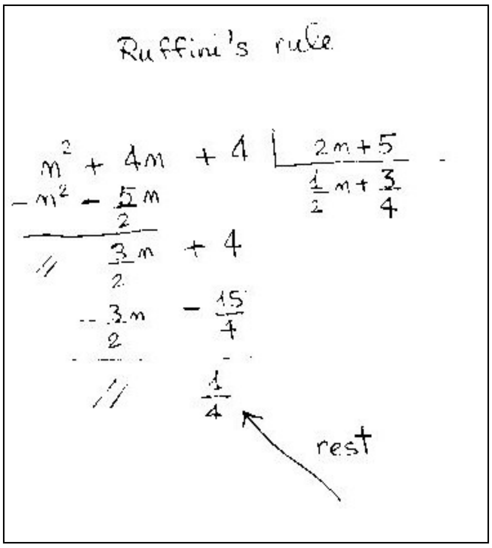


Figure 4 – Ruffini's rule

The Ruffini's rule says we have a rest, then k isn't a integer, then gcd(a, d)=1.

[1] We use the form $(2+n)^2$ and not n^2 because the integral above makes sense from 2.

In this case is satisfied the basic assumptions of the GRH.

We remember that:

$$\pi(x,a,d) = \frac{1}{\varphi(d)} \int_2^x \frac{1}{\ln t} dt + O(x^{\frac{1}{2}+\varepsilon}), \quad x \to \infty$$

In our case a=$(2+n)^2$, x=$(2+n+1)^2$. We use the form $(2+n)^2$ and not n^2 because the integral above makes sense from 2.

For simplicity we say

$$\pi(x,a,d) \sim \frac{1}{\varphi(d)} \cdot \left(\left[\frac{t}{\ln t} \right]_a^x \right)$$

For example we obtain a table of values.

π(x, a, d)	value
π(9,4,5)	0,302672
π(16,9,7)	0,279117
π(25,16,9)	0,332651
π(36,25,11)	0,22793
π(49,36,13)	0,212043
π(64,49,15)	0,310915
…	

Table 4 - π(x, a, d)

They are values ranging but never null values. Moreover when x grows, p grows and also **d grows**, which increases the probability of existence of prime numbers in the interval $(2+n)^2$ and $(2+n+1)^2$.

Now we remember the **Bertrand's postulate: "** If *n* is a positive integer greater than 1, then there is always a prime number *p* with $n < p < 2n$".

Now for *Bertand's postulate* the distance is: *d1 = 2n – n = n* and for *Legendre's conjecture d2 =2n+5, so d2>d1.*

Then a simple conclusion is that in the interval $(2+n)^2$ and $(2+n+1)^2$ there is at least a prime number.

Appendix

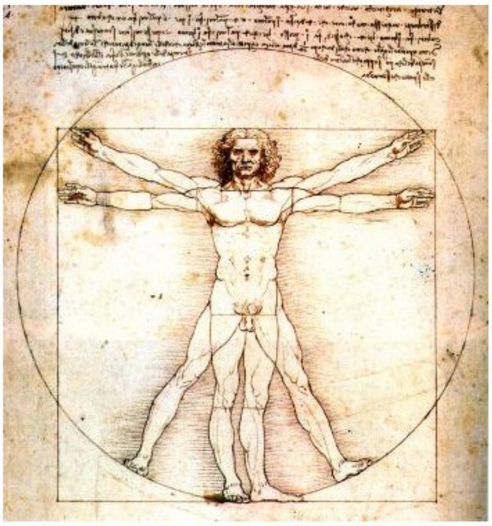
```
/*******************************************************
* ListLandauPrimes(n)
* It returns the Landau’s primes up to n on a file
*
* GetLandauPrimes(n)
* It counts the Landau’s primes up to n
*
* R. Turco
* PARI/GP
*******************************************************/

{ListLandauPrimes(n) = local(i, j);

  j=0;
  for(i=1,n,
     if( isprime(i*i + 1), write1("c:\\srcpari\\landau.txt"," ", i*i + 1));
     if( isprime(i*i + 1), j=j+1);
     if( i*i+1 > n, break;);
  );
  print1(" #Landau's prime up to ", n, " : ", j);

}
{GetLandauPrimes(n) = local(i,j);

  j=0;
  for(i=1,n,
     if( isprime(i*i + 1), j=j+1);
     if( i*i+1 > n, break;);
  );
  return(j);
}
```



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Sites

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