On the possible applications of some theorems concerning the Number Theory to the various mathematical aspects and sectors of String Theory I

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Abstract

The aim of this paper is that of show the further and possible connections between the p-adic and adelic strings and Lagrangians with Riemann zeta function with some problems, equations and theorems in Number Theory.
In the **Section 1**, we have described some equations and theorems concerning the quadrature- and mean-convergence in the Lagrange interpolation. In the **Section 2**, we have described some equations and theorems concerning the difference sets of sequences of integers. In the **Section 3**, we have showed some equations and theorems regarding some problems of a statistical group theory (symmetric groups) and in the **Section 4**, we have showed some equations and theorems concerning the measure of the non-monotonicity of the Euler Phi function and the related Riemann zeta function.
In the **Section 5**, we have showed some equations concerning the p-adic and adelic strings, the zeta strings and the Lagrangians for adelic strings
In conclusion, in the **Section 6**, we have described the mathematical connections concerning the various sections previously analyzed. Indeed, in the **Section 1**, **2** and **3**, where are described also various theorems on the prime numbers, we have obtained some mathematical connections with the Ramanujan's modular equations, thence with the modes corresponding to the physical vibrations of the bosonic and supersymmetric strings and also with p-adic and adelic strings. Principally, in the **Section 3**, where is frequently used the Hardy-Ramanujan stronger asymptotic formula and are described some theorems concerning the prime numbers. With regard the **Section 4**, we have obtained some mathematical connections between some equations concerning the Euler Phi function, the related Riemann zeta function and the zeta strings and field Lagrangians for p-adic sector of adelic string (**Section 5**). Furthermore, in the **Sections 1, 2, 3** and **4**, we have described also various mathematical expressions regarding some frequency connected with the exponents of the Aurea ratio, i.e. with the exponents of the number $\Phi = \frac{\sqrt{5}+1}{2}$. We consider important remember that the number 7 of the various exponents is related to the compactified dimensions of the M-theory.

1. On some equations and theorems concerning the quadrature- and mean-convergence in the Lagrange interpolation. [1]

Let

$$B \equiv \begin{Bmatrix} x_1^{(1)} & & & \\ x_1^{(2)} & x_2^{(2)} & & \\ \vdots & & & \\ x_1^{(n)} & x_2^{(n)} & & x_n^{(n)} \end{Bmatrix} \quad (1.1)$$

be an aggregate of points, where for every n

$$1 \geq x_1^{(n)} > x_2^{(n)} > \ldots > x_n^{(n)} \geq -1. \quad (1.2)$$

Let $f(x)$ be defined in the interval $[-1,+1]$ (hence also the value of the Aurea ratio 0.618033987). We define the n^{th} Lagrange-parabola of $f(x)$ with respect to B, as the polynomial of degree $\leq n-1$, which takes at the points $x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)}$ the values $f\left(x_1^{(n)}\right), f\left(x_2^{(n)}\right), \ldots, f\left(x_n^{(n)}\right)$. We denote this polynomial by $L_n(f)$ and we sometimes omit to indicate its dependence upon x and B. It is known, that

$$L_n(f) = \sum_{i=1}^{n} f\left(x_i^{(n)}\right) l_i^{(n)}(x) \equiv \sum_{i=1}^{n} f(x_i) l_i(x). \quad (1.3)$$

The functions $l_i(x)$ called the fundamental functions of the interpolation, are polynomials of degree $n-1$ and if

$$\omega(x) \equiv \omega_n(x) = \prod_{i=1}^{n} (x - x_i) \quad (1.4)$$

then

$$l_i(x) = \frac{\omega(x)}{\omega'(x_i)(x - x_i)}. \quad (1.5)$$

It is known that if $\psi(x)$ is a polynomial of the m^{th} degree, then

$$L_{m+k}(\psi) \equiv \psi(x) \qquad k = 1,2,\ldots \quad \text{(with } k \text{ also prime number)} \quad (1.6)$$

When $\psi(x) \equiv 1$, we obtain from (1.6) and (1.3)

$$\sum_{i=1}^{n} l_i(x) \equiv 1. \quad (1.7)$$

Mean convergence requires for any bounded and R integrable function $f(x)$

$$\lim_{n\to\infty}\int_{-1}^{1}[f(x)-L_n(f)]^2dx=0. \quad (1.8)$$

THEOREM I.

Let $p(x)$ *be a function such that*

$$p(x)\geq M>0 \qquad -1\leq x\leq +1, \quad (1.9) \qquad \int_{-1}^{1}p(x)dx \quad (1.10) \quad \textit{exists.}$$

It is known that there is an infinite sequence of polynomials $\omega_0(x),\omega_1(x),\ldots$ *where the degree of* $\omega_n(x)$ *is* n *with*
$\int_{-1}^{1}\omega_n(x)\omega_m(x)p(x)dx\neq 0$ *if* $n=m$, $\int_{-1}^{1}\omega_n(x)\omega_m(x)p(x)dx=0$ *if* $n\neq m$; *coefficient of* x^n *in* $\omega_n(x)=1$.
As known $\omega_n(x)$ *has in* $[-1,+1]$*(hence also the value of the Aurea ratio 0.618033987)* n *different real roots. Then our relation (1.8) is true for any matrix formed of these roots*. Or more generally,

THEOREM Ia.

Let $\omega_n(x)$ *be the above polynomials,* A_n *and* B_n *constants such that the equation*

$$R_n(x)\equiv x^n+\ldots\equiv\omega_n(x)+A_n\omega_{n-1}(x)+B_n\omega_{n-2}(x)=0 \quad (1.11)$$

may have in $[-1,+1]$*(hence also the value of the Aurea ratio 0.618033987)* n *different real roots and* $B_n\leq 0$*; then (1.8) holds also for the matrices formed by these roots.*
We prove Theorem Ia by proving the relation

$$\lim_{n\to\infty}\int_{-1}^{1}[f(x)-L_n(f)]^2p(x)dx=0, \quad (1.12)$$

which will be shown to be a consequence of $p(x)\geq M$ and of the existence of $\int_{-1}^{1}p(x)dx$. From (1.12) it follows by (1.9) that

$$0\leq\int_{-1}^{1}[f(x)-L_n(f)]^2dx\leq\frac{1}{M}\int_{-1}^{1}[f(x)-L_n(f)]^2p(x)dx, \quad (1.13)$$

and this by (1.12) establishes Theorem Ia.

COROLLARY OF THEOREM Ia.

For all bounded and R *integrable* $f(x)$ *we have for the matrices given in Theorem Ia*

$$\lim\int_{-1}^{1}|f(x)-L_n(f)|dx=0 \quad (1.14)$$

and a fortiori there is quadrature convergence for these matrices.

THEOREM II.

If the sequence

$$C(n)=\sum_{k=1}^{n}\int_{-1}^{1} l_k(x)^2\,dx$$

is unbounded as $n\to\infty$*, there exists a continuous* $f_6(x)$ *such that for our matrix*

$$\overline{\lim_{n\to\infty}}\int_{-1}^{1}[f_6(x)-L_n(f_6)]^2\,dx=+\infty .$$

We have to prove (1.12) for the fundamental points given by the roots of the $R_n(x)$ polynomials of (1.11). First we prove that

$$\int_{-1}^{1} l_i(x)p(x)dx\ge 0 \qquad i=1,2,\ldots,n \text{ (with } n \text{ also prime number)} \qquad (1.15)$$

and

$$\sum_{i=1}^{n}\int_{-1}^{1} l_i(x)^2\,p(x)dx\le\int_{-1}^{1} p(x)dx . \quad (1.16)$$

Consider the expression

$$\int_{-1}^{1}\left[l_i(x)^2-l_i(x)\right]p(x)dx .$$

But $l_i(x)^2-l_i(x)=R_n(x)F(x)$, where $F(x)$ is a polynomial of degree $(n-2)$, in which the coefficient of the highest term is evidently $1/\omega'_n(x_i)^2$. Thus if $F(x)=c_0\omega_0(x)+\ldots+\omega_{n-2}(x)/\omega'_n(x_i)^2$, by the orthogonality of the $\omega_n(x)$'s we have

$$\int_{-1}^{1}\left[l_i(x)^2-l_i(x)\right]p(x)dx=\frac{B_n}{\omega'_n(x_i)^2}\int_{-1}^{1}\omega_{n-2}(x)^2\,p(x)dx\le 0$$

i.e.

$$\int_{-1}^{1} l_i(x)^2\,p(x)dx\le\int_{-1}^{1} l_i(x)p(x)dx \quad (1.17)$$

which immediately establishes (1.15); by summation for $i=1,2,\ldots,n$ (with n also prime number) we obtain (1.16) in consequence of (1.7).
Let now Ω_4 be an aggregate in $[-1,+1]$ (hence also the value of the Aurea ratio 0.618033987) formed of closed non-overlapping intervals. We prove that

$$\sum_{\substack{i\\ x_i^{(n)}\{\Omega_4}}}\sum_{\substack{i\\ x_k^{(n)}\{\Omega_4}}}\left|\int_{-1}^{1} l_i(x)l_k(x)p(x)dx\right|\le 2\sum_{x_i^{(n)}\{\Omega_4}}\int_{-1}^{1} l_i(x)^2\,p(x)dx . \quad (1.18)$$

First we assert that for every k,i with $1\le k\le n$, $1\le i\le n$ (with n also prime number)

$$(-1)^{i+k+1} I_{ik} \equiv (-1)^{i+k+1} \int_{-1}^{1} l_i(x) l_k(x) p(x) dx \geq 0 \quad \text{if} \quad i \neq k . \quad (1.19)$$

For by (1.5)

$$I_{ik} = \frac{1}{R'_n(x_i) R'_n(x_k)} \int_{-1}^{1} \frac{R_n(x)}{(x-x_i)(x-x_k)} R_n(x) p(x) dx .$$

As $i \neq k$,

$$\frac{R_n(x)}{(x-x_k)(x-x_i)} = d_0 \omega_0(x) + \ldots + d_{n-3} \omega_{n-3}(x) + \omega_{n-2}(x).$$

Hence considering the definition of $R_n(x)$ we have

$$I_{ik} = \frac{B_n}{R'_n(x_i) R'_n(x_k)} \int_{-1}^{1} \omega_{n-2}(x)^2 p(x) dx$$

which proves (1.19), as $B_n \leq 0$ and sign $R'_n(x_i) R'_n(x_k) = (-1)^{i+k}$. Thus we have

$$\sum_{\substack{i \\ x_i^{(n)} \{ \Omega_4}} \sum_{\substack{k \\ x_k^{(n)} \{ \Omega_4}} \left| \int_{-1}^{1} l_i(x) l_k(x) p(x) dx \right| = \sum_{x_i^{(n)} \{ \Omega_4} \int_{-1}^{1} l_i(x)^2 p(x) dx - \sum_{x_i^{(n)} \{ \Omega_4} \sum_{x_k^{(n)} \{ \Omega_4} (-1)^{i+k} \int_{-1}^{1} l_i(x) l_k(x) p(x) dx =$$

$$= 2 \sum_{x_i^{(n)} \{ \Omega_4} \int_{-1}^{1} l_i(x)^2 p(x) dx - \int_{-1}^{1} \left[\sum_{x_i^{(n)} \{ \Omega_4} (-1)^i l_i(x) \right]^2 p(x) dx \leq 2 \sum_{x_i^{(n)} \{ \Omega_4} \int_{-1}^{1} l_i(x)^2 p(x) dx ;$$

thus (1.18) is proved.
If Ω_4 denotes the whole of the interval $[-1,+1]$ (hence also the value of the Aurea ratio 0.618033987), in consequence of (1.16) we have

$$\sum_{i=1}^{n} \sum_{k=1}^{n} \left| \int_{-1}^{1} l_i(x) l_k(x) p(x) dx \right| \leq 2 \int_{-1}^{1} p(x) dx . \quad (1.20)$$

Let now $f(x)$ be continuous, $\varphi(x)$ the polynomial of degree $n-1$ that gives the best approximation to it in Tschebischeff's sense for the interval $[-1,+1]$. Write

$$f(x) - \varphi(x) = \Delta(x) \quad (1.21)$$

$$\max_{|x| \leq 1} | f(x) - \varphi(x) | = E_{n-1} \quad (1.22)$$

$$I_n = \int_{-1}^{1} [f(x) - L_n(f)]^2 p(x) dx .$$

Then by (1.6) we have

$$I_n = \int_{-1}^{1} [\Delta(x) - L_n(\Delta)]^2 p(x) dx \leq 2 \int_{-1}^{1} \Delta(x)^2 p(x) dx + 2 \int_{-1}^{1} L_n(\Delta)^2 p(x) dx \equiv I'_n + I''_n . \quad (1.23)$$

We have evidently

$$I'_n \le 2E_{n-1}^2 \int_{-1}^{1} p(x)dx . \quad (1.24)$$

Further by (1.3)

$$\left|I''_n\right| = 2\left|\sum_{i=1}^{n}\sum_{k=1}^{n} \Delta(x_i)\Delta(x_k)\int_{-1}^{1} l_i(x) l_k(x) p(x)dx\right| \le 2E_{n-1}^2 \sum_{i=1}^{n}\sum_{k=1}^{n}\left|\int_{-1}^{1} l_i(x) l_k(x) p(x)dx\right| \le 4E_{n-1}^2 \int_{-1}^{1} p(x)dx$$

(1.25)

by (1.20); from (1.23), (1.24) and (1.25) we have

$$\left|I_n\right| \le 6E_{n-1}^2 \int_{-1}^{1} p(x)dx \quad (1.26)$$

which by Weierstrass' theorem establishes (1.12) for any continuous $f(x)$.

LEMMA.

Let B_1 be a matrix satisfying the following expression $\int_{-1}^{1} l_i(x)p(x)dx$, and Ω_5 be a set of a finite number of non-overlapping intervals in $[-1,+1]$ (thence also the value of the Aurea ratio 0.618033987) ; then for $n > n_0$ we have

$$\sum_{\substack{i \\ x_i^{(n)}\{\Omega_5}} \int_{-1}^{1} l_i(x)p(x)dx < 2\int_{\Omega_5} p(x)dx . \quad (1.27)$$

We easily obtain this result if we consider the function $\psi(x)$ having the value 1 for points of Ω_5 and 0 elsewhere. $\psi(x)$ is evidently bounded and R integrable, so that according to Fejér's theorem

$$\lim_{n\to\infty} \int_{-1}^{1} L_n(\psi)p(x)dx = \int_{-1}^{1} \psi(x)p(x)dx . \quad (1.28)$$

But by the definition of $\psi(x)$ we may write

$$\int_{-1}^{1} L_n(\psi)p(x)dx = \sum_{\substack{i \\ x_i^{(n)}\{\Omega_5}} \int_{-1}^{1} l_i(x)p(x)dx , \quad (1.29)$$

further

$$\int_{-1}^{1} \psi(x)p(x)dx = \int_{\Omega_5} p(x)dx . \quad (1.30)$$

The expression (1.27) is an evident consequence of (1.28), (1.29) and (1.30). Now we consider the matrix B defined as in Theorem Ia. In consequence of (1.15) the Lemma is applicable; we obtain from (1.17) and (1.27)

$$\sum_{\substack{i \\ x_i^{(n)}\{\Omega_5}} \int_{-1}^{1} l_i(x)^2 p(x)dx \le \sum_{\substack{i \\ x_i^{(n)}\{\Omega_5}} \int_{-1}^{1} l_i(x)p(x)dx < 2\int_{\Omega_5} p(x)dx\,, \quad (1.31)$$

and finally from (1.18)

$$\sum_{\substack{i \\ x_i^{(n)}\{\Omega_5}} \sum_{\substack{k \\ x_k^{(n)}\{\Omega_5}} \left| \int_{-1}^{1} l_i(x) l_k(x) p(x)dx \right| < 4\int_{\Omega_5} p(x)dx\,. \quad (1.32)$$

Let now $f(x)$ be any bounded and R integrable function. Then in virtue of the Riemann integrability, to any ε we can find a finite aggregate of non-overlapping open intervals of total length $\le \varepsilon$ such that if we exclude these intervals, the oscillation of the function is $\le \varepsilon$ at any point of the remaining aggregate Ω_6. We now define $f_7(x)$ as follows: (i) in Ω_6 let $f_7(x) \equiv f(x)$. (ii) if we denote the excluded intervals by $(p_1, q_1)\ldots(p_\nu, q_\nu)$, ($\nu$ finite), the function $f_7(x)$ is represented in (p_i, q_i) by the straight line connecting the point $(p_i, f(p_i))$ and $(q_i, f(q_i))$. Thus we define $f_7(x)$ for the whole of $[-1,+1]$ (hence also the value of the Aurea ratio 0.618033987), and its oscillation is at any point $\le \varepsilon$. But then $f_7(x)$ may be uniformly approximated by a polynomial $\varphi(x)$ to within 2ε. Let the degree of $\varphi(x)$ be $m = m(\varepsilon)$. Then we have

$$\begin{aligned} I_n \equiv \int_{-1}^{1} [f(x) - L_n(f)]^2 p(x)dx &\le 2\int_{-1}^{1} [f_7(x) - L_n(f_7)]^2 p(x)dx + 2\int_{-1}^{1} [f - f_7 - L_n(f - f_7)]^2 p(x)dx \le \\ &\le 2\int_{-1}^{1} [f_7(x) - L_n(f_7)]^2 p(x)dx + 4\int_{-1}^{1} [f - f_7]^2 p(x)dx + 4\int_{-1}^{1} L_n(f - f_7)^2 p(x)dx \equiv J'_n + J''_n + J'''_n\,, \end{aligned} \quad (1.33)$$

say. As the degree of approximation to $f_7(x)$ is 2ε, we have by (1.26) for $n > m(\varepsilon)$

$$|J'_n| \le 24\varepsilon^2 \int_{-1}^{1} p(x)dx\,. \quad (1.34)$$

Further as $f(x) - f_7(x)$ differs from 0 only upon intervals, of which the total length is $\le \varepsilon$ and as $|f(x) - f_7(x)| \le 2\max_{|x|\le 1} |f(x)| \equiv 2M$, we have

$$|J''_n| \le 16M^2 \sum_{i=1}^{\nu} \int_{p_i}^{q_i} p(x)dx\,. \quad (1.35)$$

For J'''_n we may evidently write

$$J'''_n = \sum_{i=1}^{n} \sum_{k=1}^{n} (f(x_i) - f_7(x_i))(f(x_k) - f_7(x_k)) \int_{-1}^{1} l_i(x) l_k(x) p(x)dx\,.$$

In consequence of the definition of $f_7(x)$ the terms of this sum differ from 0 only when x_i and x_k lie in intervals (p_l, q_l) and (p_μ, q_μ) respectively. Hence

$$\left|J_n'''\right| \le 4M^2 \sum_{\substack{i \\ x_i\{(p_l,q_l)}} \sum_{\substack{k \\ x_k\{(p_\mu,q_\mu)}} \left|\int_{-1}^{1} l_i(x) l_k(x) p(x) dx\right| \le 16M^2 \sum_{i=1}^{\nu} \int_{p_i}^{q_i} p(x) dx, \quad \text{with} \quad l, \mu = 1,2,\ldots,\nu \qquad (1.36)$$

by (1.32). As the total length of the range of integration is $\le \varepsilon$, it is evident by (1.33), (1.34), (1.35) and (1.36), that $I_n \to 0$ as $n \to \infty$. Hence the result. Let us write

$$S(n) = \sum_{i=1}^{n} \int_{-1}^{1} l_i(x)^2 dx,$$

and suppose this to be unbounded as $n \to \infty$. We shall prove that we can find a continuous function $f(x)$ with

$$\limsup_{n\to\infty} \int_{-1}^{1} [f(x) - L_n(f)]^2 dx = +\infty.$$

By hypothesis there exists an infinite sequence $n_1 < n_2 < \ldots$ with $S(n_1) < S(n_2) < \ldots \to \infty$. For the sake of simplicity of notation we denote by m the m^{th} element of this sequence n_m.
Let the m^{th} fundamental points be $1 \ge \xi_1^{(m)} > \xi_2^{(m)} > \ldots > \xi_m^{(m)} \ge -1$. We regard them as abscissas and to any $\xi_i^{(m)}$ we adjoin an ordinate ε_i, where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ have arbitrarily the values + 1 or – 1. Thus we have m points; we connect them obtaining a continuous function $\psi_\varepsilon(x)$ with

$$|\psi_\varepsilon(x)| \le 1 \quad \text{for} \quad -1 \le x \le +1 \qquad (\text{thence also } x = 0.618033987\text{, i.e. the Aurea ratio}) \qquad (1.37)$$

and

$$\int_{-1}^{1} L_m(\psi_\varepsilon)^2 dx = \sum_{\mu=1}^{m} \sum_{\nu=1}^{m} \varepsilon_\mu \varepsilon_\nu \int_{-1}^{1} l_i(x) l_k(x) dx. \qquad (1.38)$$

By variation of the ε's we obtain 2^m different $\psi_\varepsilon(x)$ functions. For these functions we have by forming the sums of (1.38)

$$\frac{1}{2^m} \sum_{\varepsilon} \int_{-1}^{1} L_m(\psi_\varepsilon)^2 dx = \sum_{\nu=1}^{m} \int_{-1}^{1} l_\nu(x)^2 dx = S(m), \qquad (1.39)$$

hence we may choose ε's, so that for the corresponding $\psi_\varepsilon(x)$ which we simply denote by $\psi(x)$, we have

$$\int_{-1}^{1} L_m(\psi)^2 dx \ge S(m). \qquad (1.40)$$

According to Weierstrass, $\psi(x)$ may be approximated by a polynomial $f_m(x)$ of degree $\mu(m)$ so that

$$|f_m(x)| \le \frac{3}{2} \qquad -1 \le x \le +1 \quad (\text{hence also the value of the Aurea ratio } 0.618033987) \qquad (1.41)$$

and

$$\int_{-1}^{1} L_m(f_m)^2 dx \ge \frac{1}{2} S(m). \qquad (1.42)$$

Now we select a partial sequence $f_{m_1}, f_{m_2}, \ldots$ of sequence $f_1(x), f_2(x), \ldots$ and define a sequence of constants $c_1, c_2, \ldots$ in the following way. Let $f_{m_1}(x) = f_1(x)$ and $c_1 = 1$. Suppose m_{r-1}, that is $f_{m_{r-1}}(x)$ and c_{r-1}, already defined, then we define

$$c_r = \min\left(\frac{c_{r-1}}{4}, \frac{1}{\max\limits_{|x|\le 1} \sum\limits_{k=1}^{m_{r-1}} \left| l_k^{(m_{r-1})}(x) \right|} \right) \quad (1.43)$$

and m_r as the least integer satisfying the following conditions:

$$m_r \ge \mu(m_{r-1}) + 1 \quad (1.44)$$

$$c_r^2 \int_{-1}^{1} L_{m_r}\left(f_{m_r}\right)^2 dx - 8c_r \sqrt{2\int_{-1}^{1} L_{m_r}\left(f_{m_r}\right)^2 dx} > 4^r \; ; \quad (1.45)$$

these 2 conditions can evidently be satisfied in consequence of (1.42) and $\lim_{m\to\infty} S(m) = \infty$.
We now form with these c_r and $f_{m_r}(x)$ the function

$$f(x) = \sum_{r=1}^{\infty} c_r f_{m_r}(x). \quad (1.46)$$

By (1.43)

$$c_r \le \frac{1}{4^r} \quad (1.47)$$

and in consequence of (1.47) and (1.41) it is evident that the infinite series for $f(x)$ uniformly converges in $[-1,+1]$ i.e. $f(x)$ is continuous. Now we consider $L_{m_r}(f)$ for a fixed value ρ of r. According to (1.44)

$$L_{m_\rho}(f) = \sum_{r=1}^{\rho-1} c_r f_{m_r}(x) + \sum_{r=\rho}^{\infty} c_r L_{m_\rho}\left(f_{m_r}\right);$$

hence

$$I_{m_\rho} = \int_{-1}^{1} \left[L_{m_\rho}(f) - f \right]^2 dx = \int_{-1}^{1} \left[c_\rho L_{m_\rho}\left(f_{m_\rho}\right) + \sum_{r=\rho+1}^{\infty} c_r L_{m_\rho}\left(f_{m_r}\right) - \sum_{r=\rho}^{\infty} c_r f_{m_r}(x) \right]^2 dx . \quad (1.48)$$

But in consequence of (1.47)

$$\left| \sum_{r=\rho}^{\infty} c_r f_{m_r}(x) \right| \le \frac{3}{2}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \ldots\right) = 2 , \quad (1.49)$$

and in accordance with (1.41) and (1.43)

$$\sum_{r=\rho+1}^{\infty} c_r L_{m_\rho}\left(f_{m_r}\right) \leq \sum_{r=\rho+1}^{\infty} c_r \cdot \frac{3}{2}\sum_{\nu=1}^{m_\rho}\left|l_\nu^{(m_\rho)}(x)\right| \leq \frac{3}{2}\left(1+\frac{1}{4}+...\right)=2\,. \quad (1.50)$$

From (1.48), (1.49) and (1.50)

$$I_{m_\rho} = \int_{-1}^{1}\left[c_\rho L_{m_\rho}\left(f_{m_\rho}\right)-4\theta\right]^2 dx \quad (1.51)$$

with $|\theta| \leq 1$ (also 0.618033987, i.e. the value of the Aurea ratio).
Further

$$I_{m_\rho} > c_\rho^2 \int_{-1}^{1} L_{m_\rho}\left(f_{m_\rho}\right)^2 dx - 8c_\rho \int_{-1}^{1}\left|L_{m_\rho}\left(f_{m_\rho}\right)\right| dx - 16 > c_\rho^2 \int_{-1}^{1} L_{m_\rho}\left(f_{m_\rho}\right)^2 dx - 8c_\rho\left[2\int_{-1}^{1} L_{m_\rho}\left(f_{m_\rho}\right)^2 dx\right]^{\frac{1}{2}} - 16\,, \quad (1.52)$$

and by (1.45)

$$I_{m_\rho} > 4^\rho - 16\,. \qquad \rho = 1,2,3,... \text{ (and also the prime numbers)} \quad (1.53)$$

With regard the eqs. (1.34) and (1.52), we note that can be related with the **Aurea ratio** by the numbers 8, 16 and 24. Indeed, we have:

$$\left|J'_n\right| \leq 24\varepsilon^2 \int_{-1}^{1} p(x)dx\,;$$

$$I_{m_\rho} > c_\rho^2 \int_{-1}^{1} L_{m_\rho}\left(f_{m_\rho}\right)^2 dx - 8c_\rho \int_{-1}^{1}\left|L_{m_\rho}\left(f_{m_\rho}\right)\right| dx - 16 > c_\rho^2 \int_{-1}^{1} L_{m_\rho}\left(f_{m_\rho}\right)^2 dx - 8c_\rho\left[2\int_{-1}^{1} L_{m_\rho}\left(f_{m_\rho}\right)^2 dx\right]^{\frac{1}{2}} - 16\,;$$

$$(\Phi)^{35/7} + (\Phi)^{-7/7} + (\Phi)^{-21/7} + (\Phi)^{-42/7} = 12\,; \quad 12 \cdot 2 = 24\,; \quad 12 \cdot \frac{4}{3} = 16\,; \quad 12 \cdot \frac{2}{3} = 8\,. \quad (1.54)$$

In the expression (1.54), $\Phi = \frac{\sqrt{5}+1}{2} = 1{,}6180339887...$, and the number 7 of the various exponents is related to the compactified dimensions of the M-theory. Furthermore, we note that 8 and 24 are related with the “modes” that correspond to the physical vibrations of the bosonic strings and superstrings.

2. On some equations and theorems concerning the difference sets of sequences of integers. [2]

A set of integers $u_1 < u_2 < ...$ will be called an $\mathcal{A}$-set if its difference set does not contain the square of a positive integer. Let $A(x)$ denote the greatest number of integers that can be selected from $1,2,...,x$ to form an $\mathcal{A}$-set and let us write

$$a(x) = \frac{A(x)}{x}\,. \quad (2.1)$$

Let N be a large integer and let us write $M = \lfloor\sqrt{N}\rfloor$. Let

$$T(\alpha) = \sum_{z=1}^{[\sqrt{N}]} e(z^2\alpha) = \sum_{z=1}^{M} e(z^2\alpha). \quad (2.2)$$

Let $u_1, u_2, ..., u_{A(N)}$ be a maximal $\mathcal{A}$ -set selected from $1,2,...,N$ and let

$$F(\alpha) = \sum_{x=1}^{A(N)} e(u_x\alpha). \quad (2.3)$$

We are going to investigate the integral

$$E = \int_0^1 |F(\alpha)|^2 T(\alpha) d\alpha . \quad (2.4)$$

Obviously,

$$E = \int_0^1 F(\alpha)F(-\alpha)T(\alpha)d\alpha = \int_0^1 \sum_{y=1}^{A(N)} e(u_y\alpha) \sum_{x=1}^{A(N)} e(-u_x\alpha) \sum_{z=1}^{M} e(z^2\alpha) d\alpha = \sum_{\substack{x,y,z \\ u_y - u_x + z^2 = 0}} 1 = 0 \quad (2.5)$$

since $u_1, u_2, ..., u_{A(N)}$ is an $\mathcal{A}$ -set.

LEMMA 1

If a,b are integers such that $a \le b$, and β is an arbitrary real number (also a prime number) then

$$\left| \sum_{k=a}^{b} e(k\beta) \right| \le \min\left\{ b - a + 1, \frac{1}{2\|\beta\|} \right\}. \quad (2.6)$$

(*For $\|\beta\| = 0$, the right hand side is defined by* $\min\left\{ a, \frac{b}{0} \right\} = a$.)

LEMMA 2

Let N, p, q be integers and α a real number (also prime number) such that $(p,q) = 1$,

$N \ge 3$, (2.7) $\quad 1 \le q \le N^{1/2} / \log N$ (2.8) *and* $\quad \left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}$. (2.9) *Then*

$$|T(\alpha)| < 21 \left(\frac{N}{q} \right)^{1/2}. \quad (2.10)$$

LEMMA 3

Let N, p, q *be integers and* α, β *real numbers (also prime numbers) such that*

$$N \geq 9, \quad (2.11) \qquad (p,q)=1, \quad (2.12) \qquad 1 \leq q \leq \sqrt{N}, \quad (2.13) \qquad \alpha = \frac{p}{q} + \beta \quad (2.14)$$

and $\dfrac{\log N}{N} \leq |\beta| < \dfrac{1}{2q\sqrt{N}}$. (2.15) *Then* $|T(\alpha)| < 30\left(\dfrac{\log N}{q|\beta|}\right)^{1/2}$. (2.16)

LEMMA 4

Let N, p, q *be integers,* R, Q, α *real numbers (also prime numbers) such that* $N \geq 1$, $(p,q)=1$,

$1 \leq R \leq q \leq Q$, (2.17) $\sqrt{N} \leq Q \leq N$ (2.18) *and* $\left|\alpha - \dfrac{p}{q}\right| < \dfrac{1}{qQ}$. (2.19) *Then*

$$|T(\alpha)| < 7\left(\frac{N}{R}\right)^{1/2} + 14(Q \log N)^{1/2}. \quad (2.20)$$

LEMMA 5

For any positive integer x, $\dfrac{1}{x} \leq a(x) \leq 1$. (2.21)

LEMMA 6

For any positive integers x *and* y, *we have*

$$A(x+y) \leq A(x) + A(y), \quad (2.22) \qquad A(xy) \leq xA(y), \quad (2.23) \qquad a(xy) \leq a(y), \quad (2.24)$$

$$a(x) \leq \left(1 + \frac{y}{x}\right) a(y). \quad (2.25)$$

LEMMA 7

Let q, t, N *be positive integers,* p *integer,* α, β *real numbers (also prime numbers) such that*

$\alpha - \dfrac{p}{q} = \beta$. (2.26). *Let* $F_1(\alpha) = \dfrac{a(t)}{q^2}\left(\displaystyle\sum_{s=1}^{q^2} e\left(\frac{sp}{q}\right)\right)\left(\sum_{j=1}^{N} e(\beta j)\right)$, (2.27) *so that if* $(p,q)=1$ *then*

$$F_1(\alpha) = a(t)\sum_{j=1} e(j\alpha) \quad \textit{for} \quad q=1, \quad F_1(\alpha)=0 \quad \textit{for} \quad q>1 \quad (\textit{where } (p,q)=1). \quad (2.28)$$

Then

$$|F(\alpha) - F_1(\alpha)| \leq (a(t) - a(N))N + 2a(t)tq^2(1 + \pi|\beta|N) = H(t, N, q, \beta). \quad (2.29)$$

LEMMA 8

Let t, N be positive integers, R, Q real numbers (also prime numbers) such that

$N \ge e^8$, (2.30) t / N , (2.31) $1 \le R \le N^{1/2} / \log N$, (2.32) $2N^{1/2} < Q \le \dfrac{N}{\log N}$. (2.33)

Then

$$a^2(t)N^{3/2} < 1260a(t)(a(t)-a(N))N^{3/2}\log\log N + 12600a^2(t)tN(\log N)^{1/2}Q^{-1/2} +$$
$$+120(a(t)-a(N))^2\{7N^{3/2}R^{3/2}\log N + 20N^2(\log N)^{1/2}Q^{-1/2}R\} + 26000a^2(t)t^2$$
$$\{3N^{-1/2}(\log N)^3 R^{11/2} + 2N^2(\log N)^{1/2}Q^{-5/2}R^3\} + 140a(t)\{N^{3/2}R^{-1/2} + 2NQ^{1/2}(\log N)^{1/2}\}. \quad (2.34)$$

Let us write

$$G(\alpha) = a(t)\sum_{j=1}^{N} e(j\alpha).$$

Then

$$E = \int_0^1 |F(\alpha)|^2 T(\alpha)d\alpha = \int_0^1 |G(\alpha)|^2 T(\alpha)d\alpha + \int_0^1 \left(|F(\alpha)|^2 - |G(\alpha)|^2\right)T(\alpha)d\alpha$$

where $E = 0$ by (2.5). Hence

$$\int_0^1 |G(\alpha)|^2 T(\alpha)d\alpha = -\int_0^1 \left(|F(\alpha)|^2 - |G(\alpha)|^2\right)T(\alpha)d\alpha . \quad (2.35)$$

Here

$$\int_0^1 |G(\alpha)|^2 T(\alpha)d\alpha = \int_0^1 \left(a(t)\sum_{j=1}^{N} e(j\alpha)\right)\left(a(t)\sum_{k=1}^{N} e(-k\alpha)\right)\left(\sum_{z=1}^{[\sqrt{N}]} e(z^2\alpha)\right)d\alpha =$$
$$= a^2(t)\int_0^1 \sum_{\substack{1\le j,k\le N \\ 1\le z\le\sqrt{N}}} e((j-k+z^2)\alpha)d\alpha = a^2(t)\sum_{\substack{j-k+z^2=0 \\ 1\le j,k\le N \\ 1\le z\le\sqrt{N}}} 1 . \quad (2.36)$$

If

$$1 \le z^2 \le \frac{N}{2} - 1, \quad z > 0 , \quad (2.37)$$
$$1 \le j \le \frac{N}{2} + 1 \quad (2.38)$$

then the numbers j , $k = j + z^2$, z satisfy the conditions

$$j - k + z^2 = 0 , \quad 1 \le j , \ k \le N , \quad 1 \le z \le \sqrt{N}$$

since

$$k = j + z^2 \le \left(\frac{N}{2} + 1\right) + \left(\frac{N}{2} - 1\right) = N .$$

By (2.30), the number of the positive integers z satisfying (2.37) is at least

$$\left[\sqrt{\frac{N}{2}-1}\right] \geq \left[\sqrt{\frac{N}{2}-\frac{N}{4}}\right] = \left[\frac{\sqrt{N}}{2}\right] \geq \frac{\sqrt{N}}{2}-1 \geq \frac{\sqrt{N}}{2}-\frac{\sqrt{N}}{4}=\frac{\sqrt{N}}{4}$$

while (2.38) holds for $\left[\frac{N}{2}\right]+1>\frac{N}{2}$ integers j. Thus (2.36) yields that

$$\int_0^1 |G(\alpha)|^2 T(\alpha)d\alpha = a^2(t) \sum_{\substack{j-k+z^2=0 \\ 1\le j,k\le N \\ 1\le z\le\sqrt{N}}} 1 > a^2(t)\cdot\frac{\sqrt{N}}{4}\cdot\frac{N}{2}=\frac{1}{8}a^2(t)N^{3/2}. \quad (2.39)$$

Now, we have to give an upper estimate for the right hand side of (2.35). For $q=1,2,...,[Q]$ and $p=0,1,...,q-1$, let

$$I_{p,q}=\left(\frac{p}{q}-\frac{1}{pQ},\frac{p}{q}+\frac{1}{qQ}\right)$$

and let S denote the set of those real numbers (also prime numbers) α for which

$$-\frac{1}{Q}<\alpha\le 1-\frac{1}{Q}$$

holds and

$$\alpha\notin I_{p,q} \ \text{ for } \ 1\le q\le R, \quad 0\le p\le q-1, \quad (p,q)=1. \quad (2.40)$$

Then

$$\left|\int_0^1 \left(|F(\alpha)|^2-|G(\alpha)|^2\right)T(\alpha)d\alpha\right| = \left|\int_{-1/Q}^{1-1/Q} \left(|F(\alpha)|^2-|G(\alpha)|^2\right)T(\alpha)d\alpha\right| \le \int_{-1/Q}^{+1/Q} \left||F(\alpha)|^2-|G(\alpha)|^2\right||T(\alpha)|d\alpha +$$
$$+\sum_{q=2}^{[R]} \sum_{\substack{1\le p\le q-1 \\ (p,q)=1}} \int_{I_{p,q}} \left||F(\alpha)|^2-|G(\alpha)|^2\right||T(\alpha)|d\alpha + \int_S \left||F(\alpha)|^2-|G(\alpha)|^2\right||T(\alpha)|d\alpha = E_1+E_2+E_3. \quad (2.41)$$

Let us estimate the term E_1 at first. For any complex numbers $u,v,$ we have

$$\left||u|^2-|v|^2\right| = |u\bar{u}-v\bar{v}| = |(u-v)\bar{u}+v(\bar{u}-\bar{v})| \le |u-v||\bar{u}|+|v||\bar{u}-\bar{v}| = |u-v|(|u|+|v|)=$$
$$=|u-v|(|(u-v)+v|+|v|)\le |u-v|(|u-v|+2|v|)=|u-v|^2+2|u-v||v|. \quad (2.42)$$

Furthermore, applying Lemma 7 with $p=0,q=1,\alpha=\beta$, we have $F_1(\alpha)=G(\alpha)$ there, thus we obtain that

$$|F(\alpha)-G(\alpha)|\le H(t,N,1,\alpha). \quad (2.43)$$

The expressions (2.42) and (2.43) yield that

$$E_1=\int_{-1/Q}^{+1/Q}\left\|F(\alpha)\right|^2-\left|G(\alpha)\right|^2\left\|T(\alpha)\right|d\alpha\le\int_{-1/Q}^{+1/Q}\left|F(\alpha)-G(\alpha)\right|^2\left|T(\alpha)\right|d\alpha+2\int_{-1/Q}^{+1/Q}\left|F(\alpha)-G(\alpha)\right\|G(\alpha)\left\|T(\alpha)\right|d\alpha\le$$

$$\le\int_{-1/Q}^{+1/Q}H^2(t,N,1,\alpha)\left|T(\alpha)\right|d\alpha+2\int_{-1/Q}^{+1/Q}H(t,N,1,\alpha)\left|G(\alpha)\right\|T(\alpha)\left|d\alpha=E'_1+2E''_1.\quad(2.44)$$

Furthermore, for $|\alpha|\le\log N/N$, we use the trivial inequality

$$\left|T(\alpha)\right|=\left|\sum_{z=1}^{M}e\left(z^2\alpha\right)\right|\le\sum_{z=1}^{M}1=M\le N^{1/2},\quad(2.45)$$

while for $\log N/N<|\alpha|\le 1/Q$ ($<1/2\sqrt{N}$, by (2.33)), we apply Lemma 3. In this way, we obtain that

$$E_1^{''}<\int_{|\alpha|\le 1/N}\left\{(a(t)-a(N))N+2a(t)t\left(1+\pi\cdot\frac{1}{N}\cdot N\right)\right\}\cdot a(t)N\cdot N^{1/2}d\alpha+$$

$$+\int_{1/N<|\alpha|\le\log N/N}\left\{(a(t)-a(N))N+2a(t)t(1+\pi)|\alpha|N\right\}\cdot a(t)\frac{1}{2|\alpha|}\cdot N^{1/2}d\alpha+$$

$$+\int_{\log N/N<|\alpha|\le 1/Q}\left\{(a(t)-a(N))N+2a(t)t(1+\pi)|\alpha|N\right\}\cdot a(t)\frac{1}{2|\alpha|}\cdot 30\left(\frac{\log N}{|\alpha|}\right)^{1/2}d\alpha<$$

$$<\frac{2}{N}\left\{a(t)(a(t)-a(N))N^{5/2}+2a^2(t)t\cdot 5\cdot N^{3/2}\right\}+\frac{1}{2}a(t)(a(t)-a(N))N^{3/2}$$

$$\int_{1/N<|\alpha|\le\log N/N}\frac{1}{|\alpha|}d\alpha+2\cdot\frac{\log N}{N}\cdot a^2(t)t\cdot 5\cdot N^{3/2}+15a(t)(a(t)-a(N))N(\log N)^{1/2}\int_{\log N/N<|\alpha|\le 1/Q}\frac{1}{|\alpha|^{3/2}}d\alpha+$$

$$+30a^2(t)t\cdot 5N(\log N)^{1/2}\int_{\log N/N<|\alpha|\le 1/Q}\frac{1}{|\alpha|^{1/2}}d\alpha.\quad(2.46)$$

Here

$$\int_{1/N<|\alpha|\le\log N/N}\frac{1}{|\alpha|}d\alpha=2\log\log N,$$

$$\int_{\log N/N<|\alpha|\le 1/Q}\frac{1}{|\alpha|^{3/2}}d\alpha<2\int_{\log N/N}^{+\infty}\frac{1}{\alpha^{3/2}}d\alpha=2\cdot 2\left(\frac{\log N}{N}\right)^{-1/2}=4\left(\frac{N}{\log N}\right)^{1/2}$$

$$\int_{\log N/N<|\alpha|\le 1/Q}\frac{1}{|\alpha|^{1/2}}d\alpha<2\int_{0}^{1/Q}\frac{1}{\alpha^{1/2}}d\alpha=2\cdot 2\left(\frac{1}{Q}\right)^{1/2}=\frac{4}{Q^{1/2}}.$$

Thus with respect to (2.30), (2.33) and $a(t)\ge a(N)$ (by (2.24) and (2.31)),

$$E_1^{''}<2a(t)(a(t)-a(N))N^{3/2}+20a^2(t)tN^{1/2}+a(t)(a(t)-a(N))N^{3/2}\log\log N+10a^2(t)tN^{1/2}\log N+$$

$$+60a(t)(a(t)-a(N))N^{3/2}+600a^2(t)tN(\log N)^{1/2}Q^{-1/2}<63a(t)(a(t)-a(N))N^{3/2}\log\log N+$$

$$+30a^2(t)tN^{1/2}\log N\left\{1+20\left(\frac{N}{Q\log N}\right)^{1/2}\right\}\le 63a(t)(a(t)-a(N))N^{3/2}\log\log N+630a^2(t)tN(\log N)^{1/2}Q^{-1/2}$$

Now we are going to estimate $E_1^{'}+E_2$. If $2\le q\le Q$, $1\le p\le q-1$ then $\alpha\in I_{p,q}$ implies that

$$\|\alpha\|\ge\frac{1}{q}-\frac{1}{qQ}\ge\frac{1}{q}-\frac{1}{2q}=\frac{1}{2q}.$$

Thus for $2\le q\le Q$, $1\le p\le q-1$ and $(p,q)=1$, Lemmas 1 and 7 (where $F_1(\alpha)=0$ in this case) and the trivial inequality (2.45) yield that

$$\int_{I_{p,q}}\left||F(\alpha)|^2-|G(\alpha)|^2\right||T(\alpha)|d\alpha\le\int_{I_{p,q}}|F(\alpha)|^2|T(\alpha)|d\alpha+\int_{I_{p,q}}|G(\alpha)|^2|T(\alpha)|d\alpha\le\int_{I_{p,q}}|F(\alpha)|^2|T(\alpha)|d\alpha+$$

$$+\int_{I_{p,q}}a^2(t)\frac{1}{2}(2q)^2N^{1/2}d\alpha=\int_{I_{p,q}}|F(\alpha)|^2|T(\alpha)|d\alpha+\frac{1}{2qQ}\cdot 2a^2(t)q^2N^{1/2}\le$$

$$\le\int_{-1/qQ}^{+1/qQ}H^2(t,N,q,\beta)\left|T\left(\frac{p}{q}+\beta\right)\right|d\beta+a^2(t)N^{1/2}.$$

Hence

$$E_1^{'}+E_2\le\int_{-1/Q}^{+1/Q}H^2(t,N,1,\alpha)|T(\alpha)|d\alpha+\sum_{q=2}^{[R]}\sum_{\substack{1\le p\le q-1\\(p,q)=1}}\left\{\int_{-1/qQ}^{+1/qQ}H^2(t,N,q,\beta)\left|T\left(\frac{p}{q}+\beta\right)\right|d\beta+a^2(t)N^{1/2}\right\}\le$$

$$\le\sum_{q=1}^{[R]}\sum_{\substack{0\le p\le q-1\\(p,q)=1}}\int_{-1/qQ}^{+1/qQ}H^2(t,N,q,\beta)\left|T\left(\frac{p}{q}+\beta\right)\right|d\beta+a^2(t)R^2N^{1/2}.\quad(2.48)$$

To estimate $T\left(\frac{p}{q}+\beta\right)$, we use Lemmas 2 and 3 for $|\beta|\le\log N/N$ and $\log N/N<|\beta|\le 1/qQ$, respectively. We obtain with respect to (2.30), (2.32) and (2.33) that (for $q\le R$, $(p,q)=1$)

$$\int_{-1/qQ}^{+1/qQ}H^2(t,N,q,\beta)\left|T\left(\frac{p}{q}+\beta\right)\right|d\beta<\int_{|\beta|\le\log N/N}\left\{2(a(t)-a(N))^2N^2+16a^2(t)t^2q^4\left(1+\pi^2\beta^2N^2\right)\right\}21\left(\frac{N}{q}\right)^{1/2}d\beta+$$

$$+\int_{\log N/N<|\beta|\le 1/qQ}\left\{2(a(t)-a(N))^2N^2+16a^2(t)t^2q^4\left(1+\pi^2\beta^2N^2\right)\right\}30\left(\frac{\log N}{q|\beta|}\right)^{1/2}d\beta<$$

$$<2\frac{\log N}{N}\left\{2(a(t)-a(N))^2N^2+16a^2(t)t^2q^4\left(1+\pi^2\frac{\log^2N}{N^2}N^2\right)\right\}21\left(\frac{N}{q}\right)^{1/2}+60(a(t)-a(N))^2N^2$$

$$(\log N)^{1/2}q^{-1/2}\int_{\log N/N<|\beta|\le 1/qQ}|\beta|^{-1/2}d\beta+\int_{\log N/N<|\beta|\le 1/qQ}480a^2(t)t^2q^4\cdot 11\beta^2N^2\cdot\left(\frac{\log N}{q|\beta|}\right)^{1/2}d\beta<$$

$$<84(a(t)-a(N))^2N^{3/2}\log N\cdot q^{-1/2}+32\cdot\log N\cdot N^{-1/2}a^2(t)t^2q^{7/2}\cdot 11\cdot\log^2N\cdot 21+120(a(t)-a(N))^2N^2$$

$$(\log N)^{1/2}q^{-1/2}\int_0^{1/qQ}\beta^{-1/2}d\beta+10560a^2(t)t^2q^{7/2}N^2(\log N)^{1/2}\int_0^{1/qQ}\beta^{3/2}d\beta=84(a(t)-a(N))^2N^{3/2}\log N\cdot q^{-1/2}+$$
$$+7392a^2(t)t^2N^{-1/2}(\log N)^3q^{7/2}+120(a(t)-a(N))^2N^2(\log N)^{1/2}q^{-1/2}\cdot 2(qQ)^{-1/2}+10560a^2(t)t^2q^{7/2}N^2$$
$$(\log N)^{1/2}\cdot\frac{2}{5}(qQ)^{-5/2}=(a(t)-a(N))^2\left\{84N^{3/2}\log N\cdot q^{-1/2}+240N^2(\log N)^{1/2}Q^{-1/2}q^{-1}\right\}+a^2(t)t^2$$
$$\left\{7392N^{-1/2}(\log N)^3q^{7/2}+4224N^2(\log N)^{1/2}Q^{-5/2}q\right\}$$

since

$$(x+y)^2\le 2x^2+2y^2$$

for any real numbers (or also prime numbers) x,y. Thus (2.48) yields with respect to (2.30) and (2.32) that

$$E_1^{'}+E_2<\sum_{q=1}^{[R]}\sum_{p=0}^{q-1}\left\{(a(t)-a(N))^2\left(84N^{3/2}\log N\cdot q^{-1/2}+240N^2(\log N)^{1/2}Q^{-1/2}q^{-1}\right)+a^2(t)t^2\right.$$
$$\left.\left(7392N^{-1/2}(\log N)^3q^{7/2}+4224N^2(\log N)^{1/2}Q^{-5/2}q\right)\right\}+a^2(t)R^2N^{1/2}<\sum_{q=1}^{[R]}\left\{(a(t)-a(N))^2\right.$$
$$\left.\left(84N^{3/2}\log N\cdot q^{1/2}+240N^2(\log N)^{1/2}Q^{-1/2}\right)+a^2(t)t^2\left(7392N^{-1/2}(\log N)^3q^{9/2}+4224N^2(\log N)^{1/2}Q^{-5/2}q^2\right)\right\}+$$
$$+a^2(t)\left(N^{1/2}/\log N\right)^2N^{1/2}\le(a(t)-a(N))^2\left\{84N^{3/2}R^{3/2}\log N+240N^2(\log N)^{1/2}Q^{-1/2}R\right\}+$$
$$+a^2(t)t^2\left\{7392N^{-1/2}(\log N)^3R^{11/2}+4224N^2(\log N)^{1/2}Q^{-5/2}R^3\right\}+\frac{1}{64}a^2(t)N^{3/2}. \quad (2.49)$$

Finally, in order to estimate E_3, we use Lemma 4. Namely, if $\alpha\in S$ then there exist integers p,q such that

$$1\ge q\ge Q,\quad 0\le p\le q-1,\quad (p,q)=1$$

and

$$\left|\alpha-\frac{p}{q}\right|<\frac{1}{qQ};$$

by (2.40), q satisfies also $R<q$. Thus (2.17) and (2.19) in Lemma 4 hold. Hence, Lemma 4 yields that

$$\sup_{\alpha\in S}|T(\alpha)|\le 7\left(\frac{N}{R}\right)^{1/2}+14(Q\log N)^{1/2}.$$

Thus we obtain applying Parseval's formula that

$$E_3=\int_S\left||F(\alpha)|^2-|G(\alpha)|^2\right||T(\alpha)|d\alpha\le\sup_{\alpha\in S}|T(\alpha)|\left(\int_S|F(\alpha)|^2d\alpha+\int_S|G(\alpha)|^2d\alpha\right)<\left\{7N^{1/2}R^{-1/2}+14Q^{1/2}(\log N)^{1/2}\right\}$$
$$\left(\int_0^1|F(\alpha)|^2d\alpha+\int_0^1|G(\alpha)|^2d\alpha\right)=\left\{7N^{1/2}R^{-1/2}+14Q^{1/2}(\log N)^{1/2}\right\}\left(A(N)+a^2(t)N\right)=$$

$$= \left\{7N^{1/2}R^{-1/2} + 14Q^{1/2}(\log N)^{1/2}\right\}\left(a(N)N + a^2(t)N\right) \le \left\{7N^{1/2}R^{-1/2} + 14Q^{1/2}(\log N)^{1/2}\right\}\left(a(t)N + a(t)N\right) =$$
$$= a(t)\left\{14N^{3/2}R^{-1/2} + 28NQ^{1/2}(\log N)^{1/2}\right\} \quad (2.50)$$

by Lemma 5 and since $a(N) \le a(t)$ by (2.24) in Lemma 6 and (2.31). Collecting the results (2.35), (2.39), (2.41), (2.44), (2.47), (2.49) and (2.50) together, we obtain that

$$\frac{1}{8}a^2(t)N^{3/2} < \int_0^1 |G(\alpha)|^2 T(\alpha)d\alpha \le E_1 + E_2 + E_3 \le \left(E_1^{'} + 2E_1^{''}\right) + E_2 + E_3 = 2E_1^{''} + \left(E_1^{'} + E_2\right) + E_3 <$$
$$< 126a(t)(a(t) - a(N))N^{3/2}\log\log N + 1260a^2(t)tN(\log N)^{1/2}Q^{-1/2} + (a(t) - a(N))^2$$
$$\left\{84N^{3/2}R^{3/2}\log N + 240N^2(\log N)^{1/2}Q^{-1/2}R\right\} + a^2(t)t^2\left\{7392N^{-1/2}(\log N)^3R^{11/2} + 4224N^2\right.$$
$$\left.(\log N)^{1/2}Q^{-5/2}R^3\right\} + \frac{1}{64}a^2(t)N^{3/2} + a(t)\left\{14N^{3/2}R^{-1/2} + 28NQ^{1/2}(\log N)^{1/2}\right\}. \quad (2.51)$$

Subtracting $\frac{1}{64}a^2(t)N^{3/2}$ and then multiplying by

$$\left(\frac{1}{8} - \frac{1}{64}\right)^{-1} = \left(\frac{7}{64}\right)^{-1} = \frac{64}{7} = 9{,}142857 < 10,$$

we obtain that

$$a^2(t)N^{3/2} < 1260a(t)(a(t) - a(N))^{3/2}\log\log N + 12600a^2(t)tN(\log N)^{1/2}Q^{-1/2} + 120(a(t) - a(N))^2$$
$$\left\{7N^{3/2}R^{3/2}\log N + 20N^2(\log N)^{1/2}Q^{-1/2}R\right\} + 26000a^2(t)t^2\left\{3N^{-1/2}(\log N)^3R^{11/2} + 2N^2(\log N)^{1/2}\right.$$
$$\left.Q^{-5/2}R^3\right\} + 140a(t)\left\{N^{3/2}R^{-1/2} + 2NQ^{1/2}(\log N)^{1/2}\right\}.$$

3. On various equations and theorems regarding some problems of a statistical group theory (symmetric groups). [3]

Let S_n stand for the symmetric group with n letters, P a generic element of it and $O(P)$ its order. Then we have

THEOREM 1

For almost all P's in S_n, i.e. with the exception of $o(n!)$ P's at most, $O(P)$ is divisible by all prime powers not exceeding

$$\frac{\log n}{\log\log n}\left\{1 + 3\frac{\log\log\log n}{\log\log n} - \frac{\omega(n)}{\log\log n}\right\} \quad (3.1)$$

if only $\omega(n) \nearrow +\infty$ *arbitrarily slowly.*

Since the P's in a conjugacy class H of S_n have the same order, we may denote by $O(H)$ the common order of its elements and it is natural to ask the corresponding statistical theorem for $O(H)$. The total number of conjugacy classes in S_n is, as well known, $p(n)$, the number of partitions of n. Now, in this Section, we prove the following theorem:

THEOREM 2

For almost all classes H*, i.e. with exception of* $o(p(n))$ *classes,* $O(H)$ *is divisible by all prime powers not exceeding*

$$\frac{2\pi}{\sqrt{6}} \cdot \frac{\sqrt{n}}{\log n}\left\{1+5\frac{\log\log n}{\log n}-\frac{\omega(n)}{\log n}\right\} \quad (3.2)$$

if only $\omega(n) \nearrow +\infty$ *arbitrarily slowly.*

This is again best possible in the following strong sense.

THEOREM 3

If $\omega(n) \nearrow +\infty$ *arbitrarily slowly, then almost no classes* H *(i.e. only* $o(p(n))$ *of it) have the property that* $O(H)$ *is divisible by all primes not exceeding*

$$\frac{2\pi}{\sqrt{6}} \cdot \frac{\sqrt{n}}{\log n}\left\{1+5\frac{\log\log n}{\log n}+\frac{\omega(n)}{\log n}\right\} \quad (3.2b)$$

Now we turn to the proof of Theorem 2. Let, for $y > 0$,

$$f(y) = \prod_{v-1}^{\infty} \frac{1}{1-e^{-vy}} = \sum_{n=0}^{\infty} p(n) e^{-ny} . \quad (3.3)$$

For this we have the classical functional equation

$$f(y) = \frac{1}{\sqrt{2\pi}} \sqrt{y} f\left(\frac{4\pi^2}{y}\right) \exp\left(-\frac{y}{24}+\frac{\pi^2}{6y}\right) \quad (3.4)$$

and hence for $y \to +0$

$$f(y) = (1+o(1)) \sqrt{\frac{y}{2\pi}} \exp\left(\frac{\pi^2}{6y}\right). \quad (3.5)$$

Let $Y = Y(n) \to \infty$ with n to be determined later and let q run through all prime powers with

$$q \le Y(n). \quad (3.6)$$

Let further $p_q(n)$ be the number of all partitions of n with the property that no summand is divisible by q. Then we have for $y > 0$

$$\sum_{n=0}^{\infty} p_q(n) e^{-ny} = \prod_{q/n} \frac{1}{1-e^{-ny}} = \frac{f(y)}{f(qy)}. \quad (3.7)$$

Putting

$$\sum_{q \le Y} p_q(n) \overset{def.}{=} h_Y(n)$$

we get

$$\sum_{n=0}^{\infty} h_Y(n) e^{-ny} = \sum_{q \le Y} \frac{f(y)}{f(qy)}. \quad (3.8)$$

Using (3.5) we get for all q's in (3.8)

$$\frac{f(y)}{f(qy)} = \frac{1+o(1)}{\sqrt{q}} \exp\left\{\frac{\pi^2}{6}\left(1-\frac{1}{q}\right)\frac{1}{y}\right\} \quad (3.9)$$

if only

$$qy \to 0. \quad (3.10)$$

Hence, if y and $\frac{1}{Y}$ are sufficiently small, we have

$$\sum_{n=0}^{\infty} h_Y(n) e^{-ny} < 2\exp\left\{\frac{\pi^2}{6}\left(1-\frac{1}{q}\right)\frac{1}{y}\right\}\sum_{q \le Y}\frac{1}{\sqrt{q}} < 5\frac{\sqrt{Y}}{\log Y}\exp\left\{\frac{\pi^2}{6}\left(1-\frac{1}{Y}\right)\frac{1}{y}\right\}.$$

Putting

$$y = \frac{\pi}{\sqrt{6}} \frac{\sqrt{1-\frac{1}{Y}}}{\sqrt{n}} \overset{def}{=} \frac{\lambda}{\sqrt{n}},$$

we get

$$h_Y(n) e^{-\lambda\sqrt{n}} = h_Y(n) e^{-ny} \le \sum_{m=0}^{\infty} h_Y(m) e^{-my} < 5\frac{\sqrt{Y}}{\log Y}\exp\left\{\frac{\pi}{\sqrt{6}}\sqrt{n} - \frac{1}{Y}\sqrt{n}\right\}$$

and hence

$$h_Y(n) < 5\frac{\sqrt{Y}}{\log Y}\exp\left\{\frac{2\pi}{\sqrt{6}}\sqrt{n} - \frac{1}{Y}\sqrt{n}\right\} < 5\frac{\sqrt{Y}}{\log Y}\exp\left\{\frac{2\pi}{\sqrt{6}}\left(1-\frac{1}{2Y}\right)\sqrt{n}\right\}. \quad (3.11)$$

Using the classical formula of Hardy-Ramanujan, we have

$$p(n) \approx \frac{1}{4n\sqrt{3}} \exp\left(\frac{2\pi}{\sqrt{6}} \cdot \sqrt{n} \right) \quad (3.12)$$

which gives for all sufficiently large n,

$$h_Y(n) < 5 \cdot 8 \frac{\sqrt{Y}}{\log Y} p(n) n \exp\left\{ -\frac{\pi}{\sqrt{6}} \frac{\sqrt{n}}{Y} \right\}. \quad (3.13)$$

Now choosing

$$Y = \frac{4}{5} \cdot \frac{\pi}{\sqrt{6}} \cdot \frac{\sqrt{n}}{\log n}, \quad (3.14)$$

the restriction (3.10) is satisfied and hence (3.13) gives

$$\frac{h_Y(n)}{p(n)} \to 0 \quad \text{for} \quad n \to \infty. \quad (3.15)$$

Now, there is a one-to-one correspondence between the conjugacy classes H of S_n and partitions

$$n = m_1 n_1 + m_2 n_2 + \ldots + m_k n_k \qquad 1 \le n_1 < n_2 < \ldots < n_k \quad (3.16)$$

of n; moreover

$$O(H) = [n_1, n_2, \ldots, n_k] V. \quad (3.17)$$

Hence $O(H)$ is divisible by a prime power q if and only if q is the divisor of some summand n_j and $h_Y(n)$ is an upper bound for the number of conjugacy classes H of S_n whose order is not divisible by some prime power $q \le Y$. Hence (3.15) means that for almost all classes H the quantity $O(H)$ is divisible by all prime powers not exceeding

$$\frac{4}{5} \cdot \frac{\pi}{\sqrt{6}} \cdot \frac{\sqrt{n}}{\log n}. \quad (3.18)$$

Next we consider the divisibility of $O(H)$ by the prime powers q satisfying

$$\frac{4}{5} \cdot \frac{\pi}{\sqrt{6}} \cdot \frac{\sqrt{n}}{\log n} \le q \le \frac{10\pi}{\sqrt{6}} \cdot \frac{\sqrt{n}}{\log n}. \quad (3.19)$$

Taking into account the Euler-Legendre “Pentagonalsatz” according to which for $\operatorname{Re} z > 0$ the relation

$$\left(\frac{1}{f(z)} \right) = \prod_{v=1}^{\infty} \left(1 - e^{-vz}\right) = \sum_{k=-\infty}^{\infty} (-1)^k \exp\left(-\frac{3k^2 + k}{2} z \right) \quad (3.20)$$

holds, equation (3.7) gives the representation

$$p_q(n) = \sum\nolimits_{(k)}' (-1)^k p\left(n - \frac{3k^2+k}{2} q\right), \quad (3.21)$$

where the summation is to be extended over the k 's with

$$\frac{3k^2+k}{2} \le \frac{n}{q}. \quad (3.22)$$

Now we shall estimate the contribution of the k 's with

$$|k| > 10\frac{\sqrt{n}}{q} \quad (3.23)$$

to the sum in (3.21). Then we have

$$\frac{3k^2+k}{2} \ge k^2 \ge 10\frac{\sqrt{n}}{q}k$$

and thus

$$n - \frac{3k^2+k}{2}q \le n - 10\sqrt{n}k < \left(\sqrt{n} - 5k\right)^2;$$

since from (3.12)

$$p(n) < c\exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n}\right) \quad (3.24)$$

we have for the k 's in (3.23)

$$p\left(n - \frac{3k^2+k}{2}q\right) < c\exp\left(\frac{2\pi}{\sqrt{6}}\left(n - \frac{3k^2+k}{2}\right)q\right) < \exp\left\{\frac{2\pi}{\sqrt{6}}\left(\sqrt{n} - 5k\right)\right\}.$$

Hence

$$\left|\sum_{|k|>10\sqrt{n/q}} (-1)^k p\left(n - \frac{3k^2+k}{2}q\right)\right| < c\exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n}\right)\sum_{k>10\sqrt{n/q}} \exp\left(-\frac{10\pi}{\sqrt{6}}k\right) < cn^{-6}\exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n}\right) < cn^{-5}p(n)$$

by (3.12). Hence, from (3.21),

$$p_q(n) = \sum_{|k|\le 10\sqrt{n/q}} (-1)^k p\left(n - \frac{3k^2+k}{2}q\right) + O\left(n^{-5}\right)p(n). \quad (3.25)$$

Next we use Hardy-Ramanujan's stronger formula in the form

$$p(m)=\frac{\exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{\left(m-\frac{1}{24}\right)}\right)}{4\left(m-\frac{1}{24}\right)\sqrt{3}}\left\{1-\frac{1}{\pi}\sqrt{\frac{3}{2}}\frac{1}{\sqrt{\left(m-\frac{1}{24}\right)}}\right\}+O(1)\exp\left\{-0{,}49\frac{2\pi}{\sqrt{6}}\sqrt{m}\right\}. \quad (3.26)$$

Noticing the elementary relation

$$\exp\left\{c_1\left(\sqrt{(x-y)}-\sqrt{x}\right)\right\}\frac{x}{x-y}\cdot\frac{1-\frac{c_2}{\sqrt{(x-y)}}+O(1)\exp\left(-c_3\sqrt{(x-y)}\right)}{1-\frac{c_2}{\sqrt{x}}+O(1)\exp\left(-c_3\sqrt{x}\right)}=$$

$$=\exp\left(-\frac{c_1 y}{2\sqrt{x}}\right)\left\{1+c_4\frac{y^2}{x^{3/2}}+c_5\frac{y^3}{x^{5/2}}+c_6\frac{y^4}{x^3}+O\left(x^{-1{,}46}\right)\right\} \quad (3.27)$$

where the c_v's are positive constants and

$$0<y\le x^{0{,}51}, \quad (3.28)$$

we obtain using (3.26) for the k's in (3.25) and q's in (3.19) from (3.27) with

$$c_1=\frac{2\pi}{\sqrt{6}}, \quad x=n-\frac{1}{24}, \quad y=\frac{3k^2+k}{2}q \quad (3.29)$$

that

$$\frac{p\left(n-\frac{3k^2+k}{2}q\right)}{p(n)}=\exp\left[-\frac{3k^2+k}{2}\cdot\frac{\pi}{\sqrt{6}}\cdot\frac{q}{\sqrt{\left(m-\frac{1}{24}\right)}}\right]\times\left\{1+c_4\left(\frac{3k^2+k}{2}\right)^2\cdot\frac{q^2}{\left(n-\frac{1}{24}\right)^{3/2}}+\right.$$

$$\left.+c_5\left(\frac{3k^2+k}{2}\right)^3\frac{q^3}{\left(n-\frac{1}{24}\right)^{5/2}}+c_6\left(\frac{3k^2+k}{2}\right)^4\frac{q^4}{\left(n-\frac{1}{24}\right)^3}+O\left(n^{-1{,}46}\right)\right\}. \quad (3.30)$$

Putting this into (3.25), we get at once

$$\frac{p_q(n)}{p(n)}=\sum_{|k|\le 10\sqrt{n/q}}(-1)^k\exp\left[-\frac{3k^2+k}{2}\cdot\frac{\pi}{\sqrt{6}}\cdot\frac{q}{\sqrt{\left(n-\frac{1}{24}\right)}}\right]+c_4\frac{q^2}{\left(n-\frac{1}{24}\right)^{3/2}}\sum_{|k|\le 10\sqrt{n/q}}(-1)^k\left(\frac{3k^2+k}{2}\right)^2\times$$

$$\times\exp\left[-\frac{3k^2+k}{2}\frac{\pi}{\sqrt{6}}\cdot\frac{q}{\sqrt{\left(n-\frac{1}{24}\right)}}\right]+c_5\frac{q^3}{\left(n-\frac{1}{24}\right)^{5/2}}\sum_{|k|\le 10\sqrt{n/q}}(-1)^k\left(\frac{3k^2+k}{2}\right)^3\times$$

$$\times\exp\left(-\frac{3k^2+k}{2}\frac{\pi}{\sqrt{6}}\frac{q}{\sqrt{\left(n-\frac{1}{24}\right)}}\right)+c_6\frac{q^4}{\left(n-\frac{1}{24}\right)^3}\sum_{|k|\le 10\sqrt{n/q}}(-1)^k\left(\frac{3k^2+k}{2}\right)^4\times$$

$$\times\exp\left[-\frac{3k^2+k}{2}\frac{\pi}{\sqrt{6}}\cdot\frac{q}{\sqrt{\left(n-\frac{1}{24}\right)}}\right]+O\left(n^{-1,46}\log n\right). \quad (3.31)$$

Obviously the same error term holds completing the sum in (3.31) to $-\infty<k<+\infty$; putting

$$\sum\nolimits_{(k)}(-1)^k\left(\frac{3k^2+k}{2}\right)^\nu\exp\left(-\frac{3k^2+k}{2}\cdot\frac{\pi}{\sqrt{6}}\cdot\frac{q}{\sqrt{n-\frac{1}{24}}}\right) \quad (3.32)$$

equal to $S_\nu(n,q)$, we get

$$\frac{p_q(n)}{p(n)}=S_0(n,q)+c_4\frac{q^2}{\left(n-\frac{1}{24}\right)^{3/2}}S_2(n,q)+c_5\frac{q^3}{\left(n-\frac{1}{24}\right)^{5/2}}S_3(n,q)+c_6\frac{q^4}{\left(n-\frac{1}{24}\right)^3}S_4(n,q)+O\left(n^{-1,46}\right).$$

(3.33)

In order to investigate $S_\nu(n,q)$ we take the reciprocal of (3.4) and apply the functional equation (3.20). This gives for $y>0$

$$\sum_{k=-\infty}^{\infty}(-1)^k\exp\left(-\frac{3k^2+k}{2}y\right)=\sqrt{\frac{2\pi}{y}}\exp\left(\frac{y}{24}-\frac{\pi^2}{6y}\right)\times\sum_{k=-\infty}^{\infty}(-1)^k\exp\left(-\frac{3k^2+k}{2}\cdot\frac{4\pi^2}{y}\right) \quad (3.34)$$

and hence

$$S_0(n,q)=\sqrt{\left(2\sqrt{6}\right)}\frac{\left(n-\frac{1}{24}\right)^{1/4}}{\sqrt{q}}\exp\left\{\frac{\pi}{\sqrt{6}}\cdot\frac{q}{\sqrt{\left(n-\frac{1}{24}\right)}}-\frac{\pi}{\sqrt{6}}\frac{\sqrt{\left(n-\frac{1}{24}\right)}}{q}\right\}\left\{1+O(1)\exp\left(-4\pi\sqrt{6}\frac{\sqrt{\left(n-\frac{1}{24}\right)}}{q}\right)\right\}$$

(3.35)

For our present aims it is enough to write

$$S_0(n,q) = (1+o(1))\sqrt{\left(2\sqrt{6}\right)}\frac{n^{1/4}}{\sqrt{q}}\exp\left(-\frac{\pi}{\sqrt{6}}\cdot\frac{\sqrt{n}}{q}\right). \quad (3.36)$$

Differentiation in (3.34) leads easily to

$$S_\nu(n,q) = O\left(\log^{10} n\right)\exp\left(-\frac{\pi}{\sqrt{6}}\cdot\frac{\sqrt{n}}{q}\right) \quad (3.37)$$

and thus (3.33) together with (3.19) gives

$$p_q(n) = (1+o(1))\sqrt{\left(2\sqrt{6}\right)}\frac{n^{1/4}}{\sqrt{q}}\exp\left(-\frac{\pi}{\sqrt{6}}\cdot\frac{\sqrt{n}}{q}\right)p(n). \quad (3.38)$$

Let us differentiate the identity (3.34) ν times $(1 \le \nu \le 4)$. This is the sum of $(\nu+1)$ terms each of the form

$$p_j(y)\exp\left(\frac{y}{24}-\frac{\pi^2}{6y}\right)\sum_{(k)}(-1)^k\left(\frac{3k^2+k}{2}\right)^j\exp\left(-\frac{3k^2+k}{2}\cdot\frac{4\pi^2}{y}\right), \qquad j=0,1,\dots,\nu \quad (3.39)$$

where the $p_j(y)$'s are polynomials in $\frac{1}{\sqrt{y}}$ of degree ≤ 20 with bounded coefficients. In particular, for $j=0$, we have

$$\left\{\sqrt{\frac{2\pi}{y}}\exp\left(\frac{y}{24}-\frac{\pi^2}{6y}\right)\right\}^{(\nu)}\sum_{(k)}(-1)^k\exp\left(-\frac{3k^2+k}{2}\cdot\frac{4\pi^2}{y}\right)$$

whereas for the terms with $j \ge 1$, since the term with $k=0$ is missing from the sum, we have an upper bound

$$O\left(\log^{10} n\right)\exp\left\{-\left(\frac{\pi}{\sqrt{6}}+4\pi\sqrt{6}\right)\frac{\sqrt{n}}{q}\right\}.$$

Hence, for $1 \le \nu \le 4$ we have

$$S_\nu(n,q) = \left\{\sqrt{\frac{2\pi}{y}}\exp\left(\frac{y}{24}-\frac{\pi^2}{6y}\right)\right\}^{(\nu)}_{y=\frac{\pi}{\sqrt{6}}\cdot\frac{q}{\sqrt{(n-1/24)}}} + O\left(\log^{10} n\right)\exp\left\{-\left(\frac{\pi}{\sqrt{6}}+4\pi\sqrt{6}\right)\frac{\sqrt{n}}{q}\right\}. \quad (3.40)$$

Let

$$Y_1 = \frac{4}{5}\frac{\pi}{\sqrt{6}}\cdot\frac{\sqrt{n}}{\log n}, \qquad Y_2 = \lambda\frac{\pi}{\sqrt{6}}\cdot\frac{\sqrt{n}}{\log n}, \quad (3.41)$$

where λ will be determined later. Putting

$$h^*(n) \overset{def}{=} \sum_{Y_1 \le q \le Y_2} p_q(n) \quad (3.42)$$

gives (3.38) for all sufficiently large n's,

$$h^*(n) < 3p(n)n^{1/4} \int_{Y_1}^{Y_2} \frac{1}{\sqrt{x}} \exp\left(-\frac{\pi}{\sqrt{6}}\frac{\sqrt{n}}{x}\right) d\Theta(x) \quad (3.43)$$

where $\Theta(x)$ stands for the number of prime powers not exceeding x. Using the prime number theorem in the form

$$\Theta(x) = \operatorname{Li} x + O(x)\exp\left(-\sqrt{\log x}\right),$$

the factor of $p(n)$ in (3.43) is

$$(1+o(1))n^{1/4} \int_{Y_1}^{Y_2} \frac{1}{\sqrt{x \log x}} \exp\left(-\frac{\pi}{\sqrt{6}}\frac{\sqrt{n}}{x}\right) dx = o\left(\frac{1}{\sqrt{\log n}}\right) \int_{Y_1}^{Y_2} \exp\left(-\frac{\pi}{\sqrt{6}}\frac{\sqrt{n}}{x}\right) dx. \quad (3.44)$$

Since the last integral

$$= \frac{\pi}{\sqrt{6}} \cdot \sqrt{n} \int_{(1/\lambda)\log n}^{(5/4)\log n} \frac{1}{y^2} e^{-v} dy = o\left(\frac{n^{\frac{1}{2}-1/\lambda}}{\log^2 n}\right)$$

we have

$$\frac{h^*(n)}{p(n)} = O\left(\frac{n^{\frac{1}{2}-1/\lambda}}{\log^{5/2} n}\right) = o(1)$$

choosing

$$\lambda = 2\left(1 + 5\frac{\log\log n}{\log n} - \frac{\omega(n)}{\log n}\right) \quad (3.45)$$

if only

$$\omega(n) \nearrow +\infty$$

arbitrarily slowly.

Let again $\omega(n) \nearrow \infty$ arbitrarily slowly; further

$$X_1 = \frac{2\pi}{\sqrt{6}} \cdot \frac{\sqrt{n}}{\log n}\left\{1 + 5\frac{\log\log n}{\log n} - \frac{\omega(n)}{\log n}\right\}, \quad X_2 = \frac{2\pi}{\sqrt{6}} \cdot \frac{\sqrt{n}}{\log n}\left\{1 + 5\frac{\log\log n}{\log n} + \frac{\omega(n)}{\log n}\right\} \quad (3.46)$$

and

$$X_1 \le q_1 < q_2 < \ldots < q_l \le X_2 \quad (3.47)$$

all primes of this interval. We define the class-function $k(H)$ by

$$k(H)=\sum_{q_\nu/O(H)}{}^{(\nu)}1. \quad (3.48)$$

First we investigate

$$S_1=\sum{}_{(H)}k(H). \quad (3.49)$$

Obviously

$$S_1=\sum_{\nu=1}^{l}\sum_{q_\nu/O(H)}{}^{(H)}1=\sum_{\nu=1}^{l}p_{q_\nu}(n).$$

Using the representation (3.38), we have that

$$S_1=(1-o(1))\sqrt{\left(2\sqrt{6}\right)}p(n)n^{1/4}\sum_{\nu=1}^{l}\frac{1}{\sqrt{q_\nu}}\exp\left(-\frac{\pi}{\sqrt{6}}\frac{\sqrt{n}}{q_\nu}\right)=(1+o(1))\sqrt{\left(2\sqrt{6}\right)}p(n)n^{1/4}\int_{X_1}^{X_2}\frac{\exp\left(-\frac{\pi}{\sqrt{6}}\cdot\frac{\sqrt{n}}{x}\right)}{\sqrt{x\log x}}.$$

Since we need asymptotic formula for S_1 we can write the precedent expression also as follow:

$$S_1=(1+o(1))2\sqrt{\frac{6}{\pi}}\frac{p(n)}{\sqrt{\log n}}\int_{X_1}^{X_2}\exp\left(-\frac{\pi}{\sqrt{6}}\frac{\sqrt{n}}{x}\right)dx=(1+o(1))8\sqrt{\pi}\,p(n)\exp\left(\frac{\omega}{2}\right)(\to+\infty). \quad (3.50)$$

Next let

$$S_2=\sum{}_{(H)}k(H)^2. \quad (3.51)$$

Then

$$S_2=\sum{}_{(H)}\sum_{q_\mu/O(H)}{}^{(\mu)}\sum_{q_\nu/O(H)}{}^{(\nu)}1=S_1+\sum_{1\le\mu\ne\nu\le l}\ \sum_{\substack{q_\mu/O(H)\\ q_\nu/O(H)}}{}^{(H)}1. \quad (3.52)$$

Fixing μ and ν the inner sum is the number of such partitions of n in which no summand is divisible either by q_μ or by q_ν. With the notation of (3.20) this quantity is as easy to see

$$\text{the coefficient } e^{-nz} \text{ in } \frac{f(z)f(q_\mu q_\nu z)}{f(q_\mu z)f(q_\nu z)}. \quad (3.53)$$

Hence

$$S_2= \text{ the coefficient } e^{-nz} \text{ in } f(z)\left\{\sum_{\mu=1}^{l}\frac{1}{f(q_\mu z)}+\sum_{1\le\mu\ne\le l}\frac{f(q_\mu q_\nu z)}{f(q_\mu z)f(q_\nu z)}\right\}. \quad (3.54)$$

The function in the curly bracket is

$$\left(\sum_{\mu=1}^{l}\frac{1}{f(q_\mu z)}\right)+\left(\sum_{\mu=1}^{l}\frac{1}{f(q_\mu z)}\right)^2-\left(\sum_{\mu=1}^{l}\frac{1}{f(q_\mu z)^2}\right)+\sum_{1\le\mu\ne\nu\le l}\frac{f(q_\mu q_\nu z)-1}{f(q_\mu z)f(q_\nu z)} \quad (3.55)$$

and accordingly we split S_2 into the parts

$$S_1,\ S_2^{(1)},\ S_2^{(2)} \text{ and } S_2^{(3)}. \quad (3.56)$$

Since from (3.3) and (3.20)

$$\frac{f(z)\{f(q_\mu q_\nu z)-1\}}{f(q_\mu z)f(q_\nu z)} = \left\{\sum_{k_1=0}^{\infty} p(k_1)e^{-k_1 z}\right\}\cdot\left\{\sum_{k_2=1}^{\infty} p(k_2)\exp(-k_2 q_\mu q_\nu z)\right\}$$
$$\left\{\sum_{k_3,k_4=-\infty}^{\infty}(-1)^{k_3+k_4}\times\exp\left(-\frac{3k_3^2+k_3}{2}q_\mu+\frac{3k_4^2+k_4}{2}q_\nu\right)z\right\} \quad (3.57)$$

we have

$$S_2^{(3)} = \sum_{k_2,k_3,k_4}{}^{1}(-1)^{k_3+k_4}p(k_2)\times\sum_{1\le\mu\ne\nu\le l}\left(n-k_2q_\mu q_\nu-\frac{3k_3^2+k_3}{2}q_\mu-\frac{3k_4^2+k_4}{2}q_\nu\right) \quad (3.58)$$

where the outer summation is to be extended to all (k_2,k_3,k_4) systems with

$$k_2\ge 1 \quad q_\mu q_\nu k_2+\frac{3k_3^2+k_3}{2}q_\mu+\frac{3k_4^2+k_4}{2}q_\nu\le n. \quad (3.59)$$

Using (3.24) and (3.46) – (3.47), the inner sum in (3.58) is quite roughly

$$\sum_{1\le\mu\ne\nu\le l}(n-k_2q_\mu q_\nu-\frac{3k_3^2+k_3}{2}q_\mu-\frac{3k_4^2+k_4}{2}q_\nu)<c\sum_{1\le\mu,\nu\le l}\exp\left(\frac{2\pi}{\sqrt6}\left\{n-\frac{2\pi^2}{3}\frac{n}{\log^2 n}\right\}^{1/2}\right)<$$
$$<c\exp\left(\frac{2\pi}{\sqrt6}\sqrt n\right)\cdot\frac{n\omega(n)^2}{\log^2 n}\exp\left(-\frac{2\pi^3}{3\sqrt6}\frac{\sqrt n}{\log^2 n}\right)<cp(n)n^2\exp\left(-\frac{2\pi^3}{3\sqrt6}\cdot\frac{\sqrt n}{\log^2 n}\right). \quad (3.60)$$

Since roughly k_2 takes at most $O(\log^2 n)$-values, further k_3 and k_4 each at most $O(n^{1/4}\log n)$-values, we get from (3.60) at once

$$S_2^{(3)} = o(p(n)). \quad (3.61)$$

Next we consider $S_2^{(2)}$. Since from (3.55) and (3.20), we have

$$-\frac{f(z)}{f(q_\mu z)^2} = -\left(\frac{f(z)}{f(q_\mu z)}\right)\frac{1}{f(q_\mu z)} = \left(\sum_{m=0}^{\infty}p_{q_\mu}(m)e^{-mz}\right)\left(\sum_{(k)}(-1)^{k+1}\exp\left\{-\frac{3k^2+k}{2}q_\mu z\right\}\right)$$

we get

$$S_2^{(2)} = \sum_{\mu=1}^{l}\sum_{(k)}(-1)^{k+1}p_{q_\mu}\left(n-\frac{3k^2+k}{2}q_\mu\right). \quad (3.62)$$

The contribution of terms with $|k|>10\log n$ is absolutely

$$<\sum_{\mu=1}^{l}\sum_{10\log n\le|k|\le\sqrt{n/q_\mu}} p\left(n-\frac{3k^2+k}{2}q_\mu\right)=O(p(n)).$$

For the remaining terms in (3.62) we can apply the representation (3.33) – (3.35) – (3.37) in the form

$$p_q(n)=\sqrt{\left(2\sqrt{6}\right)}\frac{n^{1/4}}{\sqrt{q}}\,p(n)\exp\left(-\frac{\pi}{\sqrt{6}}\frac{\sqrt{n}}{q}\right)\left\{1+O\left(\frac{1}{\log n}\right)\right\}. \quad (3.63)$$

The contribution of the error term to (3.62) is absolutely

$$O(1)p(n)\sum_{\mu=1}^{l}\frac{n^{1/4}}{\sqrt{q_\mu}}\sum_{|k|\le 10\sqrt{n/q_\mu}}\exp\left(\frac{\pi}{\sqrt{6}}\cdot\frac{\left\{n-\frac{3k^2+k}{2}q_\mu\right\}^{1/2}}{q_\mu}\right)=o(p(n))$$

using (3.46). Hence from (3.62) and (3.63), we have

$$S_2^{(2)}=o(p(n))+\sqrt{\left(2\sqrt{6}\right)}\sum_{\mu=1}\frac{1}{\sqrt{q_\mu}}\sum_{k\le 10\log n}{}^{(k)}(-1)^{k+1}\times\left(n-\frac{3k^2+k}{2}q_\mu\right)^{1/4}p\left(n-\frac{3k^2+k}{2}q_\mu\right)\times$$
$$\times\exp\left(-\frac{\pi}{\sqrt{6}}\cdot\frac{\left\{n-\frac{3k^2+k}{2}q_\mu\right\}^{1/2}}{q_\mu}\right). \quad (3.64)$$

Rough estimations show that replacing

$$\left(n-\frac{3k^2+k}{2}q_\mu\right)^{1/4} \quad \text{by} \quad n^{1/4}$$

and

$$\exp\left(-\frac{\pi}{\sqrt{6}}\frac{\left\{n-\frac{3k^2+k}{2}q_\mu\right\}^{1/2}}{q_\mu}\right) \quad \text{by} \quad \exp\left(-\frac{\pi}{\sqrt{6}}\frac{\sqrt{n}}{q_\mu}\right)$$

the error is again $o(p(n))$ and hence

$$S_2^{(2)}=o(p(n))+\sqrt{\left(2\sqrt{6}\right)}n^{1/4}\sum_{\mu=1}^{l}\frac{1}{\sqrt{q_\mu}}\cdot\exp\left\{-\frac{\pi}{\sqrt{6}}\frac{\sqrt{n}}{q_\mu}\right\}\times\left(\sum_{|k|\le 10\log n}(-1)^{k+1}p\left(n-\frac{3k^2+k}{2}q_\mu\right)\right). \quad (3.65)$$

Completing the inner sum means again an error of $o(p(n))$ and using (3.21) we get

$$S_2^{(2)} = o(p(n)) - \sqrt{(2\sqrt{6})}n^{1/2}\sum_{\mu=1}^{l}\frac{p_{q_\mu}(n)}{\sqrt{q_\mu}} \times \exp\left\{-\frac{\pi}{\sqrt{6}}\frac{\sqrt{n}}{q_\mu}\right\} < o(p(n)). \quad (3.66)$$

Next we consider $S_2^{(1)}$. Using (3.20) and (3.3)

$$f(z)\left(\sum_{\mu=1}^{l}\frac{1}{f(q_\mu z)}\right)^2 = \left\{\sum_{m=0}^{\infty} p(m)e^{-mz}\right\} \times \left\{\sum_{\mu_1=1}^{l}\sum_{\mu_2=1}^{l}\sum_{k_1}\sum_{k_2}(-1)^{k_1+k_2}\exp\left(-\frac{3k_1^2+k_1}{2}q_{\mu_1} - \frac{3k_2^2+k_2}{2}q_{\mu_2}\right)z\right\}$$

and hence the representation

$$S_2^{(1)} = \sum_{\mu_1=1}^{l}\sum_{\mu_2=1}^{l}\sum_{k_1}\sum_{k_2}(-1)^{k_1+k_2} \times p\left(n - \frac{3k_1^2+k_1}{2}q_{\mu_1} - \frac{3k_2^2+k_2}{2}q_{\mu_2}\right). \quad (3.67)$$

One can see easily that the contribution of k_2's with $|k_2|m > 10\log n$ is $o(p(n))$ and hence using also (3.21)

$$S_2^{(1)} = o(p(n)) + \sum_{\mu_1=1}^{l}\sum_{\mu_2=1}^{l}\sum_{|k_2|\le 10\log n}(-1)^{k_2}\, p_{q_{\mu_1}}\left(n - \frac{3k_2^2+k_2}{2}q_{\mu_2}\right). \quad (3.68)$$

To go further, we shall need for $p_{q_{\mu_1}}(m)$ an asymptotic representation which is finer than the one in (3.38) (even the one in (3.63)).
Using (3.33) and the formula (3.40) we get

$$\frac{p_q(m)}{p(m)} = S_0(m,q) + \left\{c_4\frac{q^2}{\left(m-\frac{1}{24}\right)^{3/2}}\left(\sqrt{\frac{2\pi}{y}}\exp\left(\frac{y}{24}-\frac{\pi^2}{6y}\right)\right)'' + c_5\frac{q^3}{\left(m-\frac{1}{24}\right)^{5/2}}\left(\sqrt{\frac{2\pi}{y}}\exp\left(\frac{y}{24}-\frac{\pi^2}{6y}\right)\right)^{(3)} + \right.$$

$$\left. + c_6\frac{q^4}{\left(m-\frac{1}{24}\right)^{3}}\left(\sqrt{\frac{2\pi}{y}}\exp\left(\frac{y}{24}-\frac{\pi^2}{6y}\right)\right)^{(4)}\right\}_{y=\pi/\sqrt{6}q/\sqrt{(m-1/24)}} + O\left(m^{-1,45}\right). \quad (3.69)$$

The contribution of the error term in (3.69) to $S_1^{(2)}$ in (3.68) is seen to be by (3.46) easily $o(p(n))$.
The contribution U of $p(m)S_0(m,q)$ is by (3.35)

$$U = o(p(n)) + \sqrt{(2\sqrt{6})}\sum_{\mu_1=1}^{l}\sum_{\mu_2=1}^{l}\sum_{|k_2|\le 10\log n}(-1)^{k_2}\frac{\left(n-\frac{3k_2^2+k_2}{2}q_{\mu_2}-\frac{1}{24}\right)}{q_{\mu_2}}\times$$

$$\times\exp\left\{\frac{\pi}{\sqrt{6}}\frac{q_{\mu_2}}{\left(n-\frac{3k_2^2+k_2}{2}q_{\mu_2}-\frac{1}{24}\right)^{1/2}}-\frac{\pi}{\sqrt{6}}\frac{\left(n-\frac{3k_2^2+k_2}{2}q_{\mu_2}-\frac{1}{24}\right)^{1/2}}{q_{\mu_1}}\right\}\times p\left(n-\frac{3k_2^2+k_2}{2}q_{\mu_2}\right). \tag{3.70}$$

By the elementary formula

$$(x-y)^{1/4}\exp\left\{c\left(\frac{q}{\sqrt{(x-y)}}-\frac{\sqrt{(x-y)}}{q}\right)\right\} = x^{1/4}\exp\left\{c\left(\frac{q}{\sqrt{x}}-\frac{\sqrt{x}}{q}\right)\right\}\times$$

$$\times\left\{1+d_1\frac{yq}{x^{3/2}}+d_2\frac{y^2q^2}{x^3}+d_3\frac{y^2q}{x^{5/2}}+d_4\frac{y}{qx^{1/2}}+d_5\frac{y^2}{qx}+d_6\frac{y^2}{q^2x}+O\left(\frac{y^3q}{x^{9/2}}\right)+O\left(\frac{y^3}{qx^{5/2}}\right)\right\}, \tag{3.71}$$

valid for

$$0 < y \le x^{0,51}, \quad q < \sqrt{x}.$$

Using it with

$$c = \frac{\pi}{\sqrt{6}}, \quad x = n - \frac{1}{24}, \quad y = \frac{3k_2^2+k_2}{2}q_{\mu_2}, \quad q = q_{\mu_1},$$

we obtain

$$U = O(p(n)) + \sqrt{(2\sqrt{6})}\sum_{\mu_1=1}^{l}\frac{\left(n-\frac{1}{24}\right)^{1/4}}{q_{\mu_1}}\times\exp\left[\frac{\pi}{\sqrt{6}}\frac{q_{\mu_1}}{\sqrt{\left(n-\frac{1}{24}\right)}}-\frac{\sqrt{\left(n-\frac{1}{24}\right)}}{q_{\mu_1}}\right]\times$$

$$\times\left\{\sum_{\mu_2=1}^{l}\sum_{k_2}(-1)^{k_2}\,p\left(n-\frac{3k_2^2+k_2}{2}q_{\mu_2}\right)\right\}. \tag{3.72}$$

The sum in the curly brackets is by (3.21) (or (3.25))

$$\sum_{\mu_2=1}^{l} p_{q_{\mu_2}}(n) = S_1$$

and the sum with respect to μ_1 is

$$\frac{1}{p(n)}(1+o(1))S_1$$

by (3.33), (3.35) and (3.40). Thus using (3.50), we have

$$S_2^{(1)} = (1+o(1))\frac{1}{p(n)}S_1^2 = (1+o(1))p(n)\left\{8\sqrt{\pi}\exp\left(\frac{\omega}{2}\right)\right\}^2 . \quad (3.73)$$

Collecting (3.52), (3.56), (3.50), (3.61), (3.66) and (3.73) we get for S_2 in (3.51) the inequality

$$S_2 < (1+o(1))p(n)\left\{8\sqrt{\pi}\exp\left(\frac{\omega}{2}\right)\right\}^2 . \quad (3.74)$$

By Cebysev's inequality, it is enough to show that

$$Z \overset{def}{=} \frac{1}{p(n)}\sum_H \left\{k(H) - 8\sqrt{\pi}\exp\left(\frac{\omega}{2}\right)\right\}^2 = o(1)\exp\omega .$$

But this follows from (3.50) and (3.74) at once.

Now we describe various mathematical expressions regarding some frequency connected with the exponents of the Aurea ratio and related with various equations of this Section.
We note that for the eq. (3.19), there are the following mathematical connections with the Aurea ratio:

$$\frac{4}{5}\cdot\frac{\pi}{\sqrt{6}} = 1{,}026039864 \cong ((\Phi)^{7/7} + (\Phi)^{-7/7} + (\Phi)^{-42/7})\cdot\frac{4}{9} = (1{,}618033988 + 0{,}618033988 + 0{,}05572809)\cdot\frac{4}{9} =$$
$$= 2{,}291796\cdot\frac{4}{9} = 1{,}018576 ;$$

$$\frac{10\pi}{\sqrt{6}} = 12{,}8254983 \cong \left[(\Phi)^{28/7} + (\Phi)^{14/7} + (\Phi)^{-21/7}\right]\cdot\frac{4}{3} = (6{,}85410197 + 2{,}61803399 + 0{,}23606798)\cdot\frac{4}{3} =$$
$$= 9{,}708204\cdot\frac{4}{3} = 12{,}94427191 .$$

With regard the eqs. (3.38), (3.40) and (3.46), we have the following connections with the Aurea ratio:

$$\frac{\pi}{\sqrt{6}} = 1{,}282549 \cong (\Phi)^{28/7}\cdot\frac{1}{3} \cong 6{,}854102\cdot\frac{1}{3} = 2{,}28470066 ;$$
$$2{,}28470066 - (\Phi)^0 = 2{,}28470066 - 1 = 1{,}28470066 ;$$

$$4\pi\sqrt{6} = 30{,}78119592 \cong 20{,}56230590 + 10{,}28115295 = 30{,}84345885 ;$$
$$20{,}56230590 = \left[(\Phi)^{35/7} + (\Phi)^{14/7}\right]\cdot\frac{3}{2} = (11{,}09016994 + 2{,}61803399)\cdot\frac{3}{2} = 13{,}708204\cdot\frac{3}{2} = 20{,}56230590$$
$$13{,}708204\cdot\frac{3}{4} = 10{,}28115295 ;$$

$$\frac{2\pi}{\sqrt{6}} = 2{,}565 \cong \left[(\Phi)^{14/7} + (\Phi)^{-7/7} + (\Phi)^{-28/7} + (\Phi)^{-42/7}\right]\cdot \frac{3}{4} = (2{,}61803399 + 0{,}61803399 + 0{,}14589803 +$$
$$+ 0{,}05572809)\cdot \frac{3}{4} = 3{,}437694 \cdot \frac{3}{4} = 2{,}57827058\,;$$

We note that for the eqs. (3.50) and (3.60), we have the following connections with the Aurea ratio:

$$2\sqrt{\frac{6}{\pi}} = 2{,}763953196 = 2{,}47213595 + 0{,}29179607 = 2{,}76393202\,;$$
$$2{,}47213595 = \left[(\Phi)^{28/7} + (\Phi)^{-14/7} + (\Phi)^{-28/7} + (\Phi)^{-49/7}\right]\cdot \frac{1}{3} = (6{,}85410197 + 0{,}38196601 +$$
$$+ 0{,}14589803 + 0{,}03444185)\cdot \frac{1}{3} = 7{,}416408 \cdot \frac{1}{3} = 2{,}47213595\,;$$
$$0{,}29179607 = \left[(\Phi)^{-7/7} + (\Phi)^{-21/7} + (\Phi)^{-56/7}\right]\cdot \frac{1}{3} = (0{,}61803399 + 0{,}23606798 + 0{,}02128624)\cdot \frac{1}{3} =$$
$$= 0{,}875388 \cdot \frac{1}{3} = 0{,}29179607\,;$$

$$8\sqrt{\pi} = 14{,}17963081\,; \quad 14{,}56230590 - 0{,}38196601 = 14{,}18033989$$
$$14{,}56230590 = \left[(\Phi)^{28/7} + (\Phi)^{14/7} + (\Phi)^{-21/7}\right]\cdot \frac{3}{2} = (6{,}85410197 + 2{,}61803399 + 0{,}23606798)\cdot \frac{3}{2} =$$
$$= 9{,}708204 \cdot \frac{3}{2} = 14{,}56230590\,;$$
$$0{,}38196601 = \left[(\Phi)^{-7/7} + (\Phi)^{-28/7}\right]\cdot \frac{1}{2} = (0{,}61803399 + 0{,}14589803)\cdot \frac{1}{2} = 0{,}763932 \cdot \frac{1}{2} = 0{,}38196601\,;$$

$$\frac{2\pi^2}{3} = 6{,}579736267 \cong 6{,}47213595 + 0{,}10524494 = 6{,}57738089$$
$$6{,}47213595 = \left[(\Phi)^{21/7} + (\Phi)^{-7/7}\right]\cdot \frac{4}{3} = (4{,}23606798 + 0{,}61803399)\cdot \frac{4}{3} = 4{,}854102 \cdot \frac{4}{3} = 6{,}47213595\,;$$
$$0{,}10524494 = \left[(\Phi)^{-28/7} + (\Phi)^{-70/7} + (\Phi)^{-84/7} + (\Phi)^{-105/7}\right]\cdot \frac{2}{3} = 0{,}157867 \cdot \frac{2}{3} = 0{,}10524494\,;$$

$$\frac{2\pi^3}{3\sqrt{6}} = 8{,}438839633 = 6{,}47213595 + 1{,}96352549 = 8{,}43566144$$
$$6{,}47213595 = \left[(\Phi)^{28/7} + (\Phi)^{14/7} + (\Phi)^{-21/7}\right]\cdot \frac{2}{3} = (6{,}85410197 + 2{,}61803399 + 0{,}23606798)\cdot \frac{2}{3} =$$
$$= 9{,}708204 \cdot \frac{2}{3} = 6{,}47213595\,;$$
$$1{,}96352549 = (\Phi)^{14/7} \cdot \frac{3}{4} \cong 2{,}618034 \cdot \frac{3}{4} = 1{,}96352549\,.$$

In conclusion, for the eq. (3.72), we have the following connection with the Aurea ratio:

$$\sqrt{\left(2\sqrt{6}\right)} = 2{,}213363839 = 2{,}15737865 + 0{,}05572809 = 2{,}21310674$$

$$2{,}15737865 = \left[(\Phi)^{21/7} + (\Phi)^{-7/7}\right]\cdot\frac{4}{9} = (4{,}23606798 + 0{,}61803399)\cdot\frac{4}{9} = 4{,}854102\cdot\frac{4}{9} = 2{,}15737865;$$

$$0{,}05572809 = \left[(\Phi)^{-28/7} + (\Phi)^{-56/7}\right]\cdot\frac{1}{3} = (0{,}14589803 + 0{,}02128624)\cdot\frac{1}{3} = 0{,}167184\cdot\frac{1}{3} = 0{,}05572809.$$

We note that also here $\Phi = \frac{\sqrt{5}+1}{2} = 1{,}6180339887...$

4. On some equations and theorems concerning the measure of the non-monotonicity of the Euler Phi function and the related Riemann zeta function. [4]

Let f be a real valued arithmetic function satisfying $\lim_{n\to\infty} f(n) = +\infty$. Define another arithmetic function $F = F_f$ by setting

$$F_f(n) = \#\{j < n : f(j) \geq f(n)\} + \#\{j > n : f(j) \leq f(n)\}.$$

The size of the values assumed by the function F provides a measure of the nonmonotonicity of f. In particular, F is identically zero if and only if f is strictly increasing. Here we shall take f to be φ, Euler's function, and study the associated function F_φ, which we henceforth call F.
For $0 \leq a,b \leq \infty$, let

$$\Phi(a,b) = \#\{n \leq a : \varphi(n) \leq b\}.$$

We have

$$\#\{j < n : \varphi(j) \geq \varphi(n)\} = n - \Phi(n,\varphi(n)) + \#\{j < n : \varphi(j) = \varphi(n)\}$$
$$\#\{j > n : \varphi(j) \leq \varphi(n)\} = \Phi(\infty,\varphi(n)) - \Phi(n,\varphi(n)).$$

Thus

$$F(n) = n + \Phi(\infty,\varphi(n)) - 2\Phi(n,\varphi(n)) + \#\{j < n : \varphi(j) = \varphi(n)\}.$$

It is known that

$$\Phi(\infty, y) = \zeta y + O\left(ye^{-\sqrt{\log y}}\right),$$

where ζ denotes the constant $\zeta(2)\zeta(3)/\zeta(6) \approx 1.9436$; and

$$\Phi(x, y) = xg(y/x) + O\left(ye^{-\sqrt{\log y}}\right),$$

where g is a continuous, increasing function on $[0,1]$ which is determined by a contour integral.
We note that for the value 1,9436 we have the following mathematical connection with the Aurea ratio:

$$1{,}9436 \cong (\Phi)^{14/7} \cdot \frac{3}{4} = 2{,}61803399 \cdot \frac{3}{4} \cong 2{,}618034 \cdot \frac{3}{4} = 1{,}96352549 .$$

Moreover, g is strictly concave, as we now indicate. We have that

$$\alpha g'(\alpha) = g(\alpha) - D_\varphi(\alpha), \qquad 0 < \alpha \leq 1. \quad (4.1)$$

Here

$$D_\varphi(\alpha) = \lim_{x \to \infty} \frac{1}{x} \#\{n \leq x : \varphi(n) \leq \alpha n\}.$$

It is known that this limit exists and defines a continuous function of α. Clearly D_φ is non-decreasing. In fact, it is known to be strictly increasing on $(0,1)$. If we integrate the differential equation for g and use the fact that $g(1) = 1$, we obtain

$$g(\alpha) = \alpha + \alpha \int_\alpha^1 t^{-2} D_\varphi(t) dt ,$$

and differentiating again, and differencing, we get for $0 < u < v \leq 1$

$$g'(v) - g'(u) = -\frac{1}{v} D_\varphi(v) + \frac{1}{u} D_\varphi(u) - \int_u^v t^{-2} D_\varphi(t) dt = -\int_u^v t^{-1} dD_\varphi(t) < \{D_\varphi(u) - D_\varphi(v)\} / v < 0 . \quad (4.1b)$$

Thus g is strictly concave on $(0,1)$. Noting that

$$\#\{j < n : \varphi(j) = \varphi(n)\} \leq \Phi(\infty, \varphi(n)) - \Phi(\infty, \varphi(n) - 1) = O\left\{\varphi(n) e^{-\sqrt{\log \varphi(n)}}\right\},$$

we have

$$\frac{F(n)}{n} = 1 + \zeta \frac{\varphi(n)}{n} - 2g\left(\frac{\varphi(n)}{n}\right) + O\left\{\frac{\varphi(n)}{n} e^{-\sqrt{\log \varphi(n)}}\right\}. \quad (4.2)$$

If we set

$$h(u) = 1 + \zeta u - 2g(u) \quad (4.3)$$

and enlarge the error we obtain the asymptotic formula

$$\frac{F(n)}{n} = h(\varphi(n)/n) + O\left(e^{-\sqrt{\log n}}\right). \quad (4.4)$$

A Dirichlet series involving the Euler phi function $\varphi(n)$ is

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)},$$

where $\zeta(s)$ is the Riemann zeta function. This is derived as follows:

$$\zeta(s)\sum_{f=1}^{\infty}\frac{\varphi(f)}{f^s}=\left(\sum_{g=1}^{\infty}\frac{1}{g^s}\right)\left(\sum_{f=1}^{\infty}\frac{\varphi(f)}{f^s}\right)$$

$$\left(\sum_{g=1}^{\infty}\frac{1}{g^s}\right)\left(\sum_{f=1}^{\infty}\frac{\varphi(f)}{f^s}\right)=\sum_{h=1}^{\infty}\left(\sum_{fg=h}1\cdot\varphi(g)\right)\frac{1}{h^s}$$

$$\sum_{h=1}^{\infty}\left(\sum_{fg=h}\varphi(g)\right)\frac{1}{h^s}=\sum_{h=1}^{\infty}\left(\sum_{d|h}\varphi(d)\right)\frac{1}{h^s}$$

$$\sum_{h=1}^{\infty}\left(\sum_{d|h}\varphi(d)\right)\frac{1}{h^s}=\sum_{h=1}^{\infty}\frac{h}{h^s}$$

$$\sum_{h=1}^{\infty}\frac{h}{h^s}=\zeta(s-1).$$

We have shown that $F(n)/n\approx h(\varphi(n)/n)$. The function h attains a minimal value h_0 at an interior point u_0 of $(0,1)$. The point u_0 is unique by the strict convexity of h. Thus $F(n)/n$ is asymptotically, near its minimal value h_0 when $\varphi(n)/n$ is near u_0. Numerical data suggest that u_0 is near 1/2 and h_0 is near 1/3. We shall show that $0.473<u_0<0.475$ and $0.321<h_0<0.324$.

We note that for the value 0,473; 0,475; 0,321 and 0,324 we have the following mathematical connections with the Aurea ratio:

$$\left[(\Phi)^{-7/7}+(\Phi)^{-35/7}\right]\cdot\frac{2}{3}=(0{,}61803399+0{,}09016994)\cdot\frac{2}{3}=0{,}708204\cdot\frac{2}{3}=0{,}47213595;$$

$$(\Phi)^{-28/7}\cdot\frac{9}{4}=0{,}14589803\cdot\frac{9}{4}=0{,}32827058.$$

LEMMA 1

$$h'(0)=-\zeta,\quad h'(1)=\zeta.$$

We have by (4.3) that $h'(u)=\zeta-2g'(u)$. The estimate

$$g(u)=\zeta u+O\{\exp(-\exp 1/(ku))\}\quad(4.5)$$

implies that $g'(0)=\zeta$, and hence $h'(0)=-\zeta$. Equation (4.1) implies that $g'(1)=0$, and hence $h'(1)=\zeta$. Thus the minimum of h is achieved in the open interval $(0,1)$. We shall now establish a formula which will lead to estimates for $g(1/2)$. This will be useful because of the close connection between g and h and the proximity of u_0 to 1/2.

LEMMA 2

$$g(1/2)=\frac{1}{2}+\frac{\zeta}{6}-\left\{\left(\frac{\zeta}{4}-g\left(\frac{1}{4}\right)\right)-\left(\frac{\zeta}{8}-g\left(\frac{1}{8}\right)\right)+\left(\frac{\zeta}{16}-g\left(\frac{1}{16}\right)\right)-\ldots\right\}.\quad(4.6)$$

We estimate

$$\#\{n \le x : n \text{ odd}, \varphi(n) \le y\}.$$

We have used the generating function

$$F(s,z) \overset{def}{=} \sum_{n=1}^{\infty} n^{-s}\varphi(n)^{-z} = \prod_p \left\{1 + p^{-s}(p-1)^{-s}\left(1 + p^{-s-z} + p^{-2s-2z} + \ldots\right)\right\} =$$

$$= \prod_p \left\{1 - p^{-s-z} + p^{-s}(p-1)^{-z}\right\}\zeta(s+z) \overset{def}{=} \prod(s,z)\zeta(s+z) \quad (4.7)$$

and the function g was represented by

$$g(\alpha) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \frac{\prod(1-z,z)}{z(1-z)}\alpha^z dz\,, \quad 0 \le \alpha \le 1. \quad (4.8)$$

The formula is valid at the end points by uniform convergence of the integral. We delete the even integers and write

$$F_0(s,z) = \sum_{\substack{n=1 \\ n\cdot odd}}^{\infty} n^{-s}\varphi(n)^{-z} = \prod(s,z)\zeta(s+z)\left\{\frac{1-2^{-s-z}}{1-2^{-s-z}+2^{-s}}\right\}. \quad (4.9)$$

The functions $F(s,z)$ and $F_0(s,z)$ have the same singularities in the region

$$\{(s,z) \in C \times C : \operatorname{Re} s + z > 0\},$$

because any singularity of the new factor $\left(1-2^{-s-z}\right)/\left(1-2^{-s-z}+2^{-s}\right)$ is cancelled by a zero of $\prod(s,z)$, and the new factor has no zeros in this region. It now follows that

$$g_0(\alpha) \overset{def}{=} \lim_{x\to\infty}\frac{1}{x}\#\{n \le x : n \text{ odd}, \varphi(n) \le \alpha x\} = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\frac{\prod(1-z,z)}{z(1-z)}\alpha^z\left(1+2^z\right)^{-1}dz =$$

$$= \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\frac{\prod(1-z,z)}{z(1-z)}\left\{\left(\frac{\alpha}{2}\right)^z - \left(\frac{\alpha}{4}\right)^z + \left(\frac{\alpha}{8}\right)^z - \ldots\right\}dz = g(\alpha/2) - g(\alpha/4) + g(\alpha/8) - \ldots. \quad (4.10)$$

Now g is concave and $g(\varepsilon) \approx \zeta\varepsilon$ as $\varepsilon \to 0$. Thus the series in the formula for $g(1/2)$ is alternating with terms decreasing to zero, indeed at a geometric rate. To further exploit our formula we must first estimate $D_\varphi(t)$ for t near 0.

LEMMA 3

$$D_\varphi(t) < 12t^3, \quad 0 < t < 1.$$

By Chebychev's inequality

$$t^{-3}\#\left\{n\le x:\frac{\varphi(n)}{n}\le t\right\}=t^{-3}\sum_{\substack{n\le x\\ n/\varphi(n)\ge 1/t}}1\le\sum_{n\le x}\left(\frac{n}{\varphi(n)}\right)^3,\quad (4.11)$$

and we estimate the last sum by writing

$$(n/\varphi(n))^3=(1*\beta)(n),$$

where $*$ denotes multiplicative convolution and β is a non-negative multiplicative function satisfying $\beta(p)=\left(p^3-(p-1)^3\right)/(p-1)^3$, $\beta(p^\alpha)=0$ for all primes p and all exponents $\alpha\ge 2$.
Thus

$$\sum_{n\le x}\left(\frac{n}{\varphi(n)}\right)^3=\sum_{n\le x}\left[\frac{x}{n}\right]\beta(n)\le x\sum_{n=1}^{\infty}\frac{\beta(n)}{n}=x\prod_p\left(1+\frac{\beta(p)}{p}\right)=x\prod_p\left\{1+\frac{1}{p}\frac{p^3-(p-1)^3}{(p-1)^3}\right\}\overset{def}{=}\gamma x.\quad (4.12)$$

Now

$$\gamma=\zeta(2)^3\prod_p\left\{1+\frac{3p^2-3p+1}{p(p-1)^3}\right\}\left\{1-\frac{1}{p^2}\right\}^3=\zeta(2)^3\prod_p\left\{1+\frac{6p^4+4p^3-3p^2-p+1}{p^7}\right\}.\quad (4.13)$$

It is easy to check that for all $p\ge 3$

$$6p^4+4p^3-3p^2-p+1<7p^4.\quad (4.14)$$

We have

$$\gamma\le\zeta(2)^3\left(1+\frac{115}{128}\right)\left\{\left(1+\frac{7}{3^3}\right)\left(1+\frac{7}{5^3}\right)\left(1+\frac{7}{7^3}\right)\right\}\exp\left\{\sum_{p\ge 11}7p^{-3}\right\},\quad (4.15)$$

and

$$7\sum_{p\ge 11}p^{-3}<7\int_{10}^{\infty}t^{-3}dt=0.035.\quad (4.16)$$

We note that for the value 0.035, have the following mathematical connection with the Aurea ratio:

$$\left[(\Phi)^{-35/7}+(\Phi)^{-63/7}\right]\cdot\frac{1}{3}=(0{,}09016994+0{,}01315562)\cdot\frac{1}{3}=0{,}103326\cdot\frac{1}{3}=0{,}0344185.$$

Thus $\gamma\le 12$, and $D_\varphi(t)$ satisfies the claimed bound.
We combine the last two lemmas with numerical data of C. R. Wall on the density function D_φ to obtain upper and lower estimates for $g(1/2)$.

LEMMA 4

$$\frac{1}{2}+\frac{\zeta}{6}-0.00154<g(1/2)<\frac{1}{2}+\frac{\zeta}{6}-0.00075.\quad (4.17)$$

The alternating series representation of $g(1/2)$ leads to the inequalities

$$\frac{1}{2}+\frac{\zeta}{6}-\left\{\left(\frac{\zeta}{4}-g\left(\frac{1}{4}\right)\right)-\left(\frac{\zeta}{8}-g\left(\frac{1}{8}\right)\right)+\left(\frac{\zeta}{16}-g\left(\frac{1}{16}\right)\right)\right\}\le g(1/2)\le\frac{1}{2}+\frac{\zeta}{6}-\left\{\left(\frac{\zeta}{4}-g\left(\frac{1}{4}\right)\right)-\left(\frac{\zeta}{8}-g\left(\frac{1}{8}\right)\right)\right\} \quad (4.18)$$

The differential equation (4.1) has the solution

$$u^{-1}g(u)=\zeta-\int_0^u D_\varphi(t)t^{-2}dt\,. \quad (4.19)$$

Thence we obtain that

$$\alpha g'(\alpha)=g(\alpha)-D_\varphi(\alpha)\Rightarrow u^{-1}g(u)=\zeta-\int_0^u D_\varphi(t)t^{-2}dt\,. \quad (4.19b)$$

The constant is evaluated here by noting that $g'(0)=\zeta$. The integral converges at zero by the preceding lemma. Thus we have

$$2^{-k}\zeta-g\left(2^{-k}\right)=2^{-k}\int_0^{2^{-k}} D_\varphi(t)t^{-2}dt\,. \quad (4.20)$$

It follows that

$$\left(\frac{\zeta}{4}-g\left(\frac{1}{4}\right)\right)-\left(\frac{\zeta}{8}-g\left(\frac{1}{8}\right)\right)+\left(\frac{\zeta}{16}-g\left(\frac{1}{16}\right)\right)=\frac{1}{4}\int_{1/8}^{1/4}D_\varphi(t)\frac{dt}{t^2}+\frac{1}{8}\int_{1/16}^{1/8}D_\varphi(t)\frac{dt}{t^2}+\frac{3}{16}\int_0^{1/16}D_\varphi(t)\frac{dt}{t^2}\,. \quad (4.21)$$

We estimate the three integrals from above, using the bound of the preceding lemma for $0\le t\le 0.007$ and the upper bounds of Wall for $0.007<t\le 0.25$. We obtain the upper bound 0.00154. Similar treatment of

$$\left(\frac{\zeta}{4}-g\left(\frac{1}{4}\right)\right)-\left(\frac{\zeta}{8}-g\left(\frac{1}{8}\right)\right)$$

leads to the lower bound 0.00075.

LEMMA 5

$$2D_\varphi\left(\frac{1}{2}\right)-1+\frac{\zeta}{6}+2R=\int_{u_0}^{1/2}t^{-1}dD_\varphi(t), \quad (4.22)$$

where $0.00075<R<0.00154$.

We have by (4.19)

$$\frac{g(u_0)}{u_0}-\frac{g(1/2)}{1/2}=\int_{u_0}^{1/2}D_\varphi(t)t^{-2}dt\,. \quad (4.23)$$

From (4.3) and the fact that $h'(u_0)=0$ we get $g'(u_0)=\frac{\zeta}{2}$. Combining this with (4.1) we obtain

$$g(u_0) = u_0 \frac{\zeta}{2} + D_\varphi(u_0). \quad (4.24)$$

This expression, Lemma 4, and the preceding integral yield

$$\frac{D_\varphi(u_0)}{u_0} - 1 + \frac{\zeta}{6} + 2R = \int_{u_0}^{1/2} D_\varphi(t) t^{-2} dt. \quad (4.25)$$

Integrating by parts we get the desired expression, i.e.

$$2D_\varphi\left(\frac{1}{2}\right) - 1 + \frac{\zeta}{6} + 2R = \int_{u_0}^{1/2} t^{-1} dD_\varphi(t).$$

THEOREM 1

$$u_0 > 0.473 \quad and \quad h_0 < 0.324.$$

Starting from Lemma 5, we write

$$2D_\varphi\left(\frac{1}{2}\right) - 1 + \frac{\zeta}{6} + 2R = \int_{0.475}^{0.5} t^{-1} dD_\varphi(t) + \int_{u_0}^{0.475} t^{-1} dD_\varphi(t) \geq \frac{1}{0.5}\{D_\varphi(0.5) - D_\varphi(0.499)\} +$$
$$+ \frac{1}{0.499}\{D_\varphi(0.499) - D_\varphi(0.498)\} + \ldots + \frac{1}{0.476}\{D_\varphi(0.476) - D_\varphi(0.475)\} + \frac{1}{0.475}\{D_\varphi(0.475) - D_\varphi(u_0)\}. \quad (4.26)$$

We rearrange terms, isolating $D_\varphi(u_0)$:

$$\frac{D_\varphi(u_0)}{0.475} \geq 1 - \frac{\zeta}{6} - 2R + \left(\frac{1}{0.499} - \frac{1}{0.5}\right) D_\varphi(0.499) + \ldots + \left(\frac{1}{0.475} - \frac{1}{0.476}\right) D_\varphi(0.475). \quad (4.27)$$

If we use the upper estimate for R and the lower estimates of C. R. Wall for $D_\varphi(0.475), \ldots, D_\varphi(0.499)$ we find that $D_\varphi(u_0) > 0.3380$.

The stated inequalities follow at once from this bound. First, we have that $D_\varphi(0.473) < 0.3362$, and thus $u_0 > 0.473$. Next, it follows from equations (4.1) and (4.3) that $h_0 = 1 - 2D_\varphi(u_0)$. Thus, $h_0 < 0.324$.

We also have bounds for u_0 and h_0 in the opposite directions.

THEOREM 2

$$u_0 < 0.475 \quad and \quad h_0 > 0.321.$$

Using Lemma 5 again, we write

$$2D_\varphi\left(\frac{1}{2}\right) - 1 + \frac{\zeta}{6} + 2R = \int_{0.475}^{0.5} t^{-1} dD_\varphi(t) + \int_{u_0}^{0.475} t^{-1} dD_\varphi(t). \quad (4.28)$$

This time we express the first integral as an upper Riemann-Stieltjes sum and sum by parts to obtain

$$\int_{0.475}^{0.5} t^{-1} dD_\varphi(t) \le \frac{D_\varphi(0.5)}{0.499} + \left(\frac{1}{0.498} - \frac{1}{0.499}\right) D_\varphi(0.499) + \ldots + \left(\frac{1}{0.475} - \frac{1}{0.476}\right) D_\varphi(0.476) - \frac{D_\varphi(0.475)}{0.475}. \quad (4.29)$$

Thus

$$\int_{u_0}^{0.475} t^{-1} dD_\varphi(t) \ge \frac{D_\varphi(0.475)}{0.475} - I, \quad (4.30)$$

where

$$I = 1 - \frac{\zeta}{6} - 2R + \left(\frac{1}{0.499} - \frac{1}{0.5}\right) D_\varphi(0.5) + \ldots + \left(\frac{1}{0.475} - \frac{1}{0.476}\right) D_\varphi(0.476). \quad (4.31)$$

Thence, we obtain

$$\int_{u_0}^{0.475} t^{-1} dD_\varphi(t) \ge \frac{D_\varphi(0.475)}{0.475} - 1 + \frac{\zeta}{6} + 2R - \left(\frac{1}{0.499} - \frac{1}{0.5}\right) D_\varphi(0.5) - \ldots - \left(\frac{1}{0.475} - \frac{1}{0.476}\right) D_\varphi(0.476).$$

We estimate I from above by using the upper bounds for $D_\varphi(0.476), \ldots, D_\varphi(0.5)$ and the lower bound for R from Lemma 5. We obtain the inequality

$$\int_{u_0}^{0.475} t^{-1} dD_\varphi(t) \ge \frac{D_\varphi(0.475)}{0.475} - 0.7145, \quad (4.32)$$

from which both assertions of the theorem will follow. The bound $D_\varphi(0.475) \ge 0.33969$ implies that

$$\int_{u_0}^{0.475} t^{-1} dD_\varphi(t) > 0.0006 > 0 \quad (4.33)$$

and hence $u_0 < 0.475$. Next, since $u_0 > 0.473$, we obtain from (4.32)

$$\frac{1}{0.473} \left\{ D_\varphi(0.475) - D_\varphi(u_0) \right\} \ge \frac{D_\varphi(0.475)}{0.475} - 0.7145. \quad (4.34)$$

This inequality and the bound $D_\varphi(0.475) < 0.34166$ yield $D_\varphi(u_0) < 0.3394$. Thus, we finally obtain $h_0 = 1 - 2D_\varphi(u_0) > 0.321$.

We note that for the values 0.3380; 0,3394; 0,34166 and 0,7145 we have the following mathematical connections with the Aurea ratio:

$$\left[(\Phi)^{-7/7} + (\Phi)^{-28/7}\right] \cdot \frac{4}{9} = (0{,}61803399 + 0{,}14589803) \cdot \frac{4}{9} = 0{,}763932 \cdot \frac{4}{9} = 0{,}33952534$$

$$\left[(\Phi)^{-21/7} + (\Phi)^{-63/7} + (\Phi)^{-77/7} + (\Phi)^{-98/7}\right] \cdot \frac{4}{3} = (0{,}23606798 + 0{,}01315562 + 0{,}00502500 + 0{,}00118624) \cdot \frac{4}{3} =$$

$$= 0{,}255435 \cdot \frac{4}{3} = 0{,}34057978;$$

$$(\Phi)^{7/7} \cdot \frac{4}{9} = 1{,}61803399 \cong 1{,}618034 \cdot \frac{4}{9} = 0{,}71912622 .$$

5. **On some equations concerning the adelic strings, the zeta strings and the Lagrangians for adelic strings.** [5] [6] [7]

As is known, the scattering of two real bosonic strings in 26-dimensional space-time at the tree level can be described in terms of the path integral in 2-dimensional quantum field theory formalism as follows:

$$A_\infty(k_1,\dots,k_4) = g_\infty^2 \int \mathcal{D}X \exp\left(\frac{2\pi i}{h} S_0[X]\right) \times \prod_{j=1}^{4} \int d^2\sigma_j \exp\left[\frac{2\pi i}{h} k_\mu^{(j)} X^\mu(\sigma_j,\tau_j)\right], \quad (5.1)$$

where $\mathcal{D}X = \mathcal{D}X^0(\sigma,\tau)\mathcal{D}X^1(\sigma,\tau)\dots\mathcal{D}X^{25}(\sigma,\tau)$, $d^2\sigma_j = d\sigma_j d\tau_j$ and

$$S_0[X] = -\frac{T}{2}\int d^2\sigma \partial_\alpha X^\mu \partial^\alpha X_\mu \quad (5.2)$$

with $\alpha = 0{,}1$ and $\mu = 0{,}1{,}\dots{,}25$.
The p-adic analogue of (5.1) is

$$A_p(k_1,\dots,k_4) = g_p^2 \int \mathcal{D}X \chi_p\left(-\frac{1}{h} S_0[X]\right) \times \prod_{j=1}^{4} \int d^2\sigma_j \chi_p\left[-\frac{1}{h} k_\mu^{(j)} X^\mu(\sigma_j,\tau_j)\right]. \quad (5.3)$$

This is the p-adic string amplitude, where $\chi_p(u) = \exp(2\pi i \{u\}_p)$ is p-adic additive character and $\{u\}_p$ is the fractional part of $u \in Q_p$.
Adelic string amplitude is product of real and all p-adic amplitudes, i.e.

$$A_A(k_1,\dots,k_4) = A_\infty(k_1,\dots,k_4)\prod_p A_p(k_1,\dots,k_4). \quad (5.4)$$

In the case of the Veneziano amplitude and $(\sigma_i,\tau_j) \in \mathrm{A}(S)\times\mathrm{A}(S)$, we have

$$A_A(k_1,\dots,k_4) = g_\infty^2 \int_R |x|_\infty^{k_1k_2} |1-x|_\infty^{k_2k_3} dx \times \prod_{p\in S} g_p^2 \prod_{j=1}^{4}\int d^2\sigma_j \times \prod_{p\notin S} g_p^2 . \quad (5.5)$$

The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$\mathcal{L}_p = \frac{1}{g^2}\frac{p^2}{p-1}\left[-\frac{1}{2}\varphi p^{-\frac{\Box}{2}}\varphi + \frac{1}{p+1}\varphi^{p+1}\right], \quad (5.6)$$

where p is any prime number, $\Box = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alambertian and we adopt metric with signature (– + … +). Now we want to introduce a model which incorporates all the above p-adic string Lagrangians in a restricted adelic way. To this end, let us take the sum of the above Lagrangians $\mathcal{L}_n$ (5.6) in the form

$$L = \sum_{n\geq 1} C_n \mathcal{L}_n = \sum_{n\geq 1} \frac{n-1}{n^2}\mathcal{L}_n = \frac{1}{g^2}\left[-\frac{1}{2}\phi\sum_{n\geq 1} n^{-\frac{\Box}{2}}\phi + \sum_{n\geq 1}\frac{1}{n+1}\phi^{n+1}\right], \quad (5.7)$$

where coefficients $C_n = \dfrac{n-1}{n^2}$ are inverses of those within $\mathcal{L}_n$.

Recall that the Riemann zeta function is defined as

$$\zeta(s) = \sum_{n\geq 1}\frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}, \quad s = \sigma + i\tau, \quad \sigma > 1. \quad (5.8)$$

Employing usual expansion for the logarithmic function and definition (5.8) we can rewrite (5.7) in the form

$$L = -\frac{1}{g^2}\left[\frac{1}{2}\phi\zeta\left(\frac{\Box}{2}\right)\phi + \phi + \ln(1-\phi)\right], \quad (5.9)$$

where $|\phi| < 1$. Here $\zeta\left(\frac{\Box}{2}\right)$ acts as pseudo-differential operator in the following way:

$$\zeta\left(\frac{\Box}{2}\right)\phi(x) = \frac{1}{(2\pi)^D}\int e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk, \quad -k^2 = k_0^2 - \vec{k}^2 > 2 + \varepsilon, \quad (5.10)$$

where $\tilde{\phi}(k) = \int e^{(-ikx)}\phi(x)dx$ is the Fourier transform of $\phi(x)$. The region of integration in (5.10) is $-k^2 = k_0^2 - \vec{k}^2 > 2 + \varepsilon$, where ε is an arbitrary small positive number, and it follows from the definition of zeta function (5.8).

When the d'Alambertian is an argument of the Riemann zeta function we shall call such string a "zeta string". Consequently, the above ϕ is an open scalar zeta string. The equation of motion for the zeta string ϕ is

$$\zeta\left(\frac{\Box}{2}\right)\phi = \frac{1}{(2\pi)^D}\int_{k_0^2 - \vec{k}^2 > 2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi}, \quad (5.11)$$

which has an evident solution $\phi = 0$.

The usual crossing symmetric Veneziano amplitude is:

$$A_\infty(a,b) = g_\infty^2\int_R |x|_\infty^{a-1}|1-x|_\infty^{b-1}d_\infty x = g_\infty^2\frac{\zeta(1-a)}{\zeta(a)}\frac{\zeta(1-b)}{\zeta(b)}\frac{\zeta(1-c)}{\zeta(c)}, \quad (5.12)$$

where $a = -\alpha(s) = -\dfrac{s}{2} - 1$, $b = -\alpha(t)$, $c = -\alpha(u)$ with the condition $a + b + c = 1$, i.e. $s + t + u = -8$. In the equation (5.12), $|...|_\infty$ denotes the ordinary absolute value, R is the field of real numbers, kinematic variables $a,b,c \in C$, and ζ is the Riemann zeta function. The corresponding Veneziano amplitude for scattering of p-adic strings was introduced as p-adic analogue of the integral form of (5.12), i.e.

$$A_p(a,b) = g_p^2 \int_{Q_p} |x|_p^{a-1} |1-x|_p^{b-1} d_p x\,, \quad (5.13)$$

where Q_p is the field of p-adic numbers, $|...|_p$ is p-adic absolute value and $d_p x$ is the Haar measure on Q_p. Performing integration in (5.13) one obtains

$$A_p(a,b) = g_p^2 \frac{1-p^{a-1}}{1-p^{-a}} \frac{1-p^{b-1}}{1-p^{-b}} \frac{1-p^{c-1}}{1-p^{-c}}\,, \quad (5.14)$$

where p is any prime number.

The exact tree-level Lagrangian of the effective scalar field φ, which describes the open p-adic string tachyon, is

$$\mathcal{L}_p = \frac{m_p^D}{g_p^2} \frac{p^2}{p-1} \left[-\frac{1}{2} \varphi p^{-\frac{\square}{2m_p^2}} \varphi + \frac{1}{p+1} \varphi^{p+1} \right], \quad (5.15)$$

where p is a prime, $\square = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alambertian and we adopt the metric with signature (– + … +). The equation (5.15) can be rewritten in the following form

$$\mathcal{L}_p = \frac{m_p^D}{g_p^2} \frac{1}{|p|_p (1-|p|_p)} \left[-\frac{1}{2} \varphi |p|_p^{\frac{\square}{2m_p^2}} \varphi + \frac{|p|_p}{|p|_p+1} \varphi^{\frac{|p|_p+1}{|p|_p}} \right], \quad (5.16)$$

where prime p is treated as a p-adic number. Using p-adic norm of p in (5.16) gives real prime in (5.15).

It is worth noting that prime number p in (5.15) and (5.16) can be replaced by any natural number $n \geq 2$ and such expressions also make sense. Now we want to introduce a Lagrangian which incorporates all the above Lagrangians (5.15), with p replaced by $n \in N$. To this end, we take the sum of all Lagrangians $\mathcal{L}_n$ in the form

$$L = \sum_{n=1}^{+\infty} C_n \mathcal{L}_n = \sum_{n=1}^{+\infty} C_n \frac{m_n^D}{g_n^2} \frac{n^2}{n-1} \left[-\frac{1}{2} \phi n^{-\frac{\square}{2m_n^2}} \phi + \frac{1}{n+1} \phi^{n+1} \right], \quad (5.17)$$

whose explicit realization depends on particular choice of coefficients C_n, string masses m_n and coupling constants g_n.

We have considered three cases for coefficients C_n in (5.17): (i) $C_n = \frac{n-1}{n^{2+h}}$, where h is a real parameter; (ii) $C_n = \frac{n^2-1}{n^2}$; and (iii) $C_n = \mu(n)\frac{n-1}{n^2}$, where $\mu(n)$ is the Mobius function. For the case (i), we obtain the following Lagrangian

$$L_h = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi \zeta\left(\frac{\Box}{2m^2} + h\right)\phi + \mathrm{A}\mathcal{C}\sum_{n=1}^{+\infty} \frac{n^{-h}}{n+1}\phi^{n+1} \right], \quad (5.18)$$

where $\mathrm{A}\mathcal{C}$ denotes analytic continuation. For the case (ii), the corresponding Lagrangian is

$$L = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi \left\{ \zeta\left(\frac{\Box}{2m^2} - 1\right) + \zeta\left(\frac{\Box}{2m^2}\right) \right\}\phi + \frac{\phi^2}{1-\phi} \right]. \quad (5.19)$$

With regard the case with the Mobius function $\mu(n)$, recall that the Mobius function is defined for all positive integers and has values 1, 0, – 1 depending on factorization of n into prime numbers p. Its explicit definition as follows:

$$\mu(n) = 0, \;\; n = p^2 m; \qquad \mu(n) = (-1)^k, \;\; n = p_1 p_2 \dots p_k, \;\; p_i \neq p_j; \qquad \mu(n) = 1, \;\; n = 1, \;(k = 0). \quad (5.20)$$

Since the inverse Riemann zeta function can be defined as

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s}, \quad s = \sigma + i\tau, \quad \sigma > 1, \quad (5.21)$$

then the corresponding Lagrangian is

$$L = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi \frac{1}{\zeta\left(\frac{\Box}{2m^2}\right)}\phi + \int_0^{\phi} \mathcal{M}(\phi)\, d\phi \right], \quad (5.22)$$

where $\mathcal{M}(\phi) = \sum_{n=1}^{+\infty} \mu(n)\phi^n = \phi - \phi^2 - \phi^3 - \phi^5 + \phi^6 - \phi^7 + \phi^{10} - \phi^{11} - \dots$
Thence, we can rewrite the eq. (5.22) also

$$L = \frac{m^D}{g^2}\left[-\frac{1}{2}\phi \frac{1}{\zeta\left(\frac{\Box}{2m^2}\right)}\phi + \int_0^{\phi} \sum_{n=1}^{+\infty} \mu(n)\phi^n\, d\phi \right]. \quad (5.22b)$$

With regard the multiplicative approach, our starting point is again p-adic Lagrangian (5.15) with equal masses, i.e. $m_p^2 = m^2$ for every p. It is useful to rewrite (5.15), first in the form,

$$\mathcal{L}_p = \frac{m^D}{g_p^2} \frac{p^2}{p^2 - 1}\left\{ -\frac{1}{2}\varphi \left[p^{-\frac{\Box}{2m^2}+1} + p^{-\frac{\Box}{2m^2}} \right]\varphi + \varphi^{p+1} \right\} \quad (5.23)$$

and then, by addition and subtraction of φ^2, as

$$\mathcal{L}_p = \frac{m^D}{g_p^2}\frac{p^2}{p^2-1}\left\{\frac{1}{2}\varphi\left[\left(1-p^{-\frac{\Box}{2m^2}+1}\right)+\left(1-p^{-\frac{\Box}{2m^2}}\right)\right]\varphi-\varphi^2\left(1-\varphi^{p-1}\right)\right\}. \quad (5.24)$$

Now we introduce a Lagrangian for the entire p-adic sector by taking products

$$\prod_p g_p^2 = C\,,\ \ \prod_p \frac{1}{1-p^{-2}}\,,\ \ \prod_p\left(1-p^{-\frac{\Box}{2m^2}+1}\right),\ \ \prod_p\left(1-p^{-\frac{\Box}{2m^2}}\right)\ \text{ and }\ \prod_p\left(1-\varphi^{p-1}\right) \quad (5.25)$$

in (5.24) at the corresponding places. Then this new Lagrangian becomes

$$\mathcal{L} = \frac{m^D}{C}\zeta(2)\left\{\frac{1}{2}\phi\left[\zeta^{-1}\left(\frac{\Box}{2m^2}-1\right)+\zeta^{-1}\left(\frac{\Box}{2m^2}\right)\right]\phi-\phi^2\prod_p\left(1-\phi^{p-1}\right)\right\}, \quad (5.26)$$

where $\zeta^{-1}(s)=1/\zeta(s)$ and new scalar field is denoted by ϕ. For the coupling constant g_p there are two interesting possibilities: (1) $g_p^2=\frac{p^2}{p^2-1}$, what yields $\zeta(2)/C=1$ in (5.26), and (2) $g_p=|r|_p$, where r may be any non zero rational number and it gives $|r|_\infty\prod_p|r|_p=1$. Both these possibilities are consistent with the adelic product formula

$$A_\infty(a,b)\prod_p A_p(a,b)=g_\infty^2\prod_p g_p^2\,.$$

Let us rewrite (5.26) in the simple form

$$\mathcal{L}=\frac{1}{2}\phi\left[\zeta^{-1}\left(\frac{\Box}{2}-1\right)+\zeta^{-1}\left(\frac{\Box}{2}\right)\right]\phi-\phi^2\Phi(\phi), \quad (5.27)$$

with $m=1$ and $\Phi(\phi)=\mathrm{A}\mathcal{C}\prod_p\left(1-\phi^{p-1}\right)$, where $\mathrm{A}\mathcal{C}$ denotes analytic continuation of infinite product $\prod_p\left(1-\phi^{p-1}\right)$, which is convergent if $|\phi|_\infty<1$ (for example $|\phi|_\infty=0{,}618033987$). One can easily see that $\Phi(0)=1$ and $\Phi(-1)=0$.
For (5.27), the corresponding equation of motion is

$$\left[\zeta^{-1}\left(\frac{\Box}{2}-1\right)+\zeta^{-1}\left(\frac{\Box}{2}\right)\right]\phi=2\phi\Phi(\phi)+\phi^2\Phi'(\phi), \quad (5.28)$$

and has $\phi=0$ as a trivial solution. In the weak-field approximation $(\phi(x)<<1)$, equation (5.28) becomes

$$\left[\zeta^{-1}\left(\frac{\Box}{2}-1\right)+\zeta^{-1}\left(\frac{\Box}{2}\right)\right]\phi=2\phi\,. \quad (5.29)$$

Note that the above operator-valued zeta function can be regarded as a pseudo-differential operator. Then (5.27) and (5.28) are transformed to the integral form. Mass spectrum of M^2 is determined by solutions of equation

$$\zeta^{-1}\left(\frac{M^2}{2}-1\right)+\zeta^{-1}\left(\frac{M^2}{2}\right)=2. \quad (5.30)$$

There are infinitely many tachyon solutions, which are below largest one $M^2 \approx -3.5$.
The potential follows from $-\mathcal{L}$ at $\Box = 0$, i.e.

$$V(\phi)=[7+\Phi(\phi)]\phi^2, \quad (5.31)$$

since $\zeta(-1)=-1/12$ and $\zeta(0)=-1/2$. This potential has local minimum $V(0)=0$ and values $V(\pm 1)=7$.

6. Mathematical connections.

Now, we describe some possible mathematical connections. We take some equations of **Section 1**. With regard the eq. (1.16), we note that can be related with the eq. (5.3) of **Section 5** concerning the p-adic strings and with the fundamental equation regarding the Palumbo-Nardelli model, hence we have the following connections:

$$\sum_{i=1}^{n}\int_{-1}^{1} l_i(x)^2 p(x)dx \le \int_{-1}^{1} p(x)dx \Rightarrow$$
$$\Rightarrow g_p^2\int \mathcal{D}X\chi_p\left(-\frac{1}{h}S_0[X]\right)\times\prod_{j=1}^{4}\int d^2\sigma_j\chi_p\left[-\frac{1}{h}k_\mu^{(j)}X^\mu(\sigma_j,\tau_j)\right]\Rightarrow$$
$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$
$$=\int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(|F_2|^2\right)\right]. \quad (6.1)$$

With regard the eq. (1.34), it can be related with the eqs. (5.3), (5.5) concerning the adelic strings, with the Palumbo-Nardelli model and with the Ramanujan modular function concerning the number 24, i.e. with the "modes" corresponding to the physical vibrations of the bosonic strings. Thence, we obtain the following connections:

$$|J'_n| \le 24\varepsilon^2\int_{-1}^{1} p(x)dx \Rightarrow g_p^2\int \mathcal{D}X\chi_p\left(-\frac{1}{h}S_0[X]\right)\times\prod_{j=1}^{4}\int d^2\sigma_j\chi_p\left[-\frac{1}{h}k_\mu^{(j)}X^\mu(\sigma_j,\tau_j)\right]\Rightarrow$$
$$\Rightarrow g_\infty^2\int_R |x|_\infty^{k_1k_2}|1-x|_\infty^{k_2k_3}dx\times\prod_{p\in S}g_p^2\prod_{j=1}^{4}\int d^2\sigma_j\times\prod_{p\notin S}g_p^2\Rightarrow$$
$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr(G_{\mu\nu}G_{\rho\sigma})f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$
$$=\int_0^\infty \frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_v\left(|F_2|^2\right)\right]\Rightarrow$$

$$\Rightarrow \frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (6.2)$$

The eqs. (1.45) and (1.52) can be related with the eq. (5.3), with the Palumbo-Nardelli model and with the Ramanujan modular function concerning the number 8, i.e. with the "modes" that correspond to the physical vibrations of the superstrings. Thence, we obtain the following connections:

$$c_r^2\int_{-1}^{1}L_{m_r}\left(f_{m_r}\right)^2 dx-8c_r\sqrt{2\int_{-1}^{1}L_{m_r}\left(f_{m_r}\right)^2 dx}>4^r\Rightarrow$$

$$\Rightarrow g_p^2\int \mathcal{D}X\chi_p\left(-\frac{1}{h}S_0[X]\right)\times\prod_{j=1}^{4}\int d^2\sigma_j\chi_p\left[-\frac{1}{h}k_\mu^{(j)}X^\mu\left(\sigma_j,\tau_j\right)\right]\Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$

$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right]$$

$$\Rightarrow \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (6.3)$$

$$I_{m_\rho}>c_\rho^2\int_{-1}^{1}L_{m_\rho}\left(f_{m_\rho}\right)^2 dx-8c_\rho\int_{-1}^{1}\left|L_{m_\rho}\left(f_{m_\rho}\right)\right|dx-16>c_\rho^2\int_{-1}^{1}L_{m_\rho}\left(f_{m_\rho}\right)^2 dx-8c_\rho\left[2\int_{-1}^{1}L_{m_\rho}\left(f_{m_\rho}\right)^2 dx\right]^{\frac{1}{2}}-16\Rightarrow$$

$$\Rightarrow g_p^2\int \mathcal{D}X\chi_p\left(-\frac{1}{h}S_0[X]\right)\times\prod_{j=1}^{4}\int d^2\sigma_j\chi_p\left[-\frac{1}{h}k_\mu^{(j)}X^\mu\left(\sigma_j,\tau_j\right)\right]\Rightarrow$$

$$\Rightarrow -\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$

$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right]$$

$$\Rightarrow \frac{1}{3}\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (6.4)$$

Now we take some equations of **Section 2**. With regard the eqs. (2.48) and (2.51), we note that can be related with the eq. (5.3) of **Section 5** concerning the p-adic strings, with the Palumbo-Nardelli model and with the Ramanujan modular functions concerning the number 8 and 24, i.e. with the "modes" that correspond to the physical vibrations of the bosonic strings and superstrings. Thence, we obtain the following connections:

$$84(a(t)-a(N))^2 N^{3/2}\log N\cdot q^{-1/2}+32\cdot\log N\cdot N^{-1/2}a^2(t)t^2q^{7/2}\cdot 11\cdot\log^2 N\cdot 21+120(a(t)-a(N))^2 N^2$$
$$(\log N)^{1/2}q^{-1/2}\int_0^{1/qQ}\beta^{-1/2}d\beta+10560a^2(t)t^2q^{7/2}N^2(\log N)^{1/2}\int_0^{1/qQ}\beta^{3/2}d\beta\Rightarrow$$
$$\Rightarrow g_p^2\int \mathcal{D}X\chi_p\left(-\frac{1}{h}S_0[X]\right)\times\prod_{j=1}^{4}\int d^2\sigma_j\chi_p\left[-\frac{1}{h}k_\mu^{(j)}X^\mu(\sigma_j,\tau_j)\right]\Rightarrow$$
$$\Rightarrow\frac{4\left[anti\log\dfrac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\dfrac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (6.5)$$

$$\frac{1}{8}a^2(t)N^{3/2}<\int_0^1|G(\alpha)|^2T(\alpha)d\alpha\le E_1+E_2+E_3\le\left(E_1^{'}+2E_1^{''}\right)+E_2+E_3=2E_1^{''}+\left(E_1^{'}+E_2\right)+E_3<$$
$$<126a(t)(a(t)-a(N))N^{3/2}\log\log N+1260a^2(t)tN(\log N)^{1/2}Q^{-1/2}+(a(t)-a(N))^2$$
$$\{84N^{3/2}R^{3/2}\log N+240N^2(\log N)^{1/2}Q^{-1/2}R\}+a^2(t)t^2\{7392N^{-1/2}(\log N)^3R^{11/2}+4224N^2$$
$$(\log N)^{1/2}Q^{-5/2}R^3\}+\frac{1}{64}a^2(t)N^{3/2}+a(t)\{14N^{3/2}R^{-1/2}+28NQ^{1/2}(\log N)^{1/2}\}\Rightarrow$$
$$\Rightarrow g_p^2\int \mathcal{D}X\chi_p\left(-\frac{1}{h}S_0[X]\right)\times\prod_{j=1}^{4}\int d^2\sigma_j\chi_p\left[-\frac{1}{h}k_\mu^{(j)}X^\mu(\sigma_j,\tau_j)\right]\Rightarrow$$
$$\Rightarrow\frac{1}{3}\cdot\frac{4\left[anti\log\dfrac{\int_0^\infty \frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\dfrac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (6.6)$$

We note that, various numbers contents in the eqs. (2.48) and (2.51) are connected with 8 and 24 and with various Fibonacci's numbers. Indeed, we have:

$84=12\cdot 7(2+5)$; $240=24\cdot 2\cdot 5=8\cdot 30=8\cdot 2\cdot 3\cdot 5$; $1260=12\cdot 21\cdot 5$;
$4224=24\cdot 8\cdot 2\cdot 11(3+8)$; $7392=24\cdot 4\cdot 7(2+5)\cdot 11(3+8)=24\cdot 28\cdot 11$; $10560=24\cdot 11\cdot 5\cdot 2^3$;
$126=3^2\cdot 14$; (Note that: 14 = 1 + 5 + 8; 28 = 2 + 5 + 21; 126 = 3 + 34 + 89).
Also here, as in the **Section 1**, we note that the numbers 8 and 24 can be related with the **Aurea ratio**. Indeed, we have:

$$(\Phi)^{35/7}+(\Phi)^{-7/7}+(\Phi)^{-21/7}+(\Phi)^{-42/7}=12;\quad 12\cdot 2=24;\quad 12\cdot\frac{4}{3}=16;\quad 12\cdot\frac{2}{3}=8.$$

Also here $\Phi=\frac{\sqrt{5}+1}{2}=1{,}6180339887...$

Now we take some equations of **Section 3**. With regard the eqs. (3.26) and (3.33), we note that can be related with the Palumbo-Nardelli model, and with the Ramanujan modular functions concerning the number 24, i.e. the modes of bosonic strings. We have:

$$\frac{\exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{\left(m-\frac{1}{24}\right)}\right)}{4\left(m-\frac{1}{24}\right)\sqrt{3}}\left\{1-\frac{1}{\pi}\sqrt{\frac{3}{2}}\frac{1}{\sqrt{\left(m-\frac{1}{24}\right)}}\right\}+O(1)\exp\left\{-0{,}49\frac{2\pi}{\sqrt{6}}\sqrt{m}\right\}\Rightarrow$$

$$\Rightarrow\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\Rightarrow$$

$$\Rightarrow-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$

$$=\int_0^\infty\frac{1}{2\kappa_{10}^2}\int d^{10}x(-G)^{1/2}e^{-2\Phi}\left[R+4\partial_\mu\Phi\partial^\mu\Phi-\frac{1}{2}\left|\tilde{H}_3\right|^2-\frac{\kappa_{10}^2}{g_{10}^2}Tr_\nu\left(\left|F_2\right|^2\right)\right]\qquad(6.7)$$

$$S_0(n,q)+c_4\frac{q^2}{\left(n-\frac{1}{24}\right)^{3/2}}S_2(n,q)+c_5\frac{q^3}{\left(n-\frac{1}{24}\right)^{5/2}}S_3(n,q)+c_6\frac{q^4}{\left(n-\frac{1}{24}\right)^3}S_4(n,q)+O\left(n^{-1,46}\right)\Rightarrow$$

$$\Rightarrow\frac{4\left[anti\log\frac{\int_0^\infty\frac{\cos\pi txw'}{\cosh\pi x}e^{-\pi x^2w'}dx}{e^{-\frac{\pi^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}\Rightarrow$$

$$\Rightarrow-\int d^{26}x\sqrt{g}\left[-\frac{R}{16\pi G}-\frac{1}{8}g^{\mu\rho}g^{\nu\sigma}Tr\left(G_{\mu\nu}G_{\rho\sigma}\right)f(\phi)-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]=$$

$$= \int_0^\infty \frac{1}{2\kappa_{10}^2} \int d^{10}x(-G)^{1/2} e^{-2\Phi} \left[R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2}\left|\tilde{H}_3\right|^2 - \frac{\kappa_{10}^2}{g_{10}^2} Tr_\nu \left(\left|F_2\right|^2 \right) \right] \quad (6.8)$$

With regard the eqs. (3.43), (3.44) and (3.50), we note that can be related with the eqs. (5.11) and (5.22) of the **Section 5**, i.e. with the equations concerning the zeta strings and the zeta-nonlocal Lagrangians. Furthermore, the eq. (3.50) is also connected with the Ramanujan modular function concerning the number 8, i.e. the modes regarding the physical vibrations of the superstrings.

Thence, we have the following mathematical connections:

$$h^*(n) < 3p(n)n^{1/4} \int_{Y_1}^{Y_2} \frac{1}{\sqrt{x}} \exp\left(-\frac{\pi}{\sqrt{6}} \frac{\sqrt{n}}{x} \right) d\Theta(x) \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D} \int_{k_0^2 - \vec{k}^2 > 2+\varepsilon} e^{ixk} \zeta\left(-\frac{k^2}{2} \right) \tilde{\phi}(k) dk = \frac{\phi}{1-\phi} \Rightarrow$$

$$\Rightarrow \frac{m^D}{g^2} \left[-\frac{1}{2}\phi \frac{1}{\zeta\left(\frac{\Box}{2m^2} \right)} \phi + \int_0^\phi \mathcal{M}(\phi) d\phi \right]. \quad (6.9)$$

$$(1+o(1))n^{1/4} \int_{Y_1}^{Y_2} \frac{1}{\sqrt{x \log x}} \exp\left(-\frac{\pi}{\sqrt{6}} \frac{\sqrt{n}}{x} \right) dx = o\left(\frac{1}{\sqrt{\log n}} \right) \int_{Y_1}^{Y_2} \exp\left(-\frac{\pi}{\sqrt{6}} \frac{\sqrt{n}}{x} \right) dx \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D} \int_{k_0^2 - \vec{k}^2 > 2+\varepsilon} e^{ixk} \zeta\left(-\frac{k^2}{2} \right) \tilde{\phi}(k) dk = \frac{\phi}{1-\phi} \Rightarrow$$

$$\Rightarrow \frac{m^D}{g^2} \left[-\frac{1}{2}\phi \frac{1}{\zeta\left(\frac{\Box}{2m^2} \right)} \phi + \int_0^\phi \mathcal{M}(\phi) d\phi \right]. \quad (6.10)$$

$$(1+o(1))2\sqrt{\frac{6}{\pi}} \frac{p(n)}{\sqrt{\log n}} \int_{X_1}^{X_2} \exp\left(-\frac{\pi}{\sqrt{6}} \frac{\sqrt{n}}{x} \right) dx = (1+o(1))8\sqrt{\pi}\, p(n) \exp\left(\frac{\omega}{2} \right) (\rightarrow +\infty) \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D} \int_{k_0^2 - \vec{k}^2 > 2+\varepsilon} e^{ixk} \zeta\left(-\frac{k^2}{2} \right) \tilde{\phi}(k) dk = \frac{\phi}{1-\phi} \Rightarrow$$

$$\Rightarrow \frac{m^D}{g^2} \left[-\frac{1}{2}\phi \frac{1}{\zeta\left(\frac{\Box}{2m^2} \right)} \phi + \int_0^\phi \mathcal{M}(\phi) d\phi \right] \Rightarrow$$

$$\Rightarrow \frac{1}{3} \cdot \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4}w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (6.11)$$

Now we take some equations of **Section 4**. With regard the eqs. (4.19), (4.21), (4.22), (4.26), (4.29) and (4.32), we note that can be related with various equations concerning the zeta strings and the zeta-nonlocal Lagrangians of **Section 5**. For example, we have the following mathematical connections:

$$u^{-1}g(u) = \zeta - \int_0^u D_\varphi(t) t^{-2} dt \Rightarrow \frac{1}{(2\pi)^D} \int_{k_0^2 - \vec{k}^2 > 2+\varepsilon} e^{ixk} \zeta\left(-\frac{k^2}{2}\right) \tilde{\phi}(k) dk = \frac{\phi}{1-\phi} \Rightarrow$$

$$\Rightarrow \frac{m^D}{g^2}\left[-\frac{1}{2}\phi \frac{1}{\zeta\left(\frac{\Box}{2m^2}\right)}\phi + \int_0^\phi \mathcal{M}(\phi) d\phi\right]. \quad (6.12)$$

With regard the eq. (4.21), it can be related also with the number 8, i.e. with the “modes” that correspond to the physical vibrations of the superstrings:

$$\left(\frac{\zeta}{4} - g\left(\frac{1}{4}\right)\right) - \left(\frac{\zeta}{8} - g\left(\frac{1}{8}\right)\right) + \left(\frac{\zeta}{16} - g\left(\frac{1}{16}\right)\right) = \frac{1}{4}\int_{1/8}^{1/4} D_\varphi(t) \frac{dt}{t^2} + \frac{1}{8}\int_{1/16}^{1/8} D_\varphi(t) \frac{dt}{t^2} + \frac{3}{16}\int_0^{1/16} D_\varphi(t) \frac{dt}{t^2} \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D} \int_{k_0^2 - \vec{k}^2 > 2+\varepsilon} e^{ixk} \zeta\left(-\frac{k^2}{2}\right) \tilde{\phi}(k) dk = \frac{\phi}{1-\phi} \Rightarrow$$

$$\Rightarrow \frac{m^D}{g^2}\left[-\frac{1}{2}\phi \frac{1}{\zeta\left(\frac{\Box}{2m^2}\right)}\phi + \int_0^\phi \mathcal{M}(\phi) d\phi\right] \Rightarrow$$

$$\Rightarrow \frac{1}{3} \cdot \frac{4\left[anti\log \frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x} e^{-\pi x^2 w'} dx}{e^{-\frac{\pi t^2}{4}w'} \phi_{w'}(itw')}\right] \cdot \frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)} + \sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (6.13)$$

$$2D_\varphi\left(\frac{1}{2}\right) - 1 + \frac{\zeta}{6} + 2R = \int_{0.475}^{0.5} t^{-1} dD_\varphi(t) + \int_{u_0}^{0.475} t^{-1} dD_\varphi(t) \geq \frac{1}{0.5}\{D_\varphi(0.5) - D_\varphi(0.499)\} +$$

$$+ \frac{1}{0.499}\{D_\varphi(0.499) - D_\varphi(0.498)\} + \ldots + \frac{1}{0.476}\{D_\varphi(0.476) - D_\varphi(0.475)\} + \frac{1}{0.475}\{D_\varphi(0.475) - D_\varphi(u_0)\} \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} \Rightarrow$$

$$\Rightarrow \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\frac{1}{\zeta\left(\frac{\square}{2m^2}\right)}\phi + \int_0^\phi \mathcal{M}(\phi)d\phi\right]. \quad (6.14)$$

$$\int_{u_0}^{0.475} t^{-1}dD_\varphi(t) \geq \frac{D_\varphi(0.475)}{0.475} - 0.7145 \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} \Rightarrow$$

$$\Rightarrow \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\frac{1}{\zeta\left(\frac{\square}{2m^2}\right)}\phi + \int_0^\phi \mathcal{M}(\phi)d\phi\right]. \quad (6.15)$$

With regard the eq. (6.11), it can be related also with the frequency connected with the Aurea ratio by the value $\frac{\pi}{\sqrt{6}}$. Indeed, we can write this equation as follow:

$$(1+o(1))2\sqrt{\frac{6}{\pi}}\frac{p(n)}{\sqrt{\log n}}\int_{X_1}^{X_2}\exp\left(-\left(\left((\Phi)^{28/7}\cdot\frac{1}{3}\right)-1\right)\frac{\sqrt{n}}{x}\right)dx = (1+o(1))8\sqrt{\pi}\,p(n)\exp\left(\frac{\omega}{2}\right)(\to +\infty) \Rightarrow$$

$$\Rightarrow \frac{1}{(2\pi)^D}\int_{k_0^2-\vec{k}^2>2+\varepsilon} e^{ixk}\zeta\left(-\frac{k^2}{2}\right)\tilde{\phi}(k)dk = \frac{\phi}{1-\phi} \Rightarrow$$

$$\Rightarrow \frac{m^D}{g^2}\left[-\frac{1}{2}\phi\frac{1}{\zeta\left(\frac{\square}{2m^2}\right)}\phi + \int_0^\phi \mathcal{M}(\phi)d\phi\right] \Rightarrow$$

$$\Rightarrow \frac{1}{3}\cdot\frac{4\left[anti\log\frac{\int_0^\infty \frac{\cos \pi txw'}{\cosh \pi x}e^{-\pi x^2 w'}dx}{e^{-\frac{\pi t^2}{4}w'}\phi_{w'}(itw')}\right]\cdot\frac{\sqrt{142}}{t^2 w'}}{\log\left[\sqrt{\left(\frac{10+11\sqrt{2}}{4}\right)}+\sqrt{\left(\frac{10+7\sqrt{2}}{4}\right)}\right]}. \quad (6.16)$$

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