WORKING PAPER

DIPARTIMENTO DI ECONOMIA PUBBLICA

Working Paper n.121

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Intertemporal Equilibrium and Walras' Theory of Capital: a Projection Based Approach

Roma, Maggio 2009

SAPIENZA UNIVERSITA' DI ROMA

INTERTEMPORAL EQUILIBRIUM AND WALRAS' THEORY OF CAPITAL: A PROJECTION BASED APPROACH*

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April 22, 2009

Abstract

In this paper we analyze the intertemporal competitive equilibrium of a walrasian model of capital accumulation. We prove the existence of equilibria by generalizing a result of Todd (1979). We overcome the indeterminacy in savings allocation to multiple types of capital goods by introducing a decreasing-return-to-scale storage technology. We finally verify that, for stored capital goods, equality of rates of returns is satisfied in equilibrium.

JEL Classification: B21, C60, C62, D51

Keywords: Walras, Capital Goods, Activity Analysis, General Equilibrium

1 Introduction

Walras' models of pure exchange and production economies provided for decades the underpinning of contemporary general equilibrium theory (see

*We would like to thank an anonymous referee for useful comments.

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Walras (1926), lectures III and IV). However, the original walrasian theory of savings and capital accumulation (see Walras (1926), lecture V, which the two previous lectures were meant to be instrumental to) has been left outside of the mainstream economic analysis. References to it, if any at all, have been made to criticize its controversial aspects.

Indeed, some of the features of Walras theory seem to lend support to this attitude. First of all, because of the absence of an explicit temporal indexation of the variables, the time-frame of the original Walras' theory is left to the reader's interpretation. In particular, it remains an open question whether the model that describes the walrasian theory of capital accumulation is static (that is, related to a single period) or dynamic, and, in the latter case, if it pertains to the short run or long run.

Moreover, since consumers can only invest their savings in capital goods, and are interested in the total return of their savings, capital goods are perceived as perfect substitutes. It follows that it was impossible for Walras to formalize savings decisions by means of well defined individual capital good' (demand) functions.

Hence, in order to close the general equilibrium system and ensure consistency of individuals choices, Walras *imposed* an explicit condition of equality of capital goods' rates of return. Since consistency also requires equality of expectations on these rates, it is also necessary to impose static expectations on prices for services of capital goods throughout the whole planning period.

However, in equilibrium there is no guarantee that these conditions are always met with equality for all capital goods. Thus, it is customary to conclude that the original walrasian model is not coherent and thus it is ill-suited for a long run equilibrium theory (see, for instance, Eatwell (1987) and Garegnani (1990)).[1]

In this paper our goal is threefold: first, to revise the walrasian theory of savings and capital using a contemporary approach. Second, to overcome the afore-mentioned controversial aspects and finally, to show that it can represent a very general, coherent and microfunded theory of long run competitive equilbrium. In order to do so, we propose a reinterpretation of the original walras' theory of capital accumulation in a context where:

- time is explicitly taken into account in the formulation of the model;

[1] On these aspects, see also Impicciatore and Rossi (1982) and Montesano (2008).

- consumers' choices of capital goods can be represented by well defined (demand) functions, so that aggregation of savings in the fictitious good E (see Walras (1926), lecture V) is no longer required;
- equality of capital goods' rates of return emerges *endogenously* in equilibrium.

Finally, a comment on the approach we take to proving the existence of an intertemporal competitive equilibrium is in order. As we must work with a linear technology, equilibrium prices must guarantee non-positive profits in the production sector. Therefore, they must lie in a subset of the unitary simplex. Therefore, in this framework the equilibrium set is different from that demand functions are defined on. To overcome the problems related to a direct application of the Brower (or Kakutani) theorem, we use a projection criterion. However, since we assume strongly monotone preferences, the standard approach for the application of this criterion (see Todd (1979) and Kehoe (1980)) has to be extended to take into account prices on the boundary of the price simplex, where the net demand is not defined.

The paper is structured as follows: in section 2 we briefly discuss the relation to the existing literature. The environment is described in section 3. In section 4 the storage technology is discussed at length. Consumers' optimal choices are derived in section 5. The production sector is formalized in section 6. In section 7 we define the notion of intertemporal equilibrium and we state the main result about the existence of equilibria. The characterization of the equilibrium rates of returns is examined in section 8. Finally, the proof of the existence of intertemporal competitive equilibria is relegated to the appendix.

2 Relation to the literature

The first rigorous analysis of the walrasian model of capital accumulation is due to Morishima (1964), Morishima (1977). Then, this theory has been revisited in a temporary equilibrium framework (see Diewert (1977)), and in the same temporal configuration as Diewert's model under the assumption that capital goods are owned by the firms (see Impicciatore and Rossi (1982)).

In contrast, in what follows we introduce a complete array of Arrow-Debreu forward markets (intertemporal equilibrium market structure). In particular, we assume that there exist both current and forward markets for consumption and capital goods, as well as for labor and capital goods services.

Moreover, we follow in Walras' footstep by assuming that each consumer has initial endowments of capital goods whose services are supplied to the production sector in the current period. Consistently with the original walrasian approach, we also assume that consumers can purchase currently produced capital goods in order to sell their services (on forward markets) in the future period. So, note that in our model consumers own the capital goods.

Since agents' actions are taken at two different points in time, we introduce a novel feature: we assume that consumers have to store capital goods in order to supply their services to the production sector in the next period. We stress that this hypothesis has two main advantages: on the one hand it is a realistic assumption. On the other hand, the storage technology is a device that enables us to derive continuous demand functions for capital goods. It also enables us to overcome the indeterminacy of savings across multiple types of capital goods. Ultimately, equality of capital goods' rates of return emerges *endogenously* in equilibrium rather than being posited at the outset.

3 The environment

Time is discrete, and the horizon is finite with two periods, labelled $t = 0, 1$. In each period there are C perishable consumption goods indexed by $c = 1, 2, ..., C$, and denoted by the vector $x_t = (..., x_{t,c}, ...)$. In each period there are J labor/leisure services indexed by $j = 1, 2, ..., J$, and denoted by the vector $l_t = (..., l_{t,j}, ...)$; there are M capital goods indexed by $m = 1, 2, ..., M$, and denoted by the vector $k_t = (..., k_{t,m}, ...)$, as well as a consistent number of capital goods' services.

Goods and services are traded on two markets. Currently produced consumption and capital goods, along with labor/leisure services and capital services from existing capital goods (inherited from the past), are traded on current markets. Consumption goods to be produced at $t = 1$, labor/leisure services at $t = 1$, and services of currently produced capital goods, that will be available at $t = 1$, are traded on forward markets.

There is a finite number of consumers indexed by $h = 1, ..., H$. We assume that capital goods are not consumed, nor do they affect agents' preferences.

Hence, consumers' preferences are defined on the consumption set $X_h = \mathbb{R}_+^{2(C+J)}$, and are represented by an utility function $u_h : X_h \to \mathbb{R}$. We posit the following assumption:

Assumption 1.3.: *For each $h \in \{1, 2, ..., H\}$, $u_h : X_h \to \mathbb{R}$ is continuous, strictly increasing, and strictly quasi-concave. Moreover, for each $h \in \{1, 2, ..., H\}$, for each $l \in \mathbb{R}_+^{2J}$,* and each $\bar{u}_h \in \mathbb{R}$, *we have that:*

$$\left\{x \in \mathbb{R}_+^{2C} : u_h(x, l) = \bar{u}_h\right\} \subset \mathbb{R}_{++}^{2C}.$$

We stick to the original Walras' approach by assuming that at $t = 0$ each consumer has initial endowments of labor/leisure services $\bar{l}_0^h \in \mathbb{R}_+^J \setminus \{0\}$ and capital goods $\bar{k}^h \in \mathbb{R}_+^M \setminus \{0\}$. The following relations hold:

$$\sum_h \bar{l}_0^h = \bar{l}_0 \in \mathbb{R}_{++}^J, \text{ and } \sum_h \bar{k}^h = \bar{k} \in \mathbb{R}_{++}^M$$

Similarly, we assume that at $t = 1$ each consumer is endowed with labor/leisure services $\bar{l}_1^h \in \mathbb{R}_+^J \setminus \{0\}$ that satisfy:

$$\sum_h \bar{l}_1^h = \bar{l}_1 \in \mathbb{R}_{++}^J.$$

At $t = 0$ services from owned capital goods are inelastically supplied. Labor/leisure services $\left(l_0^h, l_1^h\right)$, consumption goods $\left(x_0^h, x_1^h\right)$, and currently produced capital goods k^h are supplied and demanded according to the utility and profit maximization problems that will be studied in sections 5 and 6.

Since, at $t = 1$, capital goods are required to undertake production, capital goods purchased at $t = 0$ are stored for one period; at $t = 1$ their services along with labor/leisure services, which have been sold at $t = 0$ on forward markets, are supplied to the production sector. Moreover, consumption goods, purchased at $t = 0$ on forward markets, are delivered from the production sector.

In what follows $p = (p_0, p_1) \in \mathbb{R}_+^{2C}$ denotes current and future consumption goods prices, $w = (w_0, w_1) \in \mathbb{R}_+^{2J}$ denotes current and future labor/leisure services prices, $v = (v_0, v_1) \in \mathbb{R}_+^{2M}$ denotes current and future capital goods services prices, and $q \in \mathbb{R}_+^M$ denotes the price vector of capital goods produced at $t = 0$. We let:

$$\pi = (p_0, p_1, q, w_0, v_0, w_1, v_1) \in \mathbb{R}^N_+$$

denote the list of Arrow-Debreu prices, with $N = 2(C + J) + 3M$.

There is a representative firm. This is without loss in generality, given the assumption of constant returns-to-scale (see section 6). The firm operates both at $t = 0$ and $t = 1$, using labor and capital services as inputs. At $t = 0$ the firm produces both consumption and capital goods. Notice, though, that at $t = 1$ the firm only produces consumption goods. This is so because consumers do not wish to consume capital goods, and $t = 1$ is the last period (finite time-horizon). We let $y^0 \in \mathbb{R}^{C+2M+J}$ denote a current production plan, and $y^1 \in \mathbb{R}^{C+M+J}$ denote a future production plan. Therefore, $y = (y^0, y^1) \in \mathbb{R}^N$ denotes an intertemporal production plan.

For the sake of simplicity, we assume that capital goods used in the production process totally depreciate at the end of each period.

4 The storage technology

Unlike in the original Walras' approach, we introduce a storage technology. This assumption is not only reasonable, but it also enables us to derive continuous demand functions for capital goods. Moreover, by virtue of the storage technology assumption, we can pin down *endogenously* a no-arbitrage condition on capital goods returns (see section 8). This stands in contrast to the original Walras' formulation in which the no-arbitrage condition was *imposed* and not derived.

Notice that storage is required only for currently produced and purchased capital goods, since capital goods existing at $t = 0$ totally depreciate at the end of the period, after production is undertaken.

We assume that consumers are endowed with individual storage technologies, formalized as follows:

Definition 1.4. *For each $h \in \{1, 2, ..., H\}$ and each capital good $m \in \{1, 2, ..., M\}$, the storage function $\sigma^h_m : \mathbb{R}_+ \to \mathbb{R}_+$ maps any feasible quantity of the capital good currently purchased and stored into the quantity of services available for supply to the production sector at $t = 1$.*

In compact notation, the storage technologies are denoted by:

$$\sigma^h(k^h) = \left(\sigma^h_1(k^h_1), ..., \sigma^h_m(k^h_m), ..., \sigma^h_M(k^h_M)\right) .$$

We assume that the storage technology exhibits decreasing returns-to-scale. That is, increasing the quantity of stored capital goods leads to a less than proportional increase in the quantity of services available for sale at $t = 1$. Formally, each function σ_m^h is assumed satisfy the following assumption:

Assumption 1.4. (STRICT CONCAVITY AND SATURATION): *For each* $h \in \{1, 2, ..., H\}$ *and each* $m \in \{1, 2, ..., M\}$, *there exists a* $\hat{k}_m^h > 0$ *such that* $\sigma_m^h : \left[0, \hat{k}_m^h\right] \to \mathbb{R}_+$ *is continuous, strictly concave, strictly increasing, and differentiable on* $\left(0, \hat{k}_m^h\right)$*; moreover,* $\sigma_m^h(0) = 0$, $\sigma_m^h(k_m^h) \leqslant k_m^h$, *and*

$$\lim_{k_m^h \to \left(\hat{k}_m^h\right)^-} \frac{\sigma_m^h\left(\hat{k}_m^h\right) - \sigma_m^h\left(k_m^h\right)}{\hat{k}_m^h - k_m^h} = 0$$

Remark 1.4. *The assumption that* $\sigma_m^h(k_m^h) \leqslant k_m^h$, *for each* $0 \leq k_m^h \leq \hat{k}_m^h$, *captures the idea that we are dealing with a storage technology. The assumption of concavity is standard and mathematically handy. The existence of a saturation point,* $\hat{k}_m^h$, *is justified in terms of factors, such as warehouses, with fixed and limited capacity. Given the time-frame of the model, we assume that it is not possible to adjust capacity at* $t = 0$. *Clearly, there is no incentive to relax the capacity constraint at* $t = 1$. *Hence, we do not formalize these underlying factors of production, although they can be invoked to motivate the above assumption.*

5 The consumer's problem

In the remainder of the paper, $\bar{k}_h$ also denotes the quantity of services that household h can supply from initial endowments of capital goods. We let $x^h = (x_0^h, x_1^h) \in \mathbb{R}_+^{2C}$ and $l^h = (l_0^h, l_1^h) \in \mathbb{R}_+^{2J}$.

Each consumer takes the price vector π as given, and chooses a bundle $(x^h, l^h, k^h) \in \mathbb{R}_+^{2(C+J)+M}$ that maximizes his utility. The consumer is subject to the storage capacity constraint, and the budget constraint. The latter is given by a single inequality by virtue of the institutional arrangement in place (Arrow-Debreu forward markets). Formally, each consumer seeks the solution to the following problem:

$$\max_{(x^h, l^h, k^h) \geq 0} u^h(x^h, l^h)$$

$$s.t. : \tag{1.5}$$

$$\begin{aligned} p \cdot x^h + w \cdot l^h + q \cdot k^h &\leqslant v_0 \cdot \bar{k}^h + w \cdot \bar{l}^h + v_1 \cdot \sigma^h(k^h) \, , \\ 0 &\leq k^h \leq \hat{k}^h \end{aligned}$$

where $\hat{k}^h = \left(\hat{k}_1^h, ..., \hat{k}_M^h \right)$. Recall that, since by assumption services $\bar{k}^h$ are inelastically supplied to the market, income $v_0 \cdot \bar{k}^h$ depends only on market prices. Income from future capital services depends not only on their prices, v_1, but also on prices of currently purchased capital goods, since q influences the quantity of capital goods that consumers purchase and store. Clearly, maximization problem (1.5) can be rearranged as follows:

$$\max_{(x^h, l^h, k^h) \geq 0} u^h(x^h, l^h)$$

$$s.t. : \tag{2.5}$$

$$\begin{aligned} p \cdot x^h + w \cdot l^h &\leqslant v_0 \cdot \bar{k}^h + w \cdot \bar{l}^h + v_1 \cdot \sigma^h(k^h) - q \cdot k^h \, , \\ 0 &\leq k^h \leq \hat{k}^h \end{aligned}$$

By virtue of strict monotonicity of preferences (Assumption 1.3.), it should be clear that the above program (2.5) can be solved in two steps: first, given (q, v_1), choose k^h so as to maximize $v_1 \cdot \sigma^h(k^h) - q \cdot k^h$ subject to $0 \leq k^h \leq \hat{k}^h$; then, choose (x^h, l^h) so as to maximize u^h given (p, w, v_0) and the optimal choice of purchased capital goods.

5.1 Demand for capital goods

In this paragraph we study the first step in the consumers' problem. As mentioned above, given (q, v_1) the generic consumer chooses k^h to solve the following problem:

$$\max_{\hat{k}^h \geqslant k^h \geqslant 0} v_1 \cdot \sigma^h(k^h) - q \cdot k^h \tag{1.5.1}$$

By virtue of Assumption 1.4., it is easy to see that the relevant necessary and sufficient conditions for an interior optimum are:

$$v_1 \odot \nabla\sigma^h(k^h) - q = 0 \tag{2.5.1}$$

where the symbol $\odot$ denotes the component-wise product of two vectors, and $\nabla\sigma^h(k^h)$ is the vector of derivatives $\mathrm{d}\sigma^h_m/\mathrm{d}k^h_m$ for each h and m. When $(q, v_1) \in \mathbb{R}^{2M}_+ \setminus \{0\}$, the solution to the above problem is a function to be denoted $\psi^h : \mathbb{R}^{2M}_+ \setminus \{0\} \longrightarrow \mathbb{R}^M$, which is continuous by the Berge's maximum theorem and Assumption 1.4. When $v_1^m = q_m = 0$ for some m, the demand for capital good m is actually multivalued as in this case k^h_m can take any value in the interval $[0, \hat{k}^h_m]$. To deal with this fact, we introduce the function $\hat{k}^h = \Psi^h(q, v_1) : \mathbb{R}^{2M}_+ \longrightarrow \mathbb{R}^M$ defined as follows: for ever m,

$$\hat{k}^h_m = \Psi^h_m(q, v_1) \ , \tag{4.5.1}$$

where $\Psi^h_m(q, v_1)$ is the continuous extension of ψ^h_m to $\mathbb{R}^{2M}_+$, whose existence it guaranteed since all assumptions of Theorem 4.7 in Nikaido (1968, p.72) are satisfied. In what follows, we will refer to Ψ^h as the virtual demand of capital. Let us now define:

$$F^h(q, v_1) := \max_{0 \leq k^h \leq \hat{k}^h} \left[v_1 \cdot \sigma^h(k^h) - q \cdot k^h \right] \tag{3.5.1}$$

and notice that, in any case, F^h defined in (3.5.1) is continuous, homogeneous of degree one and $F^h \geq 0$.

5.2 Demand for consumption goods and labor/leisure

In the second step of maximization problem (2.5), given $F^h(q, v_1)$ and (p, w, v_0), the generic consumer chooses (x^h, l^h) to solve:

$$
\begin{aligned}
\max_{(x^h, l^h) \geq 0} \quad & u^h(x^h, l^h) \\
\text{s.t.} \quad & p \cdot x^h + w \cdot l^h \leqslant v_0 \cdot \bar{k}^h + w \cdot \bar{l}^h + F^h(v_1, q) \,. \qquad (1.5.2)
\end{aligned}
$$

By virtue of strong monotonicity of preferences, prices for consumption goods and labor/leisure services must be strictly positive. Moreover, the budget constraint holds with equality. Hence, hereafter we shall restrict attention to $(p, w) \gg 0$. Since in this case the budget set is compact and convex (see Lemma 1.5.2. below), the utility function is continuous and strictly quasi concave (see Assumption 1.3.), and F is homogeneous of degree one, the above problem has a solution $\left(x^h(\pi), l^h(\pi)\right)$ with properties that are summarized in the following lemma:

Lemma 1.5.2. *For every $h \in \{1, 2, ..., H\}$, the demand function $\left(x^h(\pi), l^h(\pi)\right)$ is continuous and homogeneous of degree zero.*

Proof: by virtue of Assumption 1.3. and Berge's maximum theorem, it will suffice to show that the budget constraint correspondence is compact valued and continuous. The homogeneity of degree zero of the demand functions is a straightforward consequence of the homogeneity of degree one of F^h (see above). Our proof draws on Ok (2007), but we slightly adapt it to take into account two features of the budget constraint in problem (1.5.2): not all prices are necessarily strictly positive; a continuous function of prices, F^h, appears in the right-hand-side member of the budget constraint.

Consider problem (1.5.2) above, and define:

$$A = \left\{\pi \in \mathbb{R}^N_+ : (p_0, p_1, w_0, w_1) \gg 0\right\}.$$

Let $B : A \twoheadrightarrow \mathbb{R}^{2(C+J)}_+$ be the budget constraint correspondence. We first show that B is upper hemi-continuous and compact valued as follows (see Aliprantis and Border, 2006, Theorem 17.20). Pick an arbitrary $\pi \in A$, so $(p, q, w_0, v_0, w_1, v_1) = \pi \in \mathbb{R}^N_+$ with $(p_0, p_1, w_0, w_1) \gg 0$. Let $\left\{\pi_n, \left(x^h_n, l^h_n\right)\right\}$ satisfy $(\pi_n) \subseteq A$, $\lim_{n\to\infty} \pi_n = \pi$, and $\left(x^h_n, l^h_n\right) \in B\,(\pi_n)$ for each n, that is:

$$p_n \cdot x^h_n + w_n \cdot l^h_n \leq v_{0n} \cdot \bar{k}^h + w_n \cdot \bar{l}^h + F^h(v_{1_n}, q_n), \text{ with } \left(x^h_n, l^h_n\right) \geq 0, \text{ for each } n.$$

Since (p_n^i) converges to $p^i > 0$, for $i = 1, 2, ..., 2C$, and (w_n^j) converges to $w^j > 0$, for $j = 1, 2, ..., 2J$, then we must have $p_*^i = \inf\{p_n^i : n \in \mathbb{N}\} > 0$ for each i, and also $w_*^j = \inf\{w_n^j : n \in \mathbb{N}\} > 0$ for each j. Since F^h is continuous, then:

$$\lim_{n\to\infty} I_n = v_{0n} \cdot \bar{k}^h + w_n \cdot \bar{l}^h + F^h(v_{1_n}, q_n) = I = v_0 \cdot \bar{k}^h + w \cdot \bar{l}^h + F^h(v_1, q)$$

with $I > 0$ (since $w \gg 0$ and $F^h \geq 0$). Hence, $I^* = \sup\{I_n : n \in \mathbb{N}\} < \infty$. Now note that, for each n, we have:

$$p_* \cdot x_n^h + w_* \cdot l_n^h \leq p_n \cdot x_n^h + w_n \cdot l_n^h \leq I_n \leq I^*$$

therefore, for each n, $\left(x_n^h, l_n^h\right) \in S$, where:

$$S = \left\{(x, y) \in \mathbb{R}_+^{2(C+J)} : p_* \cdot x + w_* \cdot y \leq I^*\right\}.$$

Clearly S is closed and bounded, and therefore there exists a subsequence, say $\left(x_{n_k}^h, l_{n_k}^h\right)$, converging to some $\left(x^h, l^h\right) \in \mathbb{R}_+^{2(C+J)}$. But then, by virtue of continuity, we get:

$$p \cdot x^h + w \cdot l^h = \lim_{k\to\infty} \left(p_{n_k} \cdot x_{n_k}^h + w_{n_k} \cdot l_{n_k}^h\right) \leq \lim_{k\to\infty} I_{n_k} = I$$

that is, $\left(x^h, l^h\right) \in B(\pi)$. Thus, B is upper hemicontinuous and compact valued on A.

Next we prove that B is lower hemicontinuous on A. Pick an arbitrary $\pi \in A$, so $(p, q, w_0, v_0, w_1, v_1) = \pi \in \mathbb{R}_+^N$ with $(p_0, p_1, w_0, w_1) \gg 0$. Assume, by way of obtaining a contradiction, that B is not lower hemicontinuous at π. Thus, there exists an open set in $\mathbb{R}_+^{2(C+J)}$, say O, with $B(\pi) \cap O \neq \varnothing$, such that for every open neighborhood of π in A, say T, there exists a $x \in T$ satisfying $B(x) \cap O = \varnothing$. But then we can find a sequence (π_n) that converges to π in A, and such that $B(\pi_n) \cap O = \varnothing$ for each n. Now pick any $\left(x^h, l^h\right) \in B(\pi) \cap O$. Since O is open in $\mathbb{R}_+^{2(C+J)}$, it follows that $\lambda\left(x^h, l^h\right) \in B(\pi) \cap O$ for $\lambda \in (0, 1)$ close enough to 1. We distinguish two cases: (i) $\left(x^h, l^h\right) = 0$. (ii) $\left(x^h, l^h\right) \in \mathbb{R}_+^{2(C+J)} \setminus \{0\}$. In case (i), for each n we have:

$$p_n \cdot x^h + w_n \cdot l^h = 0 \leq v_{0n} \cdot \bar{k}^h + w_n \cdot \bar{l}^h + F^h(v_{1_n}, q_n)$$

Hence $\left(x^h, l^h\right) \in B\left(\pi_n\right) \cap O$, which is a contradiction. In case (ii), since $(p_0, p_1, w_0, w_1) \gg 0$ and $\left(x^h, l^h\right) \in \mathbb{R}_+^{2(C+J)} \backslash \{0\}$, then:

$$\begin{array}{rcl} (p, w) \cdot \lambda\left(x^h, l^h\right) & = & \lambda\left(p \cdot x^h + w \cdot l^h\right) < \\ p \cdot x^h + w \cdot l^h & \leq & v_0 \cdot \bar{k}^h + w \cdot \bar{l}^h + F^h(v_1, q) \end{array}$$

Therefore, by virtue of continuity, for n large enough we must have:

$$(p_n, w_n) \cdot \lambda\left(x^h, l^h\right) < v_{0_n} \cdot \bar{k}^h + w_n \cdot \bar{l}^h + F^h(v_{1_n}, q_n)$$

that is, $\lambda\left(x^h, l^h\right) \in B\left(\pi_n\right) \cap O$, which is a contradiction.■

For each $h \in \{1, 2, ..., H\}$, we now define the virtual net demand function as follows:

$$\begin{array}{rcl} \hat{z}_h & : & \left\{\pi \in \mathbb{R}_+^N : (p, w) \gg 0\right\} \longrightarrow \mathbb{R}^N, \text{ with} \\ \hat{z}_h & = & (x^h(\pi), \hat{k}^h(q, v_1), l_0^h(\pi) - \bar{l}_0^h, -\bar{k}^h, l_1^h(\pi) - \bar{l}_1^h, -\sigma^h(\hat{k}^h(q, v_1))), \\ \text{with } \hat{k}^h & = & \Psi^h(q, v_1), \end{array}$$

where we recall that $N = 2(C+J)+3M$. Clearly, when $q \gg 0$, we have that $\Psi^h(q, v_1) = \psi^h(q, v_1)$ and virtual net demand $\hat{z}_h$ coincides with net demand $z_h(\pi)$. Since preferences are strongly monotone, the budget set in problem (1.5.2) holds with equality. Thus, for each $\hat{z}_h = \hat{z}_h(\pi)$ we have that $\pi \cdot \hat{z}_h = 0$. Summing across all consumers yields:

$$\pi \cdot \sum_h \hat{z}_h = \pi \cdot \hat{z} = 0\,, \tag{2.5.2}$$

which represents the version of the Walras' law suitable for the economy at hand. It should be noted that homogeneity of degree zero of ψ^h and Lemma 1.5.2. imply that for each $h \in \{1, 2, ..., H\}$, $\hat{z}_h$ is homogeneous of degree zero. Finally, we remark that if π° is a price vector with some zero element corresponding to the sub-components p and/or w, then $\hat{z}_h(\pi^\circ)$, hence $z(\pi^\circ)$, is not well defined. In this case, the usual boundary conditions apply.

6 The production sector

The production sector is characterized by a fixed-coefficients technology. Current and future consumption goods, and current capital goods are produced using current and future services, respectively. Following Walras (1926), we assume that there is a single activity for each good which is produced. In addition to the production activities, we suppose that there exists one *free disposal* activity for each good and service and that the technology satisfies the assumption of irreversibility. Let $Q = 4(C+M)+2J$. The production sector can be represented by the following $N \times Q$ matrix:

$$B = \left[\begin{array}{c|c} P & -I \end{array}\right],$$

where P is a $N \times (2C+M)$ matrix of production activities, and I is the $N \times N$ identity matrix of free disposal activities. Note that P can be further decomposed as follows:

$$P = \begin{bmatrix} I_0^C & \cdot & \cdot \\ \cdot & I_1^C & \cdot \\ \cdot & \cdot & I^M \\ W_0^C & \cdot & W_0^K \\ \cdot & W_1^C & \cdot \end{bmatrix},$$

where I_t^C is the $C \times C$ output (identity) matrix for consumption goods at period t, with $t = 0,1$. I^M is the $M \times M$ output (identity) matrix for capital goods at $t = 0$; W_t^C is the $(J+M) \times C$ matrix of input coefficient for the production of consumption goods at t, with $t = 0,1$;W_0^K is the $(J+M) \times M$ matrix of input coefficients for the production of capital goods at $t = 0$. The dots denote matrices of suitable dimension with all entries equal to zero. We also let:

$$W_t^C = \begin{bmatrix} a_t^1 & \vdots & a_t^c & \vdots & a_t^C \\ b_t^1 & \vdots & b_t^c & \vdots & b_t^C \end{bmatrix},$$

where $a_t^c = (a_t^{c,1}, ...a_t^{c,j}, ...a_t^{c,J})$ and $b_t^c = (b_t^{c,1}, ...b_t^{c,m}, ...b_t^{c,M})$ are the input coefficients for labor and capital services, respectively, used in the production of consumption good c in period t. Similarly, we let:

$$W_0^K = \begin{bmatrix} a_0^1 & \vdots & a_0^m & \vdots & a_0^M \\ b_0^1 & \vdots & b_0^m & \vdots & b_0^M \end{bmatrix},$$

where $a_0^m = (a_0^{m,1}, ...a_0^{m,j}, ...a_0^{m,J})$ and $b_0^m = (b_0^{m,1}, ...b_0^{m,m'}, ...b_0^{m,M})$ denote the input coefficients for labor and capital services, respectively, used in the production of capital good m at $t = 0$. We assume that, for all t, c, m, input coefficients are non-positive. Clearly,

$$Y = \left\{ B\bar{y} : \bar{y} \in \mathbb{R}_+^Q \right\} \tag{*}$$

is the production set of the representative firm ($\bar{y} \in \mathbb{R}_+^Q$ is a vector of non-negative activity levels). We posit two fairly natural assumptions on the activity-analysis matrix B:

Assumption 1.6. (NO OUTPUTS WITHOUT ANY INPUTS):

$$\left\{ B\bar{y} \in \mathbb{R}^N : \bar{y} \geq 0,\ B\bar{y} \geq 0 \right\} = \{0\}.$$

Assumption 2.6. (CURRENT AND FUTURE INDISPENSABILITY): *For every capital good m produced at $t = 0$, there exists some j such that $a_0^{m,j} < 0$. For every capital good m, there is some consumption good c, produced at $t = 1$, such that $b_1^{c,m} < 0$.*

It is well-known that with constant returns to scale a solution to the firm's profit maximization problem exists only for those prices for which unitary profits of each activity are non-positive. Therefore, any candidate equilibrium price vector π must satisfy the condition:

$$\pi B \leqslant 0, \tag{2.6}$$

where $0 \in \mathbb{R}^Q$. For every π satisfying (2.6) the supply of the representative firm is given by $y = B\bar{y} \in \mathbb{R}^N$, with $\bar{y} \in \mathbb{R}_+^Q$.Clearly, if π satisfies (2.6), so does $\lambda\pi$ for $\lambda > 0$. Hence, the supply correspondence is homogeneous of degree zero in π. Clearly, if some activity makes negative profits at the prevailing prices, the firm does not activate it at all. Therefore, positive activity levels can arise in equilibrium only if profits are zero.

7 Intertemporal equilibrium

We know, from the discussion in paragraphs 5 and 6, that $\hat{z}^h$ is homogeneous of degree zero, and so is the supply correspondence of the representative firm. Thus, we can normalize prices as follows:

$$\Delta_+^{N-1} := \{\pi \in \Delta^{N-1} \,:\, (p, w) \gg 0\} \tag{1.7}$$

where:

$$\Delta^{N-1} := \{\pi \in \mathbb{R}_+^N \,:\, \sum_n \pi_n = 1\}$$

is the unitary simplex in $\mathbb{R}^N$. Define now the aggregate virtual net demand function $\hat{z}$ as follows:

$$\hat{z} : \Delta_+^{N-1} \longrightarrow \mathbb{R}^N, \text{ with } \hat{z} = \sum_h \hat{z}_{h.}$$

Recall that prices must satisfy condition (2.6) in order for the representative firm's supply to be well-defined. With this in mind, we are now ready to introduce the notion of equilibrium.

Definition 1.7. *A vector* $(\pi^*, \bar{y}^*) \in \Delta_+^{N-1} \times \mathbb{R}_+^Q$ *constitutes a virtual competitive equilibrium if:* (*i*) $\hat{z} = B\bar{y}^*$; (*ii*) $\hat{z} = \sum_h \hat{z}_{h.}(\pi^*)$; (*iii*) $\pi^* B \leqslant 0$.

Remark 1.7. *Since the matrix B includes the negative identity matrix, and* $\bar{y}^* \in \mathbb{R}_+^Q$, *condition* (*i*) *asserts that supply is greater than or equal to aggregate demand on each market, and we have market clearing on any market for which the price is positive. Walras' law (see 2.5.2* above) *and condition* (*i*) *imply that* $\pi^* B\bar{y}^* = 0$. *In other words, at prices* π^*, $\bar{y}^*$ *maximizes the representative firm's profit. Clearly,* $\hat{z} = \sum_h \hat{z}_{h.}(\pi^*)$ *is the condition that each consumer optimizes at prices* π^*.

However, we are ultimately interested in actual equilibria. Therefore, we need to study when the virtual net demand coincides with the actual one, that is when $\Psi^h(q, v_1) = \psi^h(q, v_1)$. Since this case happens when $q \gg 0$, we introduce the following subset of the unitary simplex:

$$\Delta_{++}^{N-1} = \{\pi \in \Delta^{N-1} \,:\, (p, w, q) \gg 0\}, \tag{1.7}$$

and we propose the following definition:

Definition 2.7. *A vector* $(\pi^*, \bar{y}^*) \in \Delta_{++}^{N-1} \times \mathbb{R}_+^Q$ *constitutes a competitive intertemporal equilibrium if:* (i) $z = B\bar{y}^*$*;* (ii) $z = \sum_h z_{h.}(\pi^*)$; (iii) $\pi^* B \leqslant 0$.

To identify the conditions that guarantee that $q \gg 0$, the following lemma is of fundamental importance:

Lemma 1.7. *Assume that Assumptions 1.3., 1.4, and 2.6. hold. If* $\pi^* = (p^*, q^*, w_0^*, v_0^*, w_1^*, v_1^*)$ is *an virtual competitive equilibrium price vector, then* $(q^*, v_1^*) \in \mathbb{R}_{++}^{2M}$.

Proof: Suppose first, by way of obtaining a contradiction, that there is some m such that $q_m^* = 0$. Then, condition (iii) in definition 1.7. implies that

$$w_0^* \cdot a_0^m + v_0^* \cdot b_0^m \leq 0 \tag{2.7}$$

But assumption 2.6. implies that (2.7) indeed holds with strict inequality, that is $w_0^* \cdot a_0^m + v_0^* \cdot b_0^m < 0$. Therefore, capital good m is not produced (firm's profit maximization). On the other hand, we know from consumers' problem (1.5.1) and Assumption 1.4., that the aggregate demand for capital good m is strictly positive. This contradicts condition (i) in Definition 1.7. Next, suppose, by way of obtaining a contradiction, that there is some m such that $v_{1m}^* = 0$. In this case, it follows from (1.5.1) that aggregate demand for capital good m is equal to zero. Thus, it follows from Assumption 1.4. that no service of capital good m is available at $t = 1$. By Assumption 2.6., there exists some consumption good, say c, whose supply, at $t = 1$, is equal to zero. So, by virtue of condition (i) in Definition 1.7., aggregate demand for consumption good c must be equal to zero as well. But this contradicts Assumption 1.3. ■

Lemma 1.7. implies at least two important consequences. First of all, the fact that $(q, v_1) \gg 0$ implies that the rates of return on capital goods will be well-defined. On the relevance of this finding see, however, section 8. Secondly, by the same fact it follows that $\Psi^h(q, v_1) = \psi^h(q, v_1)$ for all h, so that we can readily state:

Corollary 2.7. *Assume that Assumptions 1.3., 1.4, and 2.6. hold. Then, if a virtual competitive equilibrium exists, it will be a competitive intertemporal equilibrium.*

As a consequence of the above corollary, it only remains to state our main result, whose proof is shown in the appendix:

Theorem 1.7. *Assume that Assumptions 1.3., 1.4, and 2.6. hold. Then, a virtual competitive equilibrium exists.*

Remark 1.7 *If we remove, in this intertemporal framework, the possibility of storing capital goods with a decreasing returns to scale technology, then, for a competitive equilibrium to be well defined, it is necessary to impose that all capital goods (if demanded) yield the same return. This would imply, first of all, that every consumer is indifferent about the composition of capital stock purchased, as in the original walrasian model. In the intertemporal equilibrium formulation, however, this also implies that the generic consumer is indifferent between purchasing and not purchasing these goods at all. This happens because, with complete forward markets, current expenditure must be equal to present-value returns. Hence expenditures on capital goods and revenues from capital services cancel out in the intertemporal budget constraint. Indeed, if this were not the case, consumers' demand for some capital good will be either unbounded or null.*

Introducing a storage technology enable us to overcome the afore-mentioned indeterminacies. Moreover, it makes it possible to formalize the demand for capital goods as well defined functions, thus making it possible to simplify the traditional definition of equilibrium. On the other hand, the proof of existence of equilibrium turns out to be different from the existing ones in the related literature (see e.g. Morishima (1964), Morishima (1977) and Zaghini (1993)).

8 On the rates of return

In a competitive intertemporal equilibrium, purchased and stored capital goods yield the same (marginal) rate of return to, and across, each consumer. To see this, note that in equilibrium, if any capital good is purchased and stored, then the amount of it cannot be equal to the upper bound given by the storage capacity constraint. For, if this were the case, then it would follow from Assumption 1.4.[2] and optimization problem (1.5.1) that the price of the capital good at hand would be less than or equal to zero. But this

[2]Recall that the left-derivative of the storage function at the capacity constraint is assumed to be zero.

contradicts Lemma 1.7. In other words, any purchased and stored capital good is an interior solution to problem (1.5.1) above. Therefore, from the first order necessary and sufficient conditions for an interior optimum, we get:

$$\frac{v_1^m r_m^h(k_m^h)}{q_m} = \frac{v_1^{m'} r_{m'}^h(k_{m'}^h)}{q_{m'}} = \frac{v_1^m r_m^{h'}(k_m^{h'})}{q_m} = \frac{v_1^{m'} r_{m'}^{h'}(k_{m'}^{h'})}{q_{m'}} = 1 \tag{1.8}$$

for all m, m', and $h, h^{'}$, where $r^h(k^h) := \nabla \sigma^h(k^h)$ denotes the vector of derivatives $\mathrm{d}\sigma_m^h/\mathrm{d}k_m^h$ for each h and m. It is now clear that the equality of (marginal) rates of return emerges *endogenously* in equilibrium. It is not imposed at the outset, unlike in the original Walras' approach. Furthermore, by virtue of Assumptions 1.3., 1.4., and 2.6, it's easy to see that in any intertemporal competitive equilibrium all M capital goods are produced. Therefore, the intertermporal equilibrium is characterized by uniformity of (marginal) rates of return. We can reinterpret (1.8) by rearranging it as follows:

$$\frac{v_1^m}{q_m} = \xi_m^h(k_m^h) \quad \text{for every} \quad h, m\,, \tag{2.8}$$

where $\xi_m^h(k_m^h) = 1/r_m^h(k_m^h)$ is the (real) marginal cost of capital good m to household h

It follows from problem (1.5.1) and Assumption 1.4. that $k_m > 0$ implies $v_1^m > q_m$. This result is not surprising, and stems from the hypothesis of a two-period economy and full depreciation of existing capital goods.

Another implication of our assumptions is that consumers with more efficient storage technologies for some capital good m[3] demand relatively more of capital good m.

It is dubious that our characterization of competitive equilibria conforms to the notion of long-run equilibrium emphasized by Garegnani and other neoricardian economists of the Cambridge school. However, we stress that in our model the equilibrium conditions, as well as the capitalistic structure, are a persistent property of the economy. Indeed, there is no incentive for markets to re-open in future dates.

[3]Given any capital good m, one says that consumer h is equipped with a more efficient storage technology than consumer $h^{'}$, if $\mathrm{d}\sigma_m^h/\mathrm{d}k_m^h > \mathrm{d}\sigma_m^{h^{'}}/\mathrm{d}k_m^{h^{'}}$.

Conceivably, under more general assumptions on preferences and technology, our conditions (2.8) would not hold in general. For instance, some capital goods may not be produced in equilibrium. In this case, conditions (1.8) would fail to hold. Consequently, an intertemporal competitive equilibrium would not be characterized by full equality of marginal rates of return.

This begs the following question: would it be correct to interpret the lack of equality of marginal rates of return as '*the symptom of a deeper deficiency of the walrasian conception of capital endowment: that of taking as an independent variables initial stocks of capital goods*' (see Garegnani (2005, p.423))? To answer this question, suppose that conditions (2.8), and the corresponding conditions in the original Walras' model, are taken to posit the equality of rates of profits, much like in Smith and Marshall. That is, let's say that the above conditions are to be thought of as "*the neoclassical version of the traditional version of a normal, or "natural" position of the economy ... on which economic thinking has relied since its conception*" (see Garegnani (2003)). Then, if this is the case, the above criticism would be well grounded indeed.

However, Garegnani's viewpoint of the long-run equilibrium is far apart from the walrsasian paradigm. In line with the walrasian model, which is entirely expressed in terms of demand and supply functions, equations (1.8) are needed exclusively to determine the optimal quantity of capital goods chosen by consumers. Likewise, in the original version of Walras' model equality of rates of return was *imposed* in order to determine the capital goods demanded by the consumers, due to the impossibility of deriving well defined demand functions.

Thus, we believe that, even under more general hypotheses, Definition 2.7. applies also to the long-run, when preferences and technology should pinpoint the goods which are scarce and the goods that are not scarce.

Remark 1.8. *Since we assume that capital goods totally depreciate, no arbitrage between old and new capital goods can take place. It follows that, in general, services of some capital good may be demanded in the current period and not in the future period, and vice versa.*

Remark 2.8 *Since strict monotonicity of preferences and assumption 2.6. are essentially equivalent to the assumptions that rule out free goods in a standard walrasian models of exchange and production, Lemma 1.7. implies that the nature of all goods is alike, regardless of how good are identified at the outset (as consumption or capital goods).*

9 Concluding remarks

The thrust of our contribution is that a standard Arrow-Debreu framework lends itself to encompass Walras' theory of capital accumulation as a theory of the long-run. We have argued that whether the capitalistic structure of the economy is stable or changes over time depends upon the assumptions on preferences and technology.

One of the main innovations we have introduced is the storage technology. Consumers funnel savings into investment by means of the storage functions. No indeterminacy arises in this transformation process, nor is the (quite contrived) aggregation of savings in a fictitious commodity necessary. We envision, therefore, that the storage technology can play a key role in the extension of Walras' model of capital accumulation to an infinite-horizon economy. We believe it is possible to extend the model without confinying the analysis to restrictive notions of equilibrium such as a steady-state, or stationary, equilibrium.

We have shown that the walrasian model can be formalized in terms of well-defined demand functions for all of the goods and services. This renders the proof of the existence of equilibria more in line with the state of the art in mainstream general equilibrium theory. We underscore, though, that our model veers away from a standard equilibrium model in two ways. First, not all of the goods and services traded affect consumers' preferences. Secondly, candidate equilibrium price vectors lie in a set different from the set on which the demand functions are defined. Thus, our proof of existence of equilibria resorts to techniques more involved than the methods based on Brouwer and Kakutani fixed point theorems.

Our work should be thought of as the first step toward resuming and reviving the walrasian theory of capital accumulation. We believe that it is worth conducting further research to cast Walras' theory in an overlapping-generation environment. In such a framework, a sequence of generations that overlap with one another can provide a compelling motivation for the hypotesis of given endowments of capital goods.

10 Appendix

In this section we prove Theorem 1.7. As mentioned, we build on the original approach of Todd (1979), but we extend it to the case of strongly monotone preferences. Recall that $N = 2(C+J)+3M$ and that:

$$\Delta^{N-1} = \left\{ \pi \in \mathbb{R}^N_+ : \sum_n \pi_n = 1 \right\}.$$

Let us introduce the following subset of Δ^{N-1}:

$$\Delta^{\varepsilon} = \left\{ \pi \in \Delta^{N-1} : (p, w) \geq \varepsilon \widehat{u} \right\},$$

where $\varepsilon \in (0,1)$ is a suitably chosen scalar and $\widehat{u} \in \mathbb{R}^{2(C+J)}_{++}$ is the unitary vector.[4] The trimmed simplex Δ^{ε} is clearly a convex and compact set. Consider now the following convex and compact set:

$$\Pi = \left\{ \pi \in \Delta^{N-1} : \pi B \leqslant 0 \right\},$$

and let:

$$\Pi^+ = \Pi \cap \Delta^{\varepsilon}.$$

It is fundamental to establish that Π^+, and hence Π, is a non-empty set. To this end, we propose the following:

Lemma 1.A. If assumption 1.6 holds, then there exists $\pi \in \mathbb{R}^N_{++}$ such that $\pi B \leqslant 0$.

*Proof.*The proof is based on an application of the Farkas' Lemma (see for instance Aliprantis and Border (2006, Corollary 5.85)). For each $i = 1, 2, ...N$ pick the vector $b^i = (0, ...0, 1, 0, ..., 0) \in \mathbb{R}^N_+$ where the scalar 1 is the i-th component of b^i. Given Assumption 1.6 it should be clear that the claim *"there exists a vector $\overline{y} \in \mathbb{R}^Q_+$ such that $B\overline{y} = b^i$"* is false. Therefore, by Farkas' Lemma there exists a non-zero vector $\pi^i \in \mathbb{R}^N$ that satisfies:

[4] We choose ε according to the following criterion: we compute the maximal supply for each good and service based on techonolgy and endowment of fixed factors. Then we identify the prices that make demand for goods and services greater than the above mentioned capacity limit. Finally we pick the smallest value of ε such that each market dispalys excess demand.

$$\pi^i B \leqslant 0 \text{ and } \pi^i b^i = \pi^i_i > 0. \tag{2}$$

Hence, since B includes the negative identity matrix, we must have $\pi^i \in \mathbb{R}^N_+$ with $\pi^i_i > 0$. To finish the proof define $\pi \in \mathbb{R}^N$ by $\pi = \sum_{i=1}^N \pi^i$ and note that $\pi \in \mathbb{R}^N_{++}$ and $\pi B = \sum_{i=1}^N \pi^i B \leqslant 0$. For an appropriate choice of ε we finally have that $\pi \in \Pi^+$. ■

It is straightforward to see that Π^+, as the intersection of convex and closed sets, it is convex, hence connected, and closed. Given this peculiar property of Π^+, and those relating Π, we can conclude that all the hypothesis in lemma A.2.3 in Magill and Quinzii (1996, p.121) are satisfied. Therefore there exists a continuous function $\alpha : \Pi \rightarrow [0, 1]$ such that:

$$\begin{aligned} \alpha(\pi) &= 1 \text{ if } \pi \in \Pi^+ \\ \alpha(\pi) &= 0 \text{ if } \pi \notin U \subset \{\pi \in \Pi \, : (p, w) \gg 0\}, \end{aligned}$$

where U is an open set of Π including Π^+. Next, let us consider the affine hull of Π :[5]

$$\mathcal{T} := \left\{ \tau \in \mathbb{R}^N \, : \, \sum_n \tau_n = 1 \right\}.$$

Following Todd (1979), now we introduce on this set the metric projection map $\varphi : \mathcal{T} \rightarrow \Pi.$, which associates to every $\tau \in \mathcal{T}$ the vector $\varphi(\tau) \in \Pi$ closest to τ. This function, that obviously coincides with the identity map of Π for any $\tau \in \Pi$, is continuous (see for instance Aliprantis and Border (2006, pages 247-48)). Next, consider the composition of mappings $f : \mathcal{T} \rightarrow \mathcal{T}$ defined by:

$$f(\varphi(\tau)) = \varphi(\tau) + \hat{z}(\varphi(\tau)) - ((u \cdot \hat{z}(\varphi(\tau)))/N)u,$$

where u is the unitary vector of $\mathbb{R}^N_{++}$. Clearly, this function is well defined and continuous whenever $\hat{z}(\tau)$ so is, that is, as a consequence of our assumptions, when $(p, w) \gg 0$. The map $\alpha(\varphi(\tau))$, however, allows us to continuously extend $f(\varphi(\tau))$ on $\mathcal{T}$. Thus, picking an arbitrary price vector $\widehat{\pi} \in \Pi^+$, we define by $F : \mathcal{T} \rightarrow \mathcal{T}$ the function:

[5] Given a nonempty subset S of a vector space X, the affine hull of S, is the set of all affine combinations of finitely many members of S, i.e. it is the set defined as follows: $\left\{\sum_{x \in T} \lambda(x)x : T \in P(S)\right\}$ and $\lambda \in \mathbb{R}^T$ with $\sum_{x \in T} \lambda(x) = 1$, where $P(S)$ is the class of all nonempty finite subsets of S.

$$F(\tau) = \alpha(\varphi(\tau))f(\varphi(\tau)) + (1 - \alpha(\varphi(\tau)))(\kappa\varphi(\tau) + (1-\kappa)\widehat{\pi}),$$

where $\kappa \in (0,1)$. This function is always well defined and continuous. Indeed, if $\varphi(\tau) \in \Pi^+$, then $\alpha = 1$ and $F(\tau) = f(\varphi(\tau))$, while if $f(\varphi(\tau) \notin U$ (and therefore not necessarily $(p,w) \gg 0$), then $\alpha = 0$ and $F(\tau) = \kappa\varphi(\tau)+(1-\kappa)\widehat{\pi}$. This "machinery" allows us to prove the following proposition:

Lemma 2.A $F : \mathcal{T} \to \mathcal{T}$ has a fixed point, i.e. there exists a $\tau^* \in \mathcal{T}$ such that $F(\tau^*) = \tau^*$

Proof. First note that F is continuous as it is the composition of two continuous functions. Clearly $\mathcal{T}$ is a non-empty subset of $\mathbb{R}^N$ which is a locally convex Hausdorff topological vector space. Since $\varphi : \mathcal{T} \to \Pi$ is onto, we have that $\varphi(\mathcal{T}) = \Pi$. Therefore $F(\mathcal{T}) = F(\varphi(\mathcal{T})) = F(\Pi)$. But Π is compact and F is continuous, thus $F(\mathcal{T})$ is a compact subset of $\mathcal{T}$. Now pick any compact convex subset of $\mathcal{T}$, say C, that contains $F(\mathcal{T})$. Since $F(\tau) \in C$ for each $\tau \in C$, we can invoke Corollary 1 in Tian (1991), thus concluding the proof. ■

The next propositions are decisive for the outcome we want to achieve.

Lemma 3.A Under the maintained assumptions, if $\tau^* = f(\varphi(\tau^*))$ then τ^* is a virtual competitive equilibrium price vector.

Proof. Suppose there exists a $\tau^* \in \mathcal{T}$ such that $f(\varphi(\tau^*)) = \tau^*$. We claim that $\varphi(\tau^*)$ is an equilibrium price. To see this, notice that $\varphi(\tau^*)$ solves the following constrained optimization problem:[6]

$$\min \left\{ \frac{1}{2}(\pi - \tau^*) \cdot (\pi - \tau^*) \mid \pi B \leqslant 0\,,\ \pi \cdot u = 1 \right\}.$$

From the Kuhn-Tucker theorem, it follows that there exist $y^* \in \mathbb{R}^Q_+$ and $\gamma \in \mathbb{R}$ that satisfy:

$$\tau^* - \varphi(\tau^*) = By^* + \gamma u \quad \text{with} \quad \varphi(\tau^*)By^* = 0\,. \tag{3}$$

Since $f(\varphi(\tau^*)) = \tau^*$, (3) can be rewritten as follows:

$$\hat{z}(\varphi(\tau^*)) - (u \cdot \hat{z}(\varphi(\tau^*))/N)u = By^* + \gamma u \quad \text{with} \quad \varphi(\tau^*)By^* = 0\,. \tag{4}$$

[6] Since B includes the negative identity matrix, it is straightforward to see that if a price vector π satisfies $\pi B \leqslant 0$ then the condition $\pi \geqslant 0$ is automatically satisfied.

Taking in (4) the inner product with $\varphi(\tau^*)$ and using Walras' law yields $\gamma = -(u \cdot \hat{z}(\varphi(\tau^*))/N)$ which plugged back in (4) in turn yields $\hat{z}(\varphi(\tau^*)) = By^*$. Hence $\varphi(\tau^*)$ is a virtual competitive equilibrium price vector. ■

Lemma 4.A Under the maintained assumptions τ^* is a fixed point of $F(\tau)$ if and only if τ^* is a fixed point of $f(\varphi(\tau))$.

Proof. Assume first that $\tau^* = f(\varphi(\tau^*))$.Then, since $\hat{z}(\varphi(\tau^*))$ is a well-defined function, it follows that (p^*, w^*) as elements of the vector τ^*, are positive. Moreover by lemma 3.A, $\varphi(\tau^*)$ is an equilibrium price vector. Hence it satisfies condition (ii) of definition 1.7. Therefore it must be that $\varphi(\tau^*) \in \Pi^+$ and $\alpha(\varphi(\tau^*)) = 1$. Thus, by construction, we have $f(\varphi(\tau^*)) = \tau^* = F(\tau^*)$. Next assume that $\tau^* = F(\tau^*)$. There are two cases to consider: $\varphi(\tau^*) \in \Pi^+$ and $\varphi(\tau^*) \notin U \supset \Pi^+$. We will show that only first case can arise. Assume first, by way of contradiction, that $\varphi(\tau^*) \notin U$. In this case $\alpha(\varphi(\tau^*)) = 0$, hence $F(\tau^*) = \kappa\varphi(\tau^*) + (1-\kappa)\widehat{\pi}$. But the following system of N equations in the unknown κ :

$$\kappa\varphi(\tau^*) + (1-\kappa)\widehat{\pi} - \tau^* = 0$$

has no solution except the trivial one $\varphi(\tau^*) = \widehat{\pi}$, which is clearly not possible. Indeed, to make this point clear, we remark that: (a) if $\tau^* \notin \Pi$, then $\kappa\varphi(\tau^*) + (1-\kappa)\widehat{\pi} \in \Pi$ because Π is a convex set; (b) if $\tau^* \in \Pi$, hence $\varphi(\tau^*) = \tau^*$from (1.A) we get $(\kappa - 1)\tau^* + (1-\kappa)\widehat{\pi} = 0$ that is $\tau^* = \widehat{\pi}$. But this is again impossible since $\widehat{\pi} \in \Pi^+$ and $\tau^* \notin U \supset \Pi^+$. Thus we conclude that $\tau^* \in \Pi^+$ hence $\alpha(\varphi(\tau^*)) = 1$, and therefore $\tau^* = f(\varphi(\tau^*))$. ■

We are finally ready to prove theorem 1.7.

Proof of theorem 1.7. By lemma 2.A there exists $\tau^* \in \mathcal{T}$ such that $\tau^* = F(\tau^*)$. By virtue of lemma 4.A we have that $\tau^* = f(\varphi(\tau^*))$. Finally, by lemma 3.$A$, τ^* is a virtual competitive equilibrium price vector. ■

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Pubblicato in proprio
Dipartimento di Economia Pubblica
Facoltà di Economia
Università degli Studi di Roma “La Sapienza”
Via del Castro Laurenziano 9 – 00161 Roma

ISSN 1974-2940