Goldbach, Twin Primes and Polignac Equivalent RH, the Landau's prime numbers and the Legendre's conjecture. Mathematical connections with "Aurea" section and some sectors of String Theory

Rosario Turco[1], Maria Colonnese, Michele Nardelli[2,3], Giovanni Di Maria, Francesco Di Noto, Annarita Tulumello

[2]Dipartimento di Scienze della Terra
Università degli Studi di Napoli Federico II, Largo S. Marcellino, 10
80138 Napoli, Italy

[3]Dipartimento di Matematica ed Applicazioni "R. Caccioppoli"
Università degli Studi di Napoli "Federico II" – Polo delle Scienze e delle Tecnologie
Monte S. Angelo, Via Cintia (Fuorigrotta), 80126 Napoli, Italy

Abstract

In this work the authors will explain and prove in the **Section 1** three equivalent RH, which are obtained linking G(N)/N with Li and g(N)/N with Li. Moreover the authors will show a new step function for G(N)/N and, through the GRH, a generalization for Polignac. Furthermore, will explain and prove in the **Section 2** an equivalent RH which is linked to the Landau's prime numbers and discuss on the Legendre's conjecture and on the infinity of Landau's prime numbers. In conclusion, we describe in the **Section 3** the mathematical connections with aurea ratio and in the **Section 4** with some equations concerning p-adic strings, p-adic and adelic zeta functions, zeta strings and zeta nonlocal scalar fields.

[1] Rosario Turco is an engineer at Telecom Italia (Naples) and creator of "Block Notes of Math" with the prof. Colonnese Maria of High School "De Bottis" of Torre del Greco, province of Naples, and all the other authors are part of the group ERATOSTENE of Caltanissetta (Sicily)

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1. Equivalent RH with G(x), g(x), P(x,d)

In [25] we have showed a closed form that links G(N) and π(N) or g(N) and π(N).

We says that:

$$\left|\frac{G(x)}{x} - \int_2^x \frac{dt}{t(\ln t)^2}\right| = O(x^{\frac{1}{2}+\varepsilon})$$

This is an equivalent RH (**R. Turco, M. Colonnese, ERATOSTENE group**)".

Can it be showed? Yes.

Several works [see also ERATOSTENE group] showed that:

$$G(N) \approx c\frac{N}{(\ln N)^2} \quad (1.1)$$

We don't think at the constant, then the PNT in simple form is:

$$\pi(N) \approx \frac{N}{\ln N}, \quad N \to \infty$$

From (1.1) and from PNT is: $\frac{G(N)}{N} \approx \frac{1}{(\ln N)^2}, \quad \frac{\pi(N)}{N \ln N} \approx \frac{1}{(\ln N)^2}$

Then the idea is:

$$\left|\frac{G(N)}{N} - \frac{\pi(N)}{N\ln N}\right| < KC(N), \ K=1, \ C(N)=\frac{1}{\ln N} \quad (1.2)$$

Introducing now the big O function, the previous expression becomes:

$$\left|\frac{G(N)}{N} - \frac{\pi(N)}{N\ln N}\right| = O((\ln N)^{-1}) \qquad (1.3)$$

and if instead of π(N) we introduce Li then it is:

$$\left|\frac{G(x)}{x} - \int_2^x \frac{dt}{t(\ln t)^2}\right| = O((\ln x)^{-1}) + O(x^{\frac{1}{2}+\varepsilon}) = O(x^{-\varepsilon}) + O(x^{\frac{1}{2}+\varepsilon}) \approx O(x^{\frac{1}{2}+\varepsilon}) \qquad (1.4)$$

The (1.4) is an expression in closed form and links the number of solutions to Goldbach G to Li and the (1.4) is an equivalent of RH. Then we have a function and its inverse, that can be go back to G(N) and vice versa with (1.2). We can do some calculation with the excel and set us also a rule that will automatically check the inequality ABS(A-B) < C1(N). We have "YES" if the rule is verified.

N	G(N)	A = G(N)/N	π(N)	B = π(N)/N LN(N)	ABS(A-B)	A-B	C1(N)=1/LN(N)	A-B<C1(N)?
4	1	0,25	2	0,36067376	0,11067376	-0,11067376	0,72134752	YES
6	1	0,166666667	3	0,279055313	0,112388647	-0,11238865	0,558110627	YES
8	1	0,125	4	0,240449173	0,115449173	-0,11544917	0,480898347	YES
10	2	0,2	4	0,173717793	0,026282207	0,026282207	0,434294482	YES
12	1	0,083333333	5	0,167679002	0,084345668	-0,08434567	0,402429604	YES
14	2	0,142857143	6	0,162395649	0,019538506	-0,01953851	0,378923182	YES
16	2	0,125	6	0,13525266	0,01025266	-0,01025266	0,36067376	YES
18	2	0,111111111	7	0,134546322	0,023435211	-0,02343521	0,345976256	YES
20	2	0,1	8	0,13352328	0,03352328	-0,03352328	0,333808201	YES

22	3	0,136363636	8	0,117641983	0,018721653	0,018721653	0,323515453	YES
24	3	0,125	9	0,117996743	0,007003257	0,007003257	0,31465798	YES
26	3	0,115384615	9	0,106244196	0,00914042	0,00914042	0,306927676	YES
28	2	0,071428571	9	0,096461238	0,025032666	-0,02503267	0,300101629	YES
30	3	0,1	10	0,098004701	0,001995299	0,001995299	0,294014104	YES
100	6	0,06	25	0,05428681	0,00571319	0,00571319	0,217147241	YES
200	8	0,04	46	0,043410008	0,003410008	-0,00341001	0,188739166	YES
300	21	0,07	62	0,036233266	0,033766734	0,033766734	0,175322254	YES
400	14	0,035	78	0,0325463	0,0024537	0,0024537	0,1669041	YES
500	13	0,026	95	0,030573127	0,004573127	-0,00457313	0,160911192	YES
600	32	0,053333333	102	0,026575249	0,026758084	0,026758084	0,156324996	YES
1000	28	0,028	168	0,024320491	0,003679509	0,003679509	0,144764827	YES
10000	128	0,0128	1229	0,013343698	0,000543698	-0,0005437	0,10857362	YES
100000	754	0,00754	9592	0,008331505	0,000791505	-0,00079151	0,086858896	YES
1000000*	5239	0,005239	78498	0,005681875	0,000442875	-0,00044287	0,072382414	YES
100000000	2593693	0,02593693	50847534	0,027603504	0,001666574	-0,00166657	0,05428681	YES

*with the formula of ERATOSTENE

Table 1 – G(N) and C1(N)

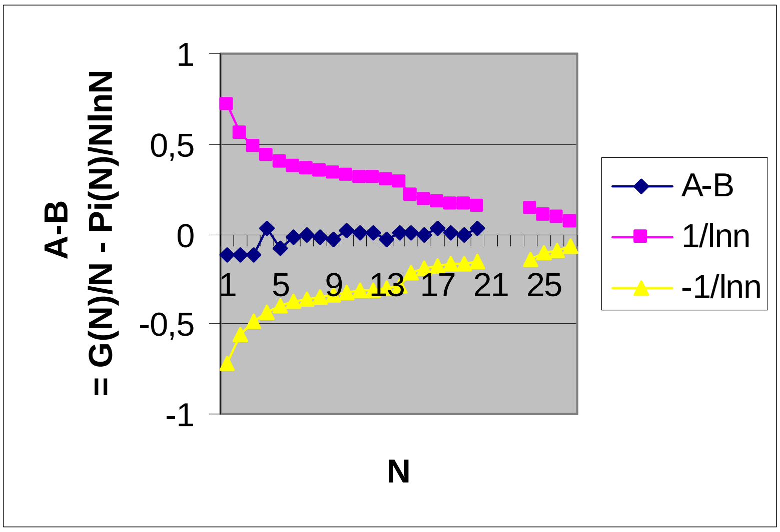

Figure 1 – bounds |1/lnN| of A-B

From (1.2) is G(N)>0; in fact G(x) make sense for x≥4. If the integral was equal to zero, we should have a positive area offset by a negative. But G (N) can not be negative, so G(N) ≠ 0.

The eq. (1.2) is almost a "tool that would have liked to Chebyshev", which brings together several concepts of probability (see [25]):

"*The difference in absolute value between the number of solutions Goldbach G(N), compared to the same number N, and the counting of prime numbers up to N compared to 'N-th prime number (~N ln N), is less than the probability that N is a prime number (~1/ln N).* "

The term of error 1/ln(N), numerically, *works well.*

In the table 1 we have also sign A-B without ABS (module) and we can see, in figure 1, A-B fluctuate around the zero. In this work we also indicate that $1/x^{½}$ *works well (See Table 2 and figure 2)* and the (1.4) is again true:

$$|\frac{G(x)}{x} - \int_2^x \frac{dt}{t(\ln t)^2}| \ = O(x^{-1/2}) + O(x^{\frac{1}{2}+\varepsilon}) \approx O(x^{\frac{1}{2}+\varepsilon})$$

In this case in the (1.2) C2(N)=1/N^{½}.

N	G(N)	A = G(N)/N	π(N)	B = π(N)/N LN(N)	ABS(A-B)	A-B	C2(N)=1/SQRT(N)	A-B<C2(N)?
4	1	0,25	2	0,36067376	0,11067376	-0,11067376	0,5	YES
6	1	0,166666667	3	0,279055313	0,112388647	-0,11238865	0,40824829	YES
8	1	0,125	4	0,240449173	0,115449173	-0,11544917	0,353553391	YES
10	2	0,2	4	0,173717793	0,026282207	0,026282207	0,316227766	YES
12	1	0,083333333	5	0,167679002	0,084345668	-0,08434567	0,288675135	YES
14	2	0,142857143	6	0,162395649	0,019538506	-0,01953851	0,267261242	YES
16	2	0,125	6	0,13525266	0,01025266	-0,01025266	0,25	YES
18	2	0,111111111	7	0,134546322	0,023435211	-0,02343521	0,23570226	YES
20	2	0,1	8	0,13352328	0,03352328	-0,03352328	0,223606798	YES
22	3	0,136363636	8	0,117641983	0,018721653	0,018721653	0,213200716	YES
24	3	0,125	9	0,117996743	0,007003257	0,007003257	0,204124145	YES
26	3	0,115384615	9	0,106244196	0,00914042	0,00914042	0,196116135	YES
28	2	0,071428571	9	0,096461238	0,025032666	-0,02503267	0,188982237	YES
30	3	0,1	10	0,098004701	0,001995299	0,001995299	0,182574186	YES
100	6	0,06	25	0,05428681	0,00571319	0,00571319	0,1	YES
200	8	0,04	46	0,043410008	0,003410008	-0,00341001	0,070710678	YES
300	21	0,07	62	0,036233266	0,033766734	0,033766734	0,057735027	YES
400	14	0,035	78	0,0325463	0,0024537	0,0024537	0,05	YES
500	13	0,026	95	0,030573127	0,004573127	-0,00457313	0,04472136	YES
600	32	0,053333333	102	0,026575249	0,026758084	0,026758084	0,040824829	YES
1000	28	0,028	168	0,024320491	0,003679509	0,003679509	0,031622777	YES
10000	128	0,0128	1229	0,013343698	0,000543698	-0,0005437	0,01	YES
100000	754	0,00754	9592	0,008331505	0,000791505	-0,00079151	0,003162278	YES
1000000	5239	0,005239	78498	0,005681875	0,000442875	-0,00044287	0,001	YES

Table 2 – G(N) and C2(N)

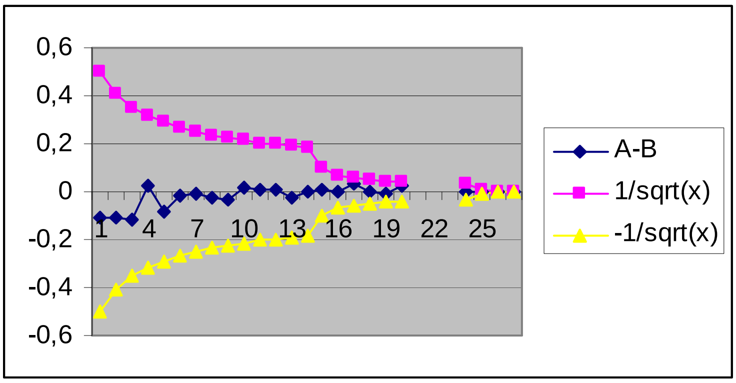

Figure 2 – bounds |1/x$^{1/2}$| of A-B

In general the term of error is 1/N$^{\varepsilon}$ where is ε=1/t with t∈R; but if t is greater than 2 , the bound is too much wide.

Another mode to see the same situation is to consider that always ABS(A-B)/C_1(N) < 1 (see figure 3).

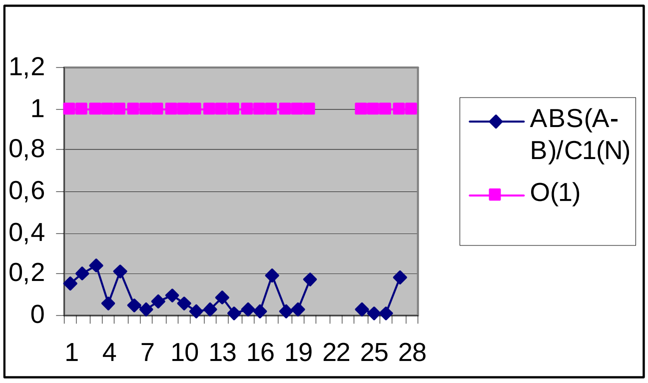

Figure 3 – ABS(A-B)/C1(N)

We can observe that *ln N* in the (1.2) is about H_n (harmonic number), but we could also introduce the von Mangoldt's function Λ(N) or we can calculate G(N)/N from (1.4) through Li/N. We observe that Li has got an expansion of type:

$$Li(x) = x/\ln x + x/(\ln x)^2 + \ldots + (n-1)! * x/(\ln x)^n + o(x/(\ln x)^n)$$

where n is the number of terms that we consider.

The report found for G (N) is good for calculating g(N) the number of pairs of twin primes:

$$\left| \frac{g(N)}{N} - \frac{\pi(N)}{N\ln N} \right| < KC(N), \quad K=1, \quad C(N)=\frac{1}{\ln N}$$

$$\left| \frac{g(x)}{x} - \int_2^x \frac{dt}{t(\ln t)^2} \right| = O(x^{\frac{1}{2}+\varepsilon}) \quad (1.5)$$

An immediate verification with some number we can make for a real conviction.

N	g(N)	A=g(N)/N	π(N)	B=π(N)/N*LN(N)	ABS(A-B)	C(N)	ABS(A-B)<C(N)?
3	0	0	2	0,606826151	0,606826151	0,910239227	YES
4	0	0	2	0,36067376	0,36067376	0,72134752	YES
5	1	0,2	3	0,372800961	0,172800961	0,621334935	YES
6	1	0,166666667	3	0,279055313	0,112388647	0,558110627	YES
7	2	0,285714286	4	0,293656196	0,00794191	0,513898342	YES
8	2	0,25	4	0,240449173	0,009550827	0,480898347	YES
9	2	0,222222222	4	0,202275384	0,019946839	0,455119613	YES
10	2	0,2	4	0,173717793	0,026282207	0,434294482	YES
11	2	0,181818182	5	0,189560178	0,007741996	0,417032391	YES
12	2	0,166666667	5	0,167679002	0,001012335	0,402429604	YES
13	3	0,230769231	6	0,179940575	0,050828656	0,389871245	YES
14	3	0,214285714	6	0,162395649	0,051890065	0,378923182	YES
15	3	0,2	6	0,147707749	0,052292251	0,369269373	YES
16	3	0,1875	6	0,13525266	0,05224734	0,36067376	YES
17	3	0,176470588	7	0,145334875	0,031135714	0,352956124	YES
18	3	0,166666667	7	0,134546322	0,032120345	0,345976256	YES
19	4	0,210526316	8	0,142999272	0,067527043	0,339623272	YES
20	3	0,15	8	0,13352328	0,01647672	0,333808201	YES
21	4	0,19047619	8	0,125127139	0,065349052	0,328458739	YES
22	4	0,181818182	8	0,117641983	0,064176199	0,323515453	YES
23	4	0,173913043	9	0,1247983	0,049114743	0,318928989	YES
24	4	0,166666667	9	0,117996743	0,048669924	0,31465798	YES
25	4	0,16	9	0,111840288	0,048159712	0,310667467	YES
26	4	0,153846154	9	0,106244196	0,047601958	0,306927676	YES
27	4	0,148148148	9	0,101137692	0,047010456	0,303413076	YES
28	4	0,142857143	9	0,096461238	0,046395905	0,300101629	YES
29	4	0,137931034	10	0,102404898	0,035526136	0,296974204	YES
30	4	0,133333333	10	0,098004701	0,035328632	0,294014104	YES
100	8	0,08	25	0,05428681	0,02571319	0,217147241	YES
200	15	0,075	46	0,043410008	0,031589992	0,188739166	YES
300	19	0,063333333	62	0,036233266	0,027100067	0,175322254	YES
400	21	0,0525	78	0,0325463	0,0199537	0,1669041	YES
500	24	0,048	95	0,030573127	0,017426873	0,160911192	YES
600	26	0,043333333	102	0,026575249	0,016758084	0,156324996	YES
1000	35	0,035	168	0,024320491	0,010679509	0,144764827	YES

Table 3 – g(N)

So even for g(N), similar considerations apply. Also (1.5) is an equivalent RH.

A step function and a generalization of Polignac

Now, we introduce the ***von Mangoldt's function*** *(also called lambda function)*:

$$\Lambda(n) = \begin{cases} \log p, & if\ n=p^k,\ \ p\ \ prime,\ k\geq 1 \\ 0, & \text{otherwise} \end{cases} \qquad (1.6)$$

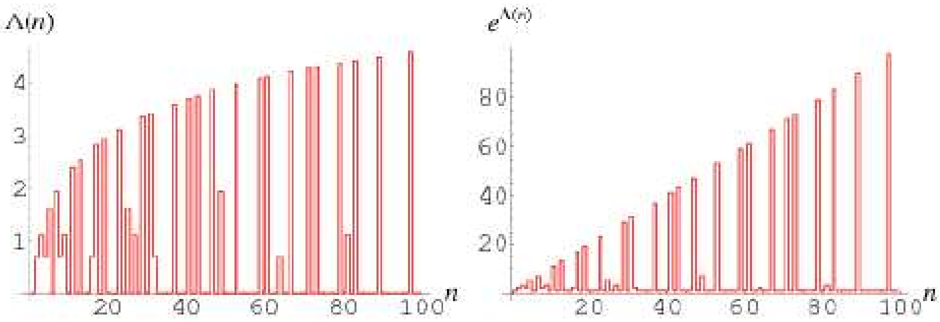

Figure 4 – von Mangoldt's function

The von Mongoldt's function isn't a multiplicative function nor an additive function. Moreover it's:

$$\log n = \sum_{d|n} \Lambda(d) \quad \text{where d | n are divisor of n}$$

<u>Example</u>

n=12

We remember $12=2^2*3$ and that the divisors of 12 are: 1, 2, 3, 4, 6, 12, then:
log 12=Λ(1)+ Λ(2)+ Λ(3)+ $\Lambda(2^2)$+ Λ(2*3)+ $\Lambda(2^2*3)$

From (1.6) it is:
log 12= 0 + log 2 + log 3 + log 2 + 0 + 0= log(2*3*2) = log 12

We know π(N) as a counting prime function (a step function):

$$\pi(N) = \sum_{p \le x} 1$$

If we introduce the von Mangoldt's function Λ(N) then *we propose* a "*step function υ(N)*" (see figure 6):

$$\nu(N) = \frac{\pi(N)}{N\ \Lambda(N)}$$

$$\frac{G(N)}{N} \sim \nu(N) \quad \text{or} \quad \lim_{x\to\infty} \frac{G(x)}{x} \Big/ \nu(x) = 1$$

For example

π(N)	**Λ(N)**	**υ(N)**	**G(N)/N**
π(10) = 4	Λ(10) = ln 10 = 2,3	υ(10) = 4/(10*2,3)=0,17	G(10)/10 = 0,1
π(30) = 10	Λ(30) = ln 30 = 3,401	υ(30) = 10/(30*3,401)=0,098≈0,1	G(30)/30 = 3/30=0,1
π(100) = 25	Λ(100) = ln 100 = 4,605	υ(100) = 25/(100*4,605)=0,054	G(100)/100 = 6/100=0,06
π(1000) = 168	Λ(1000) = ln 1000 = 6,907	υ(1000) = 168/(1000*6,907)=0,0243	G(1000)/1000 = 28/1000=0,028

The approssimations of the step function v(N) improve when N grows (see figure 5).

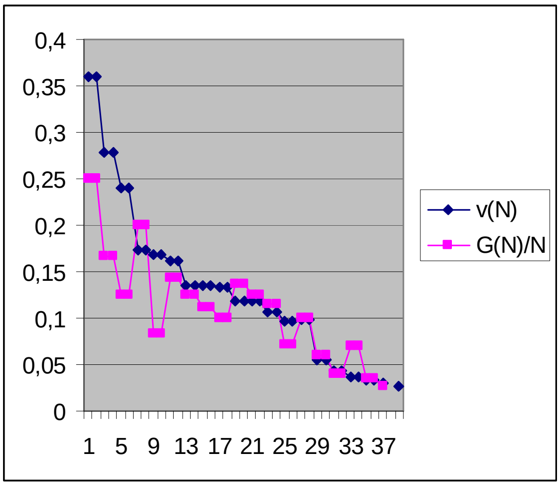

Figure 5 – step function v(N)

Can we generalize this result as *a generalization of Polignac*? Yes. If we call *P(x , d)* the number of primes ≤x and which are far d, if we remember the GRH [see 25], since it is:

$$\pi(x,a,d) = \frac{1}{\varphi(d)} \int_2^x \frac{1}{\ln t} dt + O(x^{\frac{1}{2}+\varepsilon}), \quad x \to \infty \quad (1.7)$$

where a, d are such that gcd(a, d)=1 and φ(d) is the *totient function* of Euler.

Then it is:

$$\left| \frac{P(x,d)}{x} - \frac{\pi(x,a,d)}{x \ln x} \right| < kC3(x), k = 1, C3(x) = \frac{1}{\ln x}$$

$$or \qquad (1.8)$$

$$\left| \frac{P(x,d)}{x} - \frac{1}{\varphi(d)} \int_2^x \frac{1}{t(\ln t)^2} dt \right| = O(x^{\frac{1}{2}+\varepsilon})$$

Example

x=127, a=2,d=9

gcd(a , d) =1

a + d: 11, 20, 29, 38, 47, 56, 65, 74, 83, 92, 101, 110, 119, 128 …

We have underlined the prime numbers above.

φ(9) = 6, in fact 1,2,4,5,7,8 are the numbers without nothing in common with 9,

Li ~ x/ln x

π(127-2,9) ~ [1/φ(9)]*(127/ln 127) ≈ 4,36 about 5.

But this result is also P(x, 9). In fact if we consider a=1,2,4,5,7,8 or the numbers less than 9 and without nothing in common with 9, we have six *arithmetic progressions*:

a=1, a + d: 10, 19, 28, 37, 46, 55, 64, 73, 82, 91, 100, 109, 118, 127 …
a=2, a + d: 11, 20, 29, 38, 47, 56, 65, 74, 83, 92, 101, 110, 119, 128 …
a=4, a + d: 13, 22, 31, 40, 49, 58, 67, 76, 85, 94, 103, 112, 121, 130 …
a=5, a + d: 14, 23, 32, 41, 50, 59, 68, 77, 86, 95, 104, 113, 122, 131 …
a=7, a + d: 16, 25, 34, 43, 52, 61, 70, 79, 88, 97, 106, 115, 124, 133 …
a=8, a + d: 17, 26, 35, 44, 53, 62, 71, 80, 89, 98, 107, 116, 125, 134 …

In all arithmetic progression we have prime numbers. How many are the prime numbers in each arithmetic progression?

About:

$$[1/\varphi(d)] * (x/\ln x).$$

Then the absolute value of difference [P(x, d)/x] - [π(x, a, d)/x ln x] is very little.

In fact for a = 2 is:

P(127,9)/127 = 5/127 = 0,00031
π(127, 2, 9)/(127 ln 127) = 0,007102
|[P(127, 9)/127] - [π(127, 2, 9)/127 ln 127]| = 0,00679 < 1/ln 127 = 0,206433.

We can obtain, with the integral, a value better than x/ln x; in fact (see Appendix) it is:

$$\int_2^x \frac{dt}{t \cdot \ln^2 t} = \frac{1}{\ln 2} - \frac{1}{\ln x}$$

Then it is:

π(127, 2, 9) = 1/6 * (1/ln 2 - 1/ln 127) = 0,20604
π(127, 2, 9) / 127 ln 127 = 0,000334 a similar result of P(x, d)/x
|[P(127, 9)/127] - [π(127, 2, 9)/127 ln 127]| = 0,000024 < 1/ln 127 = 0,206433.

So, also (1.8) is an equivalent RH.

Appendix

$$\int_2^x \frac{dt}{t\cdot \ln^2 t} = \int_2^x d(\ln x)\cdot \frac{dt}{\ln^2 t} = \left[\ln t \cdot \frac{1}{\ln^2 t}\right]_2^x - \int_2^x \ln t \cdot d(\frac{1}{\ln^2 t})dt =$$

$$\frac{1}{\ln x} - \frac{1}{\ln 2} - \int_2^x \ln t \cdot (\frac{-2\ln t}{\ln^4 t}\cdot \frac{1}{t})dt = \frac{1}{\ln x} - \frac{1}{\ln 2} + 2\int_2^x \frac{dt}{t\cdot \ln^2 t}$$

so :

$$\int_2^x \frac{dt}{t\cdot \ln^2 t} = \frac{1}{\ln x} - \frac{1}{\ln 2} + 2\int_2^x \frac{dt}{t\cdot \ln^2 t}$$

then :

$$\frac{1}{\ln 2} - \frac{1}{\ln x} = \int_2^x \frac{dt}{t\cdot \ln^2 t}$$

2. The Landau's prime numbers – an equivalent RH

We propose to call "*Landau's prime numbers*" the prime numbers that we can obtain from the form:

$$Landau's\ prime\ numbers : p = n^2 + 1, \text{with } p \text{ prime number}$$

In order that p is prime, n must be even but this excepts n=1 which gives 2. So we can't obtain them, for example, from any prime numbers or from any square prime numbers (this excepts for n=2).

The Landau's prime numbers *La(x)* aren't many (see Table 1), but we can infinitely produce them with a simple program (see Appendix) up to 1E+14. The formulas n^2+1 is similar at the Euler's form x^2+x+41.

x	La(x)
1E+1	3
1E+2	5
1E+3	10
1E+4	19
1E+5	51
1E+6	112
1E+7	316
1E+8	841
1E+9	2378
1E+10	6656
1E+11	18822
1E+12	54110
1E+13	156081
1E+14	456362

Table 4 – Landau's prime numbers

We propose a close form that links La(x) and π(N) or La(x) and Li(x).

We say that:

$$\left|\frac{La(x)}{x} - \int_2^x \frac{dt}{t(\ln t)^2}\right| = O(x^{\frac{1}{2}+\varepsilon}) \quad (2.1)$$

This is an equivalent RH (**R. Turco, M. Colonnese, ERATOSTENE group**)".

The (2.1) is equivalent to:

$$\left|\frac{La(N)}{N} - \frac{\pi(N)}{N\ln N}\right| < KC(N),\ K=1,\ C(N)=\frac{1}{2*\ln N} \quad (2.2)$$

Introducing now the big O function, the previous expression becomes:

$$\left|\frac{La(N)}{N} - \frac{\pi(N)}{N\ln N}\right| = O((2*\ln N)^{-1}) \qquad (2.3)$$

If instead of π(N) we introduce Li then it is:

$$\frac{La(x)}{x} - \int_2^x \frac{dt}{t(\ln t)^2}\Big| = O((2*\ln x)^{-1}) + O(x^{\frac{1}{2}+\varepsilon}) = O(2^{-1} * x^{-\varepsilon}) + O(x^{\frac{1}{2}+\varepsilon}) \approx O(x^{\frac{1}{2}+\varepsilon}) \qquad (2.4)$$

We can do some calculation with the excel and set us also a rule that will automatically check the inequality ABS(A-B) < C(N). We will have "YES" if the rule is verified.

x	La(x)	La(x)/x	π(x)	C(x)=1/LN(x)	π(x)/x ln x	ABS(A-B)	ABS(A-B)<C(x)?
10	3	0,3	4	0,434294482	0,173718	0,126282	YES
100	5	0,05	25	0,217147241	0,054287	0,004287	YES
1.000	10	0,01	168	0,144764827	0,02432	0,01432	YES
10.000	19	0,0019	1.229	0,10857362	0,013344	0,011444	YES
100.000	51	0,00051	9.592	0,086858896	0,008332	0,007822	YES
1.000.000	112	0,000112	78.498	0,072382414	0,005682	0,00557	YES
10.000.000	316	3,16E-05	664.579	0,062042069	0,004123	0,004092	YES

Table 5 – La(N) and C(N)

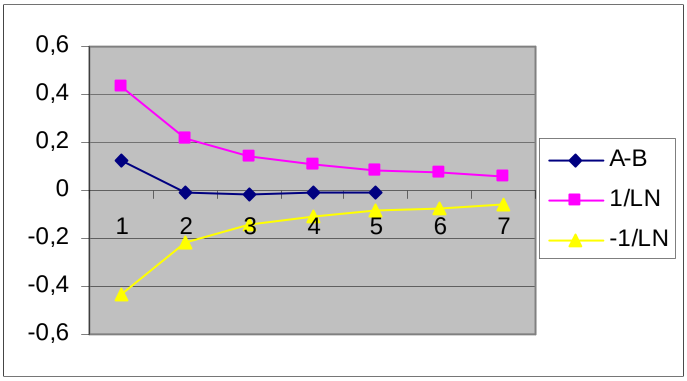

Figure 6 – bounds |1/lnN| of A-B

The term of error 1/(ln(N)) in (2.2), numerically, *works well*; while the term of error $1/\ln(N)^2$ has got an exception at N=100.000 (see table 3)

x	La(x)	La(x)/x	π(x)	C(x)=1/LN(x)^2	π(x)/x ln x	ABS(A-B)	ABS(A-B)<C(x)?
10	3	0,3	4	0,188611697	0,173718	0,126282	YES
100	5	0,05	25	0,047152924	0,054287	0,004287	YES
1.000	10	0,01	168	0,020956855	0,02432	0,01432	YES
10.000	19	0,0019	1.229	0,011788231	0,013344	0,011444	YES
100.000	51	0,00051	9.592	0,007544468	0,008332	0,007822	NO

Table 6 – 1/ln(N)^2

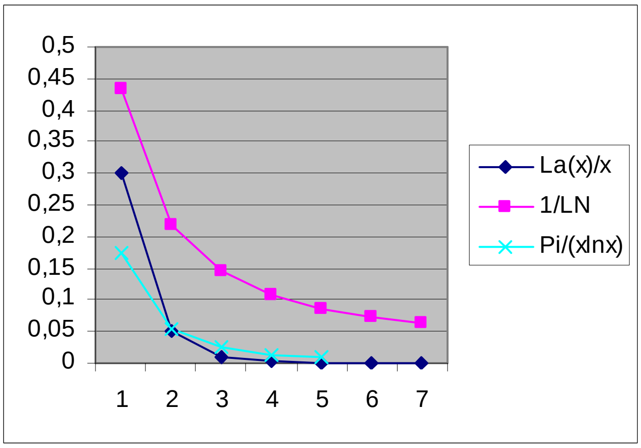

Figure 7 – behaviour of La(x)/x, π(x)/x ln x e 1/lnx

In figure 2 we see that the error is for low values of x. Another mode to see the same situation of the (2.2) is to consider that always A-B/C(N) < 1 (see figure 3).

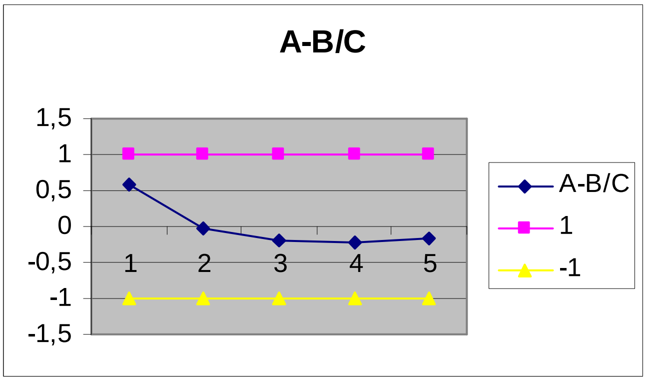

Figure 8 – A-B/C (N) < 1

Are the Landau's prime numbers infinite? Yes.

We have seen, through experimental way, that Landau's prime numbers are infinite. We can proof.

*In every arithmetic progression a, a + q, a + 2q, a + 3q, … where the positive integers a and q are coprime, there are infinitely many primes (**Dirichlet's theorem on arithmetic progressions**).*

In fact if $a=n^2$ and q=1, con n even (with exception n=1), then gcd(a, q) = 1, so we can obtain infinite prime numbers. We say that: "*Landau's prime numbers n^2+1 are prime numbers if only if $\varphi(n^2+1)=n^2$*", where φ(x) is the Euler's totient function.

Is true the Legendre's conjectures?

Let p be a prime number, then the Legendre's conjecture is:

$$\exists p : (2+n)^2 < p < (2+n+1)^2, \forall n \in N \quad (2.5)$$ [2]

In (2.5) the sign < and not ≤ derives from Lemma one.

Lemma one
A perfect square isn't a prime number.

It is a trivial lemma, which derives from *Fundamental Theorem of Arithmetic*..

Lemma two
Let x, a, d ∈ N, with $x = (2+n+1)^2$, $a = (2+n)^2$ *and* $d=x-a$, *then* $d=2n+5$ *and it is always* $\gcd(a, d)=1$.

<u>*Proof of the Lemma 2*</u>

For substitution from $d=x-a$ we obtain $d=2n+5$. Now for absurd we suppose *a* and *d* have got any common divisors such that $\gcd(a, d) \neq 1$; then *a* is, for example, a multiple of *d*:

$$a = k d, \quad k \in N \text{ (or k is integer).}$$

It is the same if we write:

$$a/d = k = [(2+n)^2]/(2n+5) = (n^2+4n+4)/2n+5$$

This is a division between two polynomials terms of 2.nd and 1.st grade.

We can use the Ruffini's rule and show that k isn't an integer, contrary to the hypothesis

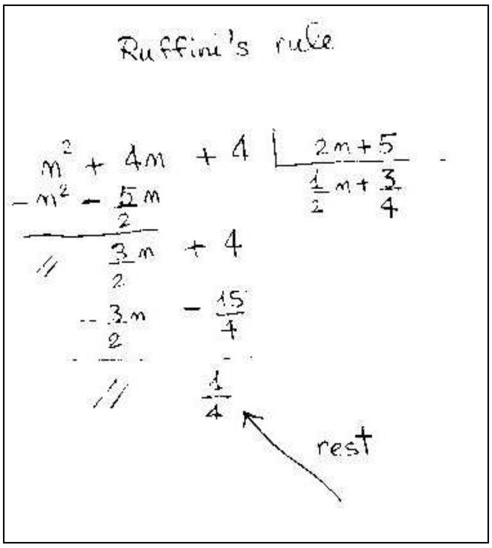

Figure 9 – Ruffini's rule

The Ruffini's rule says we have a rest, then k isn't a integer, then gcd(a, d)=1.

In this case is satisfied the basic assumptions of the GRH.

[2] We use the form $(2+n)^2$ and not n^2 because the integral above makes sense from 2.

We remember that:

$$\pi(x,a,d) = \frac{1}{\varphi(d)}\int_2^x \frac{1}{\ln t}dt + O(x^{\frac{1}{2}+\varepsilon}), \quad x \to \infty$$

In our case a=(2+n)^2, x=(2+n+1)^2. We use the form (2+n)^2 and not n^2 because the integral above makes sense from 2.

For simplicity we say

$$\pi(x,a,d) \approx \frac{1}{\varphi(d)} \cdot \left(\left[\frac{t}{\ln t} \right]_a^x \right)$$

For example we obtain a table of values.

π(x, a, d)	**value**
π(9,4,5)	0,302672
π(16,9,7)	0,279117
π(25,16,9)	0,332651
π(36,25,11)	0,22793
π(49,36,13)	0,212043
π(64,49,15)	0,310915
…	

Table 7 - π(x, a, d)

They are values ranging but never null values. Moreover when x grows, p grows and also **d grows**, which increases the probability of existence of prime numbers in the interval (2+n)^2 and (2+n+1)^2.

Now we remember the **Bertrand's postulate: "** If *n* is a positive integer greater than 1, then there is always a prime number *p* with *n* < *p* < 2*n"*.

Now for *Bertand's postulate* the distance is: *d1 = 2n – n = n* and for *Legendre's conjecture d2 =2n+5, so d2>d1.*

Then a simple conclusion is that in the interval (2+n)^2 and (2+n+1)^2 there is at least a prime number.

3. Mathematical connections with aurea ratio.

With regard the values 4,36 and 0,206433 (pages 9 and 10), we have the following mathematical connections:

$$(\Phi)^{21/7} + (\Phi)^{-30/7} = 4{,}236067977 + 0{,}127156535 = 4{,}36322\,; \quad (3.1)$$
$$(\Phi)^{-28/7} + (\Phi)^{-41/7} = 0{,}145898034 + 0{,}059693843 = 0{,}205591877 \cong 0{,}2056\,. \quad (3.2)$$

With regard the values denoted in red of Table 1 (page 4) and Table 2 (page 12), precisely 0,434294482 0,217147241 0,144764827 and 0,173717793 we have the following mathematical connections:

$$(\Phi)^{-14/7} + (\Phi)^{-43/7} = 0{,}381966011 + 0{,}052025802 \cong 0{,}4340\,; \quad (3.3)$$

$$(\Phi)^{-28/7} + (\Phi)^{-38/7} = 0{,}145898034 + 0{,}073366139 = 0{,}219264\,; \quad (3.4)$$

$$(\Phi)^{-36/7} + (\Phi)^{-41/7} = 0{,}084179515 + 0{,}059693843 = 0{,}143873358 \cong 0{,}144\,; \quad (3.5)$$

$$(\Phi)^{-34/7} + (\Phi)^{-37/7} = 0{,}096586666 + 0{,}07858706 = 0{,}1751737 \cong 0{,}175\,. \quad (3.6)$$

Note that $\Phi = \frac{\sqrt{5}+1}{2} \cong 1{,}6180339$ is the aurea ratio and that with regard the index n/7, n = 1, 2, …, $+\infty$, n = -1, -2,…, $-\infty$, while 7 is the number of the compactified dimensions of M-theory.

4. On some equations concerning p-adic strings, p-adic and adelic zeta functions, zeta strings and zeta nonlocal scalar fields. [27] [28] [29] [30] [31]

Like in the ordinary string theory, the starting point of p-adic strings is a construction of the corresponding scattering amplitudes. Recall that the ordinary crossing symmetric Veneziano amplitude can be presented in the following forms:

$$A_\infty(a,b) = g^2 \int_R |x|_\infty^{a-1} |1-x|_\infty^{b-1} dx = g^2 \left[\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} + \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)} + \frac{\Gamma(c)\Gamma(a)}{\Gamma(c+a)} \right] = g^2 \frac{\zeta(1-a)}{\zeta(a)} \frac{\zeta(1-b)}{\zeta(b)} \frac{\zeta(1-c)}{\zeta(c)} =$$

$$= g^2 \int DX \exp\left(-\frac{i}{2\pi} \int d^2\sigma \partial^\alpha X_\mu \partial_\alpha X^\mu \right) \prod_{j=1}^{4} \int d^2\sigma_j \exp\left(i k_\mu^{(j)} X^\mu \right), \quad (4.1-4.4)$$

where $\hbar = 1$, $T = 1/\pi$, and $a = -\alpha(s) = -1 - \frac{s}{2}$, $b = -\alpha(t)$, $c = -\alpha(u)$ with the condition $s + t + u = -8$, i.e. $a + b + c = 1$.
The p-adic generalization of the above expression

$$A_\infty(a,b) = g^2 \int_R |x|_\infty^{a-1} |1-x|_\infty^{b-1} dx\,,$$

is:

$$A_p(a,b) = g_p^2 \int_{Q_p} |x|_p^{a-1} |1-x|_p^{b-1} dx\,, \quad (4.5)$$

where $|\ldots|_p$ denotes p-adic absolute value. In this case only string world-sheet parameter x is treated as p-adic variable, and all other quantities have their usual (real) valuation.
Now, we remember that the Gauss integrals satisfy adelic product formula

$$\int_R \chi_\infty(ax^2 + bx) d_\infty x \prod_{p \in P} \int_{Q_p} \chi_p(ax^2 + bx) d_p x = 1, \quad a \in Q^\times, \quad b \in Q, \quad (4.6)$$

what follows from

$$\int_{Q_v} \chi_v\left(ax^2 + bx\right) d_v x = \lambda_v(a) |2a|_v^{-\frac{1}{2}} \chi_v\left(-\frac{b^2}{4a}\right), \quad v = \infty, 2, ..., p, ... \quad (4.7)$$

These Gauss integrals apply in evaluation of the Feynman path integrals

$$K_v(x'', t''; x', t') = \int_{x', t'}^{x'', t''} \chi_v\left(-\frac{1}{h}\int_{t'}^{t''} L(\dot{q}, q, t) dt\right) D_v q, \quad (4.8)$$

for kernels $K_v(x'', t''; x', t')$ of the evolution operator in adelic quantum mechanics for quadratic Lagrangians. In the case of Lagrangian

$$L(\dot{q}, q) = \frac{1}{2}\left(-\frac{\dot{q}^2}{4} - \lambda q + 1\right),$$

for the de Sitter cosmological model one obtains

$$K_\infty(x'', T; x', 0) \prod_{p \in P} K_p(x'', T; x', 0) = 1, \quad x'', x', \lambda \in Q, \; T \in Q^\times, \quad (4.9)$$

where

$$K_v(x'', T; x', 0) = \lambda_v(-8T) |4T|_v^{-\frac{1}{2}} \chi_v\left(-\frac{\lambda^2 T^3}{24} + [\lambda(x'' + x') - 2]\frac{T}{4} + \frac{(x'' - x')^2}{8T}\right). \quad (4.10)$$

Also here we have the number 24 that correspond to the Ramanujan function that has 24 "modes", i.e., the physical vibrations of a bosonic string. Hence, we obtain the following mathematical connection:

$$K_v(x'', T; x', 0) = \lambda_v(-8T) |4T|_v^{-\frac{1}{2}} \chi_v\left(-\frac{\lambda^2 T^3}{24} + [\lambda(x'' + x') - 2]\frac{T}{4} + \frac{(x'' - x')^2}{8T}\right) \Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4} w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10 + 11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10 + 7\sqrt{2}}{4}\right)}\right]}. \quad (4.10b)$$

The adelic wave function for the simplest ground state has the form

$$\psi_A(x) = \psi_\infty(x) \prod_{p \in P} \Omega\left(|x|_p\right) = \begin{cases} \psi_\infty(x), x \in Z \\ 0, x \in Q \setminus Z \end{cases}, \quad (4.11)$$

where $\Omega(|x|_p)=1$ if $|x|_p \le 1$ and $\Omega(|x|_p)=0$ if $|x|_p > 1$. Since this wave function is non-zero only in integer points it can be interpreted as discreteness of the space due to p-adic effects in adelic approach. The Gel'fand-Graev-Tate gamma and beta functions are:

$$\Gamma_\infty(a)=\int_R |x|_\infty^{a-1}\chi_\infty(x)d_\infty x=\frac{\zeta(1-a)}{\zeta(a)}, \quad \Gamma_p(a)=\int_{Q_p}|x|_p^{a-1}\chi_p(x)d_p x=\frac{1-p^{a-1}}{1-p^{-a}}, \quad (4.12)$$

$$B_\infty(a,b)=\int_R |x|_\infty^{a-1}|1-x|_\infty^{b-1}d_\infty x=\Gamma_\infty(a)\Gamma_\infty(b)\Gamma_\infty(c), \quad (4.13)$$

$$B_p(a,b)=\int_{Q_p}|x|_p^{a-1}|1-x|_p^{b-1}d_p x=\Gamma_p(a)\Gamma_p(b)\Gamma_p(c), \quad (4.14)$$

where $a,b,c \in C$ with condition $a+b+c=1$ and $\zeta(a)$ is the Riemann zeta function. With a regularization of the product of p-adic gamma functions one has adelic products:

$$\Gamma_\infty(u)\prod_{p\in P}\Gamma_p(u)=1, \quad B_\infty(a,b)\prod_{p\in P}B_p(a,b)=1, \quad u\neq 0,1, \quad u=a,b,c, \quad (4.15)$$

where $a+b+c=1$. We note that $B_\infty(a,b)$ and $B_p(a,b)$ are the crossing symmetric standard and p-adic Veneziano amplitudes for scattering of two open tachyon strings. Introducing real, p-adic and adelic zeta functions as

$$\zeta_\infty(a)=\int_R \exp(-\pi x^2)|x|_\infty^{a-1}d_\infty x=\pi^{-\frac{a}{2}}\Gamma\left(\frac{a}{2}\right), \quad (4.16)$$

$$\zeta_p(a)=\frac{1}{1-p^{-1}}\int_{Q_p}\Omega(|x|_p)|x|_p^{a-1}d_p x=\frac{1}{1-p^{-a}}, \quad \mathrm{Re}\,a>1, \quad (4.17)$$

$$\zeta_A(a)=\zeta_\infty(a)\prod_{p\in P}\zeta_p(a)=\zeta_\infty(a)\zeta(a), \quad (4.18)$$

one obtains

$$\zeta_A(1-a)=\zeta_A(a), \quad (4.19)$$

where $\zeta_A(a)$ can be called adelic zeta function. We have also that

$$\zeta_A(a)=\zeta_\infty(a)\prod_{p\in P}\zeta_p(a)=\zeta_\infty(a)\zeta(a)=\int_R \exp(-\pi x^2)|x|_\infty^{a-1}d_\infty x\cdot\frac{1}{1-p^{-1}}\int_{Q_p}\Omega(|x|_p)|x|_p^{a-1}d_p x. \quad (4.19b)$$

Let us note that $\exp(-\pi x^2)$ and $\Omega(|x|_p)$ are analogous functions in real and p-adic cases. Adelic harmonic oscillator has connection with the Riemann zeta function. The simplest vacuum state of the adelic harmonic oscillator is the following Schwartz-Bruhat function:

$$\psi_A(x)=2^{\frac{1}{4}}e^{-\pi x_\infty^2}\prod_{p\in P}\Omega(|x_p|_p), \quad (4.20)$$

whose the Fourier transform

$$\psi_A(k) = \int \chi_A(kx)\psi_A(x) = 2^{\frac{1}{4}} e^{-\pi k_\infty^2} \prod_{p\in P} \Omega\left(\left|k_p\right|_p\right) \quad (4.21)$$

has the same form as $\psi_A(x)$. The Mellin transform of $\psi_A(x)$ is

$$\Phi_A(a) = \int \psi_A(x)|x|^a d_A^\times x = \int_R \psi_\infty(x)|x|^{a-1} d_\infty x \prod_{p\in P} \frac{1}{1-p^{-1}} \int_{Q_p} \Omega\left(|x|_p\right)|x|^{a-1} d_p x = \sqrt{2}\Gamma\left(\frac{a}{2}\right)\pi^{-\frac{a}{2}}\zeta(a) \quad (4.22)$$

and the same for $\psi_A(k)$. Then according to the Tate formula one obtains (4.19).

The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$\mathcal{L}_p = \frac{1}{g^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2}}\varphi + \frac{1}{p+1}\varphi^{p+1}\right], \quad (4.23)$$

where p is any prime number, $\Box = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+...+)$. Now, we want to show a model which incorporates the p-adic string Lagrangians in a restricted adelic way. Let us take the following Lagrangian

$$L = \sum_{n\geq1} C_n \mathcal{L}_n = \sum_{n\geq1} \frac{n-1}{n^2}\mathcal{L}_n = \frac{1}{g^2}\left[-\frac{1}{2}\phi \sum_{n\geq1} n^{-\frac{\Box}{2}}\phi + \sum_{n\geq1}\frac{1}{n+1}\phi^{n+1}\right]. \quad (4.24)$$

Recall that the Riemann zeta function is defined as

$$\zeta(s) = \sum_{n\geq1}\frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}, \quad s = \sigma + i\tau, \quad \sigma > 1. \quad (4.25)$$

Employing usual expansion for the logarithmic function and definition (4.25) we can rewrite (4.24) in the form

$$L = -\frac{1}{g^2}\left[\frac{1}{2}\phi\zeta\left(\frac{\Box}{2}\right)\phi + \phi + \ln(1-\phi)\right], \quad (4.26)$$

where $|\phi| < 1$. $\zeta\left(\frac{\Box}{2}\right)$ acts as pseudodifferential operator in the following way:

$$\zeta\left(\frac{\Box}{2}\right)\phi(x) = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk, \quad -k^2 = k_0^2 - \vec{k}^2 > 2 + \varepsilon, \quad (4.27)$$

where $\tilde{\phi}(k) = \int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.

Dynamics of this field ϕ is encoded in the (pseudo)differential form of the Riemann zeta function. **When the d'Alambertian is an argument of the Riemann zeta function we shall call such string a "zeta string".** Consequently, the above ϕ is an open scalar zeta string. The equation of motion for the zeta string ϕ is

$$\zeta\left(\frac{\Box}{2}\right)\phi=\frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon}e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}, \quad (4.28)$$

which has an evident solution $\phi=0$.

For the case of time dependent spatially homogeneous solutions, we have the following equation of motion

$$\zeta\left(\frac{-\partial_t^2}{2}\right)\phi(t)=\frac{1}{(2\pi)}\int_{|k_0|>\sqrt{2}+\varepsilon}e^{-ik_0t}\zeta\left(\frac{k_0^2}{2}\right)\tilde{\phi}(k_0)dk_0=\frac{\phi(t)}{1-\phi(t)}. \quad (4.29)$$

With regard the open and closed scalar zeta strings, the equations of motion are

$$\zeta\left(\frac{\Box}{2}\right)\phi=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\sum_{n\geq 1}\theta^{\frac{n(n-1)}{2}}\phi^n, \quad (4.30)$$

$$\zeta\left(\frac{\Box}{4}\right)\theta=\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk=\sum_{n\geq 1}\left[\theta^{n^2}+\frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right], \quad (4.31)$$

and one can easily see trivial solution $\phi=\theta=0$.

The exact tree-level Lagrangian of effective scalar field φ, which describes open p-adic string tachyon, is:

$$\mathcal{L}_p=\frac{m_p^D}{g_p^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2m_p^2}}\varphi+\frac{1}{p+1}\varphi^{p+1}\right], \quad (4.32)$$

where p is any prime number, $\Box=-\partial_t^2+\nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature $(-+\ldots+)$, as above. Now, we want to introduce a model which incorporates all the above string Lagrangians (4.32) with p replaced by $n\in N$. Thence, we take the sum of all Lagrangians $\mathcal{L}_n$ in the form

$$L=\sum_{n=1}^{+\infty}C_n\mathcal{L}_n=\sum_{n=1}^{+\infty}C_n\frac{m_n^D}{g_n^2}\frac{n^2}{n-1}\left[-\frac{1}{2}\phi n^{-\frac{\Box}{2m_n^2}}\phi+\frac{1}{n+1}\phi^{n+1}\right], \quad (4.33)$$

whose explicit realization depends on particular choice of coefficients C_n, masses m_n and coupling constants g_n.

Now, we consider the following case

$$C_n=\frac{n-1}{n^{2+h}}, \quad (4.34)$$

where h is a real number. The corresponding Lagrangian reads

$$L_h=\frac{m^D}{g^2}\left[-\frac{1}{2}\phi\sum_{n=1}^{+\infty}n^{-\frac{\Box}{2m^2}-h}\phi+\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right] \quad (4.35)$$

and it depends on parameter h. According to the Euler product formula one can write

$$\sum_{n=1}^{+\infty} n^{-\frac{\Box}{2m^2}-h} = \prod_p \frac{1}{1-p^{-\frac{\Box}{2m^2}-h}}. \quad (4.36)$$

Recall that standard definition of the Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}, \quad s = \sigma + i\tau, \quad \sigma > 1, \quad (4.37)$$

which has analytic continuation to the entire complex s plane, excluding the point $s = 1$, where it has a simple pole with residue 1. Employing definition (4.37) we can rewrite (4.35) in the form

$$L_h = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\zeta\left(\frac{\Box}{2m^2}+h\right)\phi + \sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right]. \quad (4.38)$$

Here $\zeta\left(\frac{\Box}{2m^2}+h\right)$ acts as a pseudodifferential operator

$$\zeta\left(\frac{\Box}{2m^2}+h\right)\phi(x) = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+h\right)\tilde{\phi}(k)dk, \quad (4.39)$$

where $\tilde{\phi}(k) = \int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$.
We consider Lagrangian (4.38) with analytic continuations of the zeta function and the power series $\sum \frac{n^{-h}}{n+1}\phi^{n+1}$, i.e.

$$L_h = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\zeta\left(\frac{\Box}{2m^2}+h\right)\phi + AC\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right], \quad (4.40)$$

where AC denotes analytic continuation.
Potential of the above zeta scalar field (4.40) is equal to $-L_h$ at $\Box = 0$, i.e.

$$V_h(\phi) = \frac{m^D}{g^2}\left(\frac{\phi^2}{2}\zeta(h) - AC\sum_{n=1}^{+\infty}\frac{n^{-h}}{n+1}\phi^{n+1}\right), \quad (4.41)$$

where $h \neq 1$ since $\zeta(1) = \infty$. The term with ζ-function vanishes at $h = -2, -4, -6, \ldots$. The equation of motion in differential and integral form is

$$\zeta\left(\frac{\Box}{2m^2}+h\right)\phi = AC\sum_{n=1}^{+\infty} n^{-h}\varphi^n, \quad (4.42)$$

$$\frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+h\right)\tilde{\phi}(k)dk = AC\sum_{n=1}^{+\infty} n^{-h}\phi^n, \quad (4.43)$$

respectively.
Now, we consider five values of h, which seem to be the most interesting, regarding the Lagrangian (4.40): $h=0$, $h=\pm 1$, and $h=\pm 2$. For $h=-2$, the corresponding equation of motion now read:

$$\zeta\left(\frac{\Box}{2m^2}-2\right)\phi=\frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}-2\right)\tilde{\phi}(k)dk=\frac{\phi(\phi+1)}{(1-\phi)^3}. \quad (4.44)$$

This equation has two trivial solutions: $\phi(x)=0$ and $\phi(x)=-1$. Solution $\phi(x)=-1$ can be also shown taking $\tilde{\phi}(k)=-\delta(k)(2\pi)^D$ and $\zeta(-2)=0$ in (4.44).
For $h=-1$, the corresponding equation of motion is:

$$\zeta\left(\frac{\Box}{2m^2}-1\right)\phi=\frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}-1\right)\tilde{\phi}(k)dk=\frac{\phi}{(1-\phi)^2}. \quad (4.45)$$

where $\zeta(-1)=-\frac{1}{12}$.
The equation of motion (4.45) has a constant trivial solution only for $\phi(x)=0$.
For $h=0$, the equation of motion is

$$\zeta\left(\frac{\Box}{2m^2}\right)\phi=\frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}. \quad (4.46)$$

It has two solutions: $\phi=0$ and $\phi=3$. The solution $\phi=3$ follows from the Taylor expansion of the Riemann zeta function operator

$$\zeta\left(\frac{\Box}{2m^2}\right)=\zeta(0)+\sum_{n\geq 1}\frac{\zeta^{(n)}(0)}{n!}\left(\frac{\Box}{2m^2}\right)^n, \quad (4.47)$$

as well as from $\tilde{\phi}(k)=(2\pi)^D 3\delta(k)$.
For $h=1$, the equation of motion is:

$$\frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+1\right)\tilde{\phi}(k)dk=-\frac{1}{2}\ln(1-\phi)^2, \quad (4.48)$$

where $\zeta(1)=\infty$ gives $V_1(\phi)=\infty$.
In conclusion, for $h=2$, we have the following equation of motion:

$$\frac{1}{(2\pi)^D}\int_{R^D}e^{ixk}\zeta\left(-\frac{k^2}{2m^2}+2\right)\tilde{\phi}(k)dk=-\int_0^{\phi}\frac{\ln(1-w)^2}{2w}dw. \quad (4.49)$$

Since holds equality

$$-\int_0^1\frac{\ln(1-w)}{w}dw=\sum_{n=1}^{\infty}\frac{1}{n^2}=\zeta(2)$$

one has trivial solution $\phi = 1$ in (4.49).

Now, we want to analyze the following case: $C_n = \frac{n^2 - 1}{n^2}$. In this case, from the Lagrangian (4.33), we obtain:

$$L = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\left\{\zeta\left(\frac{\Box}{2m^2} - 1\right) + \zeta\left(\frac{\Box}{2m^2}\right)\right\}\phi + \frac{\phi^2}{1-\phi}\right]. \quad (4.50)$$

The corresponding potential is:

$$V(\phi) = -\frac{m^D}{g}\frac{31 - 7\phi}{24(1-\phi)}\phi^2. \quad (4.51)$$

We note that 7 and 31 are prime natural numbers, i.e. $6n \pm 1$ with $n = 1$ and 5, with 1 and 5 that are Fibonacci's numbers. Furthermore the number 24 is related to the Ramanujan function that has 24 "modes" that correspond to the physical vibrations of a bosonic string. Thence, we obtain:

$$V(\phi) = -\frac{m^D}{g}\frac{31 - 7\phi}{24(1-\phi)}\phi^2 \Rightarrow \frac{\pi\sqrt{142}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (4.51b)$$

The equation of motion is:

$$\left[\zeta\left(\frac{\Box}{2m^2} - 1\right) + \zeta\left(\frac{\Box}{2m^2}\right)\right]\phi = \frac{\phi\left[(\phi-1)^2 + 1\right]}{(\phi-1)^2}. \quad (4.52)$$

Its weak field approximation is:

$$\left[\zeta\left(\frac{\Box}{2m^2} - 1\right) + \zeta\left(\frac{\Box}{2m^2}\right) - 2\right]\phi = 0, \quad (4.53)$$

which implies condition on the mass spectrum

$$\zeta\left(\frac{M^2}{2m^2} - 1\right) + \zeta\left(\frac{M^2}{2m^2}\right) = 2. \quad (4.54)$$

From (4.54) it follows one solution for $M^2 > 0$ at $M^2 \approx 2.79m^2$ and many tachyon solutions when $M^2 < -38m^2$.

We note that the number 2.79 in connected with the ϕ and Φ, i.e. the "aureo" numbers. Indeed, we have that:

$$\left(\frac{\sqrt{5}+1}{2}\right)^2 + \left(\frac{\sqrt{5}-1}{2}\right) \cong 2.78$$

Furthermore, we have also that:

$$(\Phi)^{14/7}+(\Phi)^{-25/7}=2{,}618033989+0{,}179314566=2{,}79734$$

With regard the extension by ordinary Lagrangian, we have the Lagrangian, potential, equation of motion and mass spectrum condition that, when $C_n=\frac{n^2-1}{n^2}$, are:

$$L=\frac{m^D}{g^2}\left[\frac{\phi}{2}\left\{\frac{\Box}{m^2}-\zeta\left(\frac{\Box}{2m^2}-1\right)-\zeta\left(\frac{\Box}{2m^2}\right)-1\right\}\phi+\frac{\phi^2}{2}\ln\phi^2+\frac{\phi^2}{1-\phi}\right], \quad (4.55)$$

$$V(\phi)=\frac{m^D}{g^2}\frac{\phi^2}{2}\left[\zeta(-1)+\zeta(0)+1-\ln\phi^2-\frac{1}{1-\phi}\right], \quad (4.56)$$

$$\left[\zeta\left(\frac{\Box}{2m^2}-1\right)+\zeta\left(\frac{\Box}{2m^2}\right)-\frac{\Box}{m^2}+1\right]\phi=\phi\ln\phi^2+\phi+\frac{2\phi-\phi^2}{(1-\phi)^2}, \quad (4.57)$$

$$\zeta\left(\frac{M^2}{2m^2}-1\right)+\zeta\left(\frac{M^2}{2m^2}\right)=\frac{M^2}{m^2}. \quad (4.58)$$

In addition to many tachyon solutions, equation (4.58) has two solutions with positive mass: $M^2\approx 2.67m^2$ and $M^2\approx 4.66m^2$.
We note, also here, that the numbers 2.67 and 4.66 are related to the "aureo" numbers. Indeed, we have that:

$$\left(\frac{\sqrt{5}+1}{2}\right)^2+\frac{1}{2\cdot 5}\left(\frac{\sqrt{5}-1}{2}\right)\cong 2.6798,$$

$$\left(\frac{\sqrt{5}+1}{2}\right)^2+\left(\frac{\sqrt{5}+1}{2}\right)+\frac{1}{2\cdot 2}\left(\frac{\sqrt{5}+1}{2}\right)\cong 4.64057.$$

Furthermore, we have also that:

$$(\Phi)^{14/7}+(\Phi)^{-41/7}=2{,}618033989+0{,}059693843=2{,}6777278;$$
$$(\Phi)^{22/7}+(\Phi)^{-30/7}=4{,}537517342+0{,}127156535=4{,}6646738.$$

With regard the **Section 1** and **2**, we have the following interesting mathematical connections between the eqs. (1.5), (1.8), (2.1) and the eqs. (4.28), (4.30), (4.31), (4.43), (4.45), (4.46), (4.48) and (4.49). Indeed, with the eqs. (4.28), (4.30) and (4.31), for example, we obtain that:

$$\left|\frac{g(x)}{x}-\int_2^x\frac{dt}{t(\ln t)^2}\right|=O(x^{\frac{1}{2}+\varepsilon})\Rightarrow\frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon}e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}\Rightarrow$$

$$\Rightarrow\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\sum_{n\geq1}\theta^{\frac{n(n-1)}{2}}\phi^n\Rightarrow$$

$$\Rightarrow\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk=\sum_{n\geq1}\left[\theta^{n^2}+\frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right]; \quad (4.59)$$

$$
\begin{aligned}
&|\frac{P(x,d)}{x}-\frac{\pi(x,a,d)}{x\ln x}|<kC3(x), k=1, C3(x)=\frac{1}{\ln x} \\
&or \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow \\
&|\frac{P(x,d)}{x}-\frac{1}{\varphi(d)}\int_2^x\frac{1}{t(\ln t)^2}dt|=O(x^{\frac{1}{2}+\varepsilon})
\end{aligned}
$$

$$
\Rightarrow \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}\Rightarrow\Rightarrow\frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\sum_{n\geq 1}\theta^{\frac{n(n-1)}{2}}\phi^n\Rightarrow
$$

$$
\Rightarrow \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk=\sum_{n\geq 1}\left[\theta^{n^2}+\frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right]; \quad (4.60)
$$

$$
|\frac{\mathrm{La(x)}}{\mathrm{x}}-\int_2^{\mathrm{x}}\frac{\mathrm{dt}}{t(\ln t)^2}| = \mathrm{O}(\mathrm{x}^{\frac{1}{2}+\varepsilon})\Rightarrow\frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}\Rightarrow
$$

$$
\Rightarrow \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk=\sum_{n\geq 1}\theta^{\frac{n(n-1)}{2}}\phi^n\Rightarrow
$$

$$
\Rightarrow \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{4}\right)\tilde{\theta}(k)dk=\sum_{n\geq 1}\left[\theta^{n^2}+\frac{n(n-1)}{2(n+1)}\theta^{\frac{n(n-1)}{2}-1}\left(\phi^{n+1}-1\right)\right]. \quad (4.61)
$$

In conclusion, with eq. (4.46), for example, we have the following mathematical connections:

$$
|\frac{\mathrm{g(x)}}{\mathrm{x}}-\int_2^{\mathrm{x}}\frac{\mathrm{dt}}{t(\ln t)^2}| = \mathrm{O}(\mathrm{x}^{\frac{1}{2}+\varepsilon})\Rightarrow\frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}; \quad (4.62)
$$

$$
\begin{aligned}
&|\frac{P(x,d)}{x}-\frac{\pi(x,a,d)}{x\ln x}|<kC3(x), k=1, C3(x)=\frac{1}{\ln x} \\
&or \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow \\
&|\frac{P(x,d)}{x}-\frac{1}{\varphi(d)}\int_2^x\frac{1}{t(\ln t)^2}dt|=O(x^{\frac{1}{2}+\varepsilon})
\end{aligned}
$$

$$
\Rightarrow \frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}; \quad (4.63)
$$

$$
|\frac{\mathrm{La(x)}}{\mathrm{x}}-\int_2^{\mathrm{x}}\frac{\mathrm{dt}}{t(\ln t)^2}| = \mathrm{O}(\mathrm{x}^{\frac{1}{2}+\varepsilon})\Rightarrow\frac{1}{(2\pi)^D}\int_{R^D} e^{ixk}\zeta\left(-\frac{k^2}{2m^2}\right)\tilde{\phi}(k)dk=\frac{\phi}{1-\phi}; \quad (4.64)
$$

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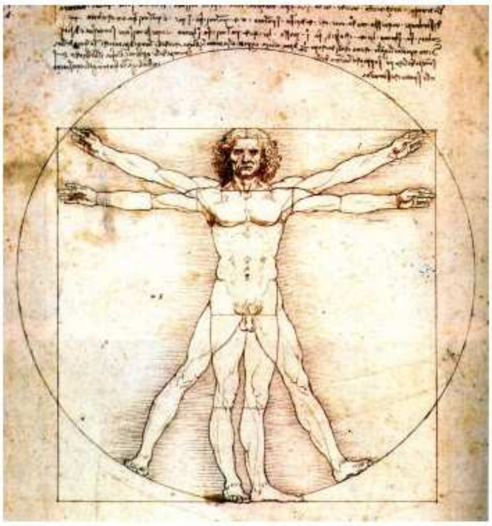

References

[1] John Derbyshire, "L'ossessione dei numeri primi: Bernhard Riemann e il principale problema irrisolto della matematica ", Bollati Boringhieri.
[2] J. B. Conrey, "The Riemann Hypothesis", Notices of the AMS, March 2003.
[3] E. C. Titchmarsh, "The Theory of the Riemann Zeta-function", Oxford University Press 2003.
[4] A. Ivic, "The Riemann Zeta-Function: Theory and Applications", Dover Publications Inc 2003.
[5] Proposta di dimostrazione della variante Riemann di Lagarias – Francesco Di Noto e Michele Nardelli – sito ERATOSTENE
[6]Test di primalità, fattorizzazione e π(N) con forme 6k±1 - Rosario Turco, Michele Nardelli, Giovanni Di Maria, Francesco Di Noto, Annarita Tulumello – CNR SOLAR Marzo 2008
[7] Fattorizzazione con algoritmo generalizzato con quadrati perfetti in ambito delle forme 6k±1" – Rosario Turco, Michele Nardelli, Giovanni Di Maria, Francesco Di Noto, Annarita Tulumello, Maria Colonnese – CNR SOLAR
[8] Semiprimi e fattorizzazione col modulo – Rosario Turco, Maria Colonnese – CNR SOLAR Maggio 2008
[9] Algoritmi per la congettura di Goldbach - G(N) reale- Rosario Turco – CNR SOLAR (2007)
[10] Il segreto della spirale di Ulam, le forme 6k±1 e il problema di Goldbach – Rosario Turco - R CNR Solar 2008 – The secret of Ulam's spiral, the forms 6k±1 and the Goldbach's problem
http://www.secamlocal.ex.ac.uk/people/staff/mrwatkin/zeta/ulam.htm
[11] Numeri primi in cerca di autore: Goldbach, numeri gemelli, Riemann, Fattorizzazione - Rosario Turco, Michele Nardelli, Giovanni Di Maria, Francesco Di Noto, Annarita Tulumello, Maria Colonnese – CNR SOLAR
[12] Teoria dei numeri e Teoria di Stringa, ulteriori connessioni Congettura (Teorema) di Polignac, Teorema di Goldston –Yldirim e relazioni con Goldbach e numeri primi gemelli" – Michele Nardelli e Francesco Di Noto – CNR SOLAR Marzo 2007;
[13] Teoremi sulle coppie di Goldbach e le coppie di numeri primi gemelli: connessioni tra Funzione zeta di Riemann, Numeri Primi e Teorie di Stringa" Nardelli Michele e Francesco Di Noto- CNRSOLAR Luglio 2007;
[14] Note su una soluzione positiva per le due congetture di Goldbach" - Nardelli Michele, Di Noto Francesco, Giovanni Di Maria e Annarita Tulumello - CNR SOLAR Luglio 2007

[15] Articoli del prof. Di Noto – sito gruppo ERATOSTENE
[16] I numeri primi gemelli e l'ipotesi di Riemann generalizzata", a cura della Prof. Annarita Tulumello
[17] Super Sintesi "Per chi vuole imparare in fretta e bene" MATEMATICA - Massimo Scorretti e Mario Italo Trioni – Avallardi
[18] Introduzione alla matematica discreta – Maria Grazia Bianchi e Anna Gillio – McGraw Hill
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[22] A computational Introduction to number theory and Algebra – Victor Shoup
[23] An Introduction to the theory of numbers – G.H. Hardy and E.M. Wright
[24] A Course in Number Theory and Crittography – Neal Koblitz
[25] Block Notes of Math – On the shoulders of giants – dedicated to Georg Friedrich Bernhard Riemann – Rosario Turco, Maria Colonnese, Michele Nardelli, Giovanni Di Maria, Francesco Di Noto, Annarita Tulumello
[26] Block Notes Matematico – Sulle spalle dei giganti – dedicato a Georg Friedrich Bernhard Riemann – Rosario Turco, Maria Colonnese, Michele Nardelli, Giovanni Di Maria, Francesco Di Noto, Annarita Tulumello – sul sito CNR Solar oppure su Database prof. Watkins Oxford
http://www.secamlocal.ex.ac.uk/people/staff/mrwatkin/zeta/tutorial.htm
[27] Branko Dragovich: "Adelic strings and noncommutativity" – arXiv:hep-th/0105103v1- 11 May 2001.
[28] Branko Dragovich: "Adeles in Mathematical Physics" – arXiv:0707.3876v1 [hep-th]– 26 Jul 2007.
[29] Branko Dragovich: "Zeta Strings" – arXiv:hep-th/0703008v1 – 1 Mar 2007.
[30] Branko Dragovich: "Zeta Nonlocal Scalar Fields" – arXiv:0804.4114v1 – [hep-th] – 25 Apr 2008.
[31] Branko Dragovich: "Some Lagrangians with Zeta Function Nonlocality" – arXiv:0805.0403 v1 – [hep-th] – 4 May 2008.

Sites

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http://xoomer.alice.it/stringtheory/