Block Notes of Math

On the Riemann Hypothesis

The conjecture "The non-trivial zeros of Riemann's zeta have all multiplicity 1" is true!

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Abstract

In this work the authors proof the conjecture.

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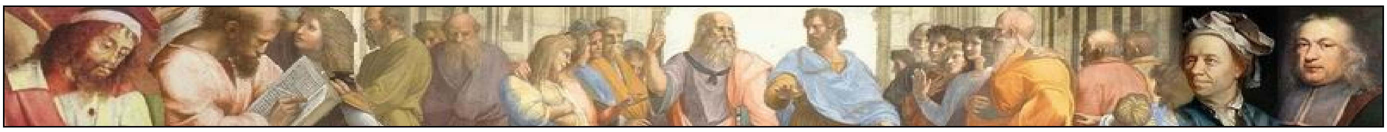

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Introduction

Riemann is today linked at two unresolved conjectures. The conjecture “The non-trivial zeros of Riemann’s zeta have all multipli city 1” is one of these.

The authors show that this conjecture is true. The proof is based on two theoretical remarks.

Remark A

We saw that Riemann defined ζ(s) as a function of complex variable s. The first step of Riemann was to extend (or to *analytically continue*) ζ(s) to all of ℂ \ {1} This can be accomplished by noticing that s=σ+it and $n^{-s} = s\int_n^\infty x^{-s-1}dx$ then:

$$\varsigma(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \sum_{n=1}^{\infty}\left(s\int_n^\infty \frac{dx}{x^{s+1}}\right) = s\sum_{n=1}^{\infty}\int_n^\infty \frac{dx}{x^{s+1}}$$

$$= s\int_1^\infty\left(\sum_{n\le x}1\right)\frac{dx}{x^{s+1}} = s\int_1^\infty \frac{[x]}{x^{s+1}}dx = s\int_1^\infty \frac{x-\{x\}}{x^{s+1}}dx \qquad (1) \qquad ([1])$$

$$= \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx,\ \sigma > 1$$

Since {x}∈[0,1), it follows that the last integral converges for σ>0 and defines a continuation of . ζ(s) to the half-plane σ=Re(s)>0. We can extend ζ(s) to a holomorphic function on all ℂ \ {1}, in fact from the last integral s=1 is a simple pole with residue 1. We note that for s real and s>0 the integral in (1) is always positive real. Then from (1) ζ(s)<0, s∈(0.1) and ζ(s)>0, s∈(1.∞).

A popular expression of Euler is:

$$\varsigma(s) = \prod_{p=prime}(1-p^{-s})^{-1}$$

$$\ln\varsigma(s) = -\sum_{p=prime}\ln(1-p^{-s}) = \sum_{p=primes}\sum_{k=1}^{\infty}\frac{p^{-ks}}{k} \qquad (2)$$

In (2) we have applied the integration of Newton linked to expression:

[1] [x] is the greatest integer ≤ x or floor of x; {x}=x-[x] is the fractional part of x.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$$

If the previous expression is integrated for x and you change the sign to bring the term 1-x to numerator, then we obtain:

$$-\log(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + ...$$

Now, we introduce the ***von Mangoldt's function*** *(also called lambda function)*:

$$\Lambda(n) = \begin{cases} \log p, & if \ n = p^k, \quad p \ \ prime, \ k \geq 1 \\ 0, & \text{otherwise} \end{cases} \qquad (3)$$

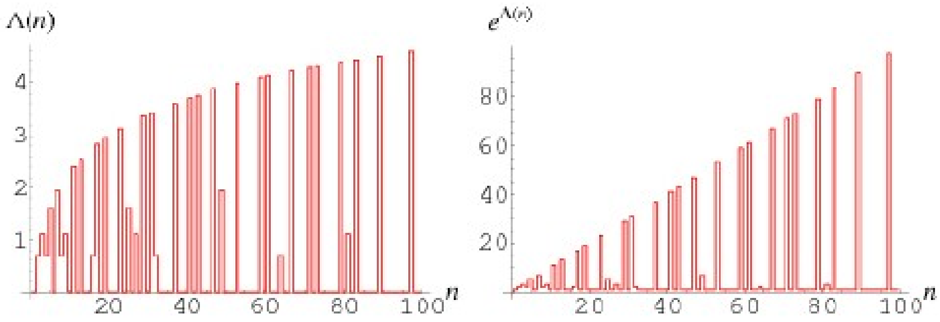


Figure 1 – von Mangoldt's function

From (2) we have:

$$p^{-ks} = \begin{cases} n^{-s}, & if \ n = p^k \\ 0, & \text{otherwise} \end{cases}$$

and if we use the rules of logarithm: $n = p^k$, $k = \log_p n = \log n / \log p$ then:

$$\frac{1}{k} = \begin{cases} \frac{\log p}{\log n} = \frac{\Lambda(n)}{\log n}, & when \ \ n = p^k \\ 0, & \text{otherwise} \end{cases}$$

Further the (2) becomes:

$$\ln \varsigma(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n \ \ n^s} = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-s} \qquad (4)$$

The (4) consents to pass from "a multiplicative problem" to "an additive problem", even if we are started from the Euler's product.

Consequently if we do the derivative of (4) then we obtain:

$$\frac{\varsigma'(s)}{\varsigma(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \qquad (5)$$

Remark B

If we have a polynomial of any degree, with real or complex variables, the search for roots is possible do it by various methods, for example:

• iterative method
• Newton's Method
• Method of Sturm's Theorem and related
• etc.

Hereafter we use only the method of Newton (see [2]).

We remember that if f(z) is a function and α is a root such that f(α) = 0 then it is possible to express for meromorphic functions $f:\mathbb{C}\to\overline{\mathbb{C}}$ a **Newton's map** N_f (z) as follows:

$$N_f(z) = z - \frac{f(z)}{f'(z)} \qquad (6)$$

Now we know that:

- If α is a simple root (multiplicity 1) of f(z) then f(α) = 0 and N_f (α) = 0, N'_f (α) = α and
$$N_f(z) - \alpha = O((z-\alpha)^2), \quad z \to \alpha$$
- If α is a root with multiplicity greather than 1 of f(z) then f(α) = 0, N_f (α) = α, $|N'_f(\alpha)|<1$ and
$$|N_f(z) - \alpha| \le C|z-\alpha|, \quad 0 < C < 1,\ z \to \alpha$$

Proof of the conjecture "The non-trivial zeros of Riemann's zeta have all multiplicity 1"

The proof is based on the Newton's Method.

In general, if we were interested at the values of the roots, it would be possible, with several iterations, to start from a value z0 and to arrive, and arrive at a n-th term that Nfn(α)=α; but we don't need to find the values of the roots but only say something about the multiplicity of these roots.

By Remark A, in the case of the Riemann's zeta the (6) becomes:

$$N_\varsigma(z) = z - \frac{\varsigma(z)}{\varsigma'(z)} \qquad (7)$$

The (7) through (5) of Remark A becomes:

$$N_\varsigma(z) = z + \frac{1}{\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}} \qquad (8)$$

Then:

$$N_\varsigma(z) = z + \frac{1}{\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^z}} \qquad (9)$$

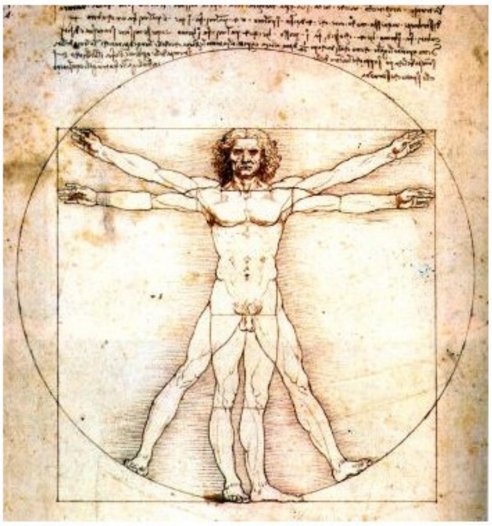
In the (9) is:
$$N_\varsigma(\alpha) \sim \alpha \qquad (10)$$

because $$\frac{1}{\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\alpha}}} << 1$$

where for the von Mongoldt's function in (9) is defined as in (3); we have a sum of functions of von Mongoldt (so the sum at denominator is not null). In addition, at the second member of (9) there are only constants, then we have:

$$N'_{\varsigma}(\alpha) = 0 \qquad (11)$$

By Remark B with (10) and (11) we conclude the conjecture "The non-trivial zeros of the Riemann's zeta have all multiplicity 1" is true!.



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Sites

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